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COEFFICIENT ESTIMATES FOR CERTAIN CLASSES OF MEROMORPHIC BI-UNIVALENT FUNCTIONS

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ABSTRACT
In this paper, we propose to investigate the coefficient estimates for two subclasses of meromorphic bi-univalent functions. Also, we estimate the upper bounds for the coefficients $|b_0|$ *and* $|b_1|$ *for functions in the defined classes. Also interesting remarks of the obtained results are discussed.*

2010 Mathematics Subject Classification: 30C45.

Keywords and Phrases: Analytic functions, univalent functions, bi-univalent functions, meromorphic functions, meromorphic bi-univalent functions.

1. INTRODUCTION

Let A denote the class of functions of the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1)$$

which are analytic in the open unit disc U = $\{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, by S we shall denote the class of all functions in A which are univalent in U.

It is well known that every function $h \in S$ has an inverse h^{-1}, defined by

$$h^{-1}(h(z)) = z \qquad (z \in U)$$

and

$$h(h^{-1}(w)) = w \quad (|w| < r_0(h);\ r_0(h) \geqq \frac{1}{4}),$$

where

$$h^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots \qquad (1.2)$$

A function $h \in S$ is said to be bi-univalent in U if both $h(z)$ and $h^{-1}(z)$ are univalent in U. Let Σ_B denote the class of bi-univalent functions in U given by (1.1). For a brief history and interesting examples of functions which are in (or which are not in) the class Σ_B, together with various other properties of the bi-univalent function class Σ_B one can refer the work of Srivastava et al. [24] and references therein. In fact, the study of the coefficient problems involving bi-univalent functions was revived recently by Srivastava et al. [24].

Various subclasses of the bi-univalent function class Σ_B were introduced and non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (1.1) were found in several recent investigations (see, for example, [1, 2, 3, 5, 6, 8, 12, 13, 20, 22, 23, 26, 28]).

The aforecited all these papers on the subject were actually motivated by the pioneering work of Srivastava et al. [24]. However, the problem to find the coefficient bounds on $|a_n|$ (n = 3,4,...) for functions $h \in \Sigma_B$ is still an open problem. Let Σ denote the class of functions of the form

$$f(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n}, \qquad (1.3)$$

which are mermorphic univalent defined in

V := $\{z : z \in \mathbb{C} \text{ and } 1 < |z| < \infty\}$.

It is well known that every function $f \in \Sigma$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in V)$$

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and

$f^{-1}(f(w)) = w \qquad (M < |w| < \infty,\ M > 0).$

Furthermore, the inverse function f^{-1} has a series expansion of the form

$$f^{-1}(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n}, \qquad (1.4)$$

Where $M < |w| < \infty$.

The coefficient problem was investigated for various interesting subclasses of the meromorphic univalent functions (see, for example [4, 10, 17]). In 1951, Springer [21] conjectured on the coefficient of the inverse of meromorphic univalent functions, latter the problem was investigated by many researchers for various subclasses (see, for example [9, 10, 11, 13, 15, 16, 18, 19, 25]).

Analogous to the bi-univalent analytic functions, a function $f \in \Sigma$ is said to be meromorphic bi-univalent if both f and f^{-1} are meromorphic univalent in V. We denote by Σ_M the class of all meromorphic bi-univalent functions in V given by (1.3).

A function f in the class Σ is said to be meromorphic bi-univalent starlike of order

α $(0 \leq \alpha < 1)$ if it satisfies the following inequalities:

$$f \in \Sigma_{\mathcal{M}}, \quad \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathcal{V} \text{ and } \Re\left(\frac{wg'(w)}{g(w)}\right) > \alpha \quad (w \in \mathcal{V}),$$

where $g(w) = f^{-1}(w)$ is the inverse of $f(z)$ whose series expansion is given by (1.4), a simple calculation shows that

$$g(w) = w - b_0 - \frac{b_1}{w} - \frac{b_2 + b_0 b_1}{w^2} - \frac{b_3 + 2b_0 b_2 + b_0^2 b_1 + b_1^2}{w^3} + \cdots. \qquad (1.5)$$

We denote by $\Sigma^*_M(\alpha)$ the class of all meromorphic bi-univalent starlike functions of order α. Similarly, a function f in the class Σ is said to be meromorphic bi-univalent strongly starlike of order α $(0 < \alpha \leq 1)$ if it satisfies the following conditions:

$$f \in \Sigma_{\mathcal{M}}, \quad \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \quad (z \in \mathcal{V} \text{ and } \left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \quad (w \in \mathcal{V})$$

where $g(w)$ is given by (1.5). We denote by $\check{\Sigma}^*_M(\alpha)$ the class of all meromorphic biunivalent strongly starlike functions of order α. The classes $\Sigma^*_M(\alpha)$ and $\check{\Sigma}^*_M(\alpha)$ were introduced and studied by Halim et al. [7].

Motivated by the works of Srivastava et al. [24] and C¸aˇglar et al. [3] and inspired by the work of Halim et al. [7], we define the following general subclasses $\Sigma^*_M(\alpha,\mu,\lambda)$ and $\check{\Sigma}^*_M(\alpha,\mu,\lambda)$ of the function class Σ_M.

Definition 1.1. A function f given by (1.3) is said to be in the class $\Sigma^*_M(\alpha,\mu,\lambda)$ if the following conditions are satisfied:

$$\Re\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} + \xi\delta z f''(z)\right) > \alpha$$

$$f \in \Sigma_M, \qquad (\mu \geqq 0,\ \lambda \geqq 1,\ \lambda > \mu;\ z \in \mathcal{V}) \qquad (1.6)$$

and

$$\Re\left((1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} + \xi\delta w g''(w)\right) > \alpha$$

$$(\mu \geqq 0,\ \lambda \geqq 1,\ \lambda > \mu;\ w \in \mathcal{V}) \qquad (1.7)$$

for some α $(0 \leq \alpha < 1)$, where g is given by (1.5).

Definition 1.2. A function f given by (1.3) is said to be in the class $\check{\Sigma}^*_M(\alpha,\mu,\lambda)$ if the following conditions are satisfied:

$$f \in \Sigma_{\mathcal{M}}, \quad \left|\arg\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} + \xi\delta z g''(z)\right)\right| < \frac{\alpha\pi}{2}$$

$$(\mu \geqq 0,\ \lambda \geqq 1,\ \lambda > \mu; z \in \mathcal{V}) \qquad (1.8)$$

and

$$\left|\arg\left((1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} + \xi\delta w g''(w)\right)\right| < \frac{\alpha\pi}{2}$$

$$(\mu \geqq 0,\ \lambda \geqq 1,\ \lambda > \mu; w \in \mathcal{V} \qquad (1.9)$$

for some α ($0 < \alpha \leq 1$), where g is given by (1.5).

It is interesting to note that, for $\lambda = 1$ and $\mu = 0$ the classes $\Sigma^*_M(\alpha,\mu,\lambda)$ and

$\breve{\Sigma}^*_M(\alpha,\mu,\lambda)$ respectively, reduces to the classes $\Sigma^*_M(\alpha)$ and $\breve{\Sigma}^*_M(\alpha)$ introduced and studied by Halim et al. [7]. In the present paper we extend the concept of bi-univalent to the classes of meromorphic functions defined on V and find estimates on the coefficients

$|b_0|$ and $|b_1|$ for functions in the above-defined classes $\Sigma^*_M(\alpha,\mu,\lambda)$ and $\breve{\Sigma}^*_M(\alpha,\mu,\lambda)$ of the function class Σ_M by employing the techniques used earlier by Halim et al. [7]. In order to derive our main results, we shall need the following lemma.

Lemma 1.3. [14] *If* $\phi \in$ P, *then* $|c_k| \leq 2$ *for each k, where* P *is the family of all functions* ϕ, *analytic in* U, *for which*

$<(\phi(z)) > 0 \qquad (z \in$ U),

where

$\phi(z) = 1 + c_1 z + c_2 z^2 + \cdots \qquad (z \in$ U).

2. Coefficient Bounds for the Function Classes $\Sigma^*_M(\alpha,\mu,\lambda)$ and $\breve{\Sigma}^*_M(\alpha,\mu,\lambda)$

We begin this section by finding the estimates on the coefficients $|b_0|$ and $|b_1|$ for functions in the class $\Sigma^*_M(\alpha,\mu,\lambda)$.

Theorem 2.1. *Let the function f(z) given by* (1.3) *be in the following class*:

$\Sigma^*_M(\alpha,\mu,\lambda) \qquad (0\ 5\ \alpha < 1; \lambda = 1; \mu = 0; \qquad \lambda > \mu).$

Then

(2.1) $$|b_0| \leqq \frac{2(1-\alpha)}{\lambda-\mu}$$

$$|b_1| \leqq 2(1-\alpha)\sqrt{\frac{(1-\mu)^2(1-\alpha)^2(2\lambda-\mu)^2}{(\lambda-\mu)^4(2\lambda-\mu-2\xi\delta)^2} + \frac{1}{(2\lambda-\mu-2\xi\delta)^2}}$$

Proof. It follows from (1.6) and (1.8) that

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} + \xi\delta z f''(z) = \alpha + (1-\alpha)p(z) \qquad (2.2)$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} + \xi\delta w g''(w) = \alpha + (1-\alpha)q(w), \qquad (2.3)$$

where $p(z)$ and $q(w)$ are functions with positive real part in V and have the following forms:

$$p(z) = 1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \cdots \qquad (2.4)$$

$$q(w) = 1 + \frac{q_1}{w} + \frac{q_2}{w^2} + \cdots$$

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Respectively Now, equating coefficients in (2.2) and (2.3), we get

$$(\mu-\lambda)b_0 = (1-\alpha)p_1 \tag{2.5}$$

$$(\mu-2\lambda+2\xi\delta)b_1 + (\mu-1)(\mu-2\lambda)\frac{b_0^2}{2} = (1-\alpha)p_2 \tag{2.6}$$

$$(\lambda-\mu)b_0 = (1-\alpha)q_1 \tag{2.7}$$

$$(2\lambda-\mu-2\xi\delta)b_1 - (\mu-1)(2\lambda-\mu)\frac{b_0^2}{2} = (1-\alpha)q_2 \tag{2.8}$$

.

From (2.5) and (2.7), we get

$$p_1 = -q_1 \tag{2.9}$$

and

$$b_0^2 = \frac{(1-\alpha)^2(p_1^2+q_1^2)}{2(\lambda-\mu)^2}. \tag{2.10}$$

Since $\mathcal{R}(p(z)) > 0$ in V, the function $p(1/z) \in$ P and hence the coefficients p_n and similarly the coefficients q_n of the function q satisfy the inequality in Lemma 1.3, therefore we readily get

$$|b_0| \leqq \frac{2-2\alpha}{\lambda-\mu}.$$

This gives the bound on $|b_0|$ as asserted in (2.1).

Next, in order to find the bound on $|b_1|$, we use (2.8) and (2.10), which yields,

$$(1-\mu)^2(2\lambda-\mu)^2b_0^4 - 4(1-\alpha)^2p_2q_2 = 4(2\lambda-\mu-2\xi\delta)^2b_1^2. \tag{2.11}$$

It follows from (2.13) that

$$b_1^2 = \frac{(1-\mu)^2(2\lambda-\mu)^2b_0^4}{4(2\lambda-\mu-2\xi\delta)^2} - \frac{(1-\alpha)^2}{(2\lambda-\mu-2\xi\delta)^2}p_2q_2.$$

Substituting the estimate obtained (2.10), and applying Lemma 1.3 once again for the coefficients p_2 and q_2, we readily get

$$|b_1| \leqq 2(1-\alpha)\sqrt{\frac{(1-\mu)^2(1-\alpha)^2(2\lambda-\mu)^2}{(\lambda-\mu)^4(2\lambda-\mu-2\xi\delta)^2} + \frac{1}{(2\lambda-\mu-2\xi\delta)^2}}.$$

This completes the proof of Theorem 2.1. ∎

Next we estimate the coefficients $|b_0|$ and $|b_1|$ for functions in the class $\check{\Sigma}^*_M(\alpha,\mu,\lambda)$

Theorem 2.2. *Let the function* $f(z)$ *given by* (1.1) *be in the following class*:

$\check{\Sigma}^*_M(\alpha,\mu,\lambda)$ $(0 < \alpha \leq 1; \lambda = 1; \mu = 0; \quad \lambda > \mu)$.

Then

$$|b_0| \leqq \frac{2\alpha}{\lambda-\mu} \tag{2.12}$$

and

$$|b_1| \leqq 2\alpha^2\sqrt{\frac{1}{(2\lambda-\mu-2\xi\delta)^2} + \frac{(1-\mu)^2(2\lambda-\mu)^2}{(\lambda-\mu)^4(2\lambda-\mu-2\xi\delta)^2}}. \tag{2.13}$$

Proof: It follows from (1.8) and (1.9) that

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$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}+\xi\delta z f''(z)=[p(z)]^{\alpha} \tag{2.14}$$

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}+\xi\delta w g''(w)=[q(w)]^{\alpha}$$

where $p(z)$ and $q(w)$ have the forms (2.4) .

Now, equating the coefficients ,

we get

$$(\mu-\lambda)b_0 = \alpha p_1 \tag{2.15}$$

$$(\mu-2\lambda+2\xi\delta)b_1+(\mu-1)(\mu-2\lambda)\frac{b_0^2}{2} = \frac{1}{2}[\alpha(\alpha-1)p_1^2+2\alpha p_2] \tag{2.16}$$

$$(\lambda-\mu)b_0 = \alpha q_1 \tag{2.17}$$

$$(2\lambda-\mu-2\xi\delta)b_1-(\mu-1)(2\lambda-\mu)\frac{b_0^2}{2} = \frac{1}{2}[\alpha(\alpha-1)q_1^2+2\alpha q_2] \tag{2.18}$$

.From (2.15) and (2.17), we find that

$$p_1 = -q_1 \tag{2.19}$$

and

$$b_0^2=\frac{\alpha^2(p_1^2+q_1^2)}{2(\lambda-\mu)^2} \tag{2.20}$$

As discussed in the proof of Theorem 2.1, applying Lemma 1.3 for the coefficients p_2 and q_2, we immediately have

$$|b_0| \leqq \frac{2\alpha}{\lambda-\mu}.$$

This gives the bound on $|b_0|$ as asserted in (2.14).

Next, in order to find the bound on $|b_1|$, by using (2.16) and (2.18), we get $2(2\lambda-\mu-2\xi\delta)^2{b_1}^2+(1-\mu)^2(2\lambda-\mu)^2\frac{b_0^4}{2} =$

$$\frac{1}{4}[\alpha^2(\alpha-1)^2(p_1^4+q_1^4)+\alpha^2(p_2^2+q_2^2)+\alpha^2(\alpha-1)(p_1^2p_2+q_1^2q_2)] \tag{2.21}$$

It follows from (2.21) and (2.20) that

$$\begin{aligned}2(2\lambda-\mu-2\xi\delta)^2b_1^2 = {} & \frac{\alpha^2(\alpha-1)^2(p_1^4+q_1^4)}{4}+\alpha^2(p_2^2+q_2^2)+\alpha^2(\alpha-1)(p_1^2p_2+q_1^2q_2)\\ & -\frac{(2\lambda-\mu)^2(1-\mu)^2\alpha^4}{8(\lambda-\mu)^4}(p_1^2+q_1^2)^2.\end{aligned}$$

Applying Lemma 1.3 once again for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|b_1| \leqq 2\alpha^2\sqrt{\frac{1}{(2\lambda-\mu-2\xi\delta)^2}+\frac{(1-\mu)^2(2\lambda-\mu)^2}{(\lambda-\mu)^4(2\lambda-\mu-2\xi\delta)^2}}.$$

This completes the proof of Theorem 2.2.

Remark 2.3. For $\delta = 0$, the results discussed in this paper are lead to the results obtained by Orhan et. al. [13]. For $\lambda = 1$ and $\mu = 0$, the bounds obtained in Theorems 2.1 and 2.2 are coincide with the outcome of [7, Theorem 1 and Theorem 2]. Similarly, various interesting consequences could be derived from our results, the details involved may be left to the reader.

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EFFECT OF MAGNETIC FIELD ON THE PERISTALTIC FLOW OF A FRACTIONAL SECOND GRADE FLUID THROUGH INCLINED CYLINDRICAL TUBE

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ABSTRACT
In this paper, we study the effect of magnetic field on the peristaltic flow of a fractional second grade fluid through inclined cylindrical is analyzed. The analytical solution for fractional parameter, magnetic field and inclined channel has been found in closed form by employing long wavelength and low Reynolds number assumptions. A discussion for pressure rise and frictional forces is provided through numerical integration. Finally, the effects of various key parameters are discussed with the help of graphs.

Keywords: Peristalsis; Fractional second grade model; Pressure; Friction force; Inclined channel ; Magnetic field; Caputo's fractional derivative.

INTRODUCTION
The study of fluid transport by means of peristaltic waves in both mechanical and physiological situations has been a subject of scientific research since the first investigation by Latham (1966). By peristaltic pumping, we mean a device for pumping fluids, generally from a region of lower pressure to one of higher pressure, by means of a contraction wave traveling along the tube. This mechanism is found in many physiological situations like urine transport from the kidney to the bladder through the ureter, movement of chyme in the gastrointestinal tract, the movements of spermatozoa in the ductus efferentesin the male reproductive tract, and ova in the female fallopian tube. Moreover, a peristaltic mechanism is involved in transporting the lump in lymphatic vessels, movement of bile in the bile duct, and the circulation of blood in small blood vessels such as arterioles, venules, and capillaries. In addition, peristaltic pumping occurs in many practical applications involving biomechanical systems. Furthermore, finger and roller pumps are frequently used for pumping corrosive or very pure materials so as to prevent direct contact of the fluid with the pump's internal surfaces.

Many modern mechanical devices have been designed on the principle of peristaltic pumping for transporting fluids without internal moving parts. The idea of peristaltic transport in mathematical point of view was first coined by Latham [1]. The initial mathematical model of peristalsis obtained by train of sinusoidal waves in an infinitely long symmetric channel or tube has been investigated by Shapiro et.al [2] and Fung and Yahi [3].Hayat et.al [4] have studied the peristaltic motion of a Jeffrey fluid under the effect of magnetic field in tube. Subba reddy et.al [5] have studied the slip Effects on the peristaltic motion of a Jeffrey fluid through a porous medium in an Asymmetric channel under the effect magnetic field. Hayat et al. [6] have studied the slip effect on peristalsis. Reddy and Venkataramana [7] have studied the Peristaltic transport of a conducting fluid through a porous medium in an asymmetric vertical channel. Several studies have made the slip effect on peristaltic transport [8, 9, 10, 11, 12]. Rathod and Sridhar [13] have studied the peristaltic flow of a couple stress fluids through a porous medium in an inclined channel. The peristaltic transport of blood under the effect of a magnetic field in non-uniform channels was studied by Mekheimer [14].

Mahmoud et.al [15] have studied the influence of heat transfer and magnetic field on peristaltic transport of a Newtonian fluid in a vertical annulus. Vajravelu *et al.* [16] made a detailed study on the effect of yield stress on peristaltic pumping of a Herschel – Bulkley fluid in an inclined tube and a channel. All these investigations are confined to hydrodynamic study of a physiological fluid obeying some yield stress model.

Rathod and Asha [17-21] have studied the peristaltic transport of a couple stress fluid in a uniform and non uniform annulus, effect of couple stress fluid and an endoscope in peristaltic motion, effect of magnetic field and an endoscope on peristaltic motion in uniform and non-uniform annulus, peristaltic transport of a magnetic fluid in a uniform and non-uniform annulus and peristaltic transport of a couple stress fluid in a uniform and non-uniform annulus through porous media.

Agrawal and Anwaruddin [22] studied the effect of magnetic field on the peristaltic flow of blood using long wavelength approximation method and observed for the flow of blood in arteries with arterial stenosis or arteriosclerosis. Hayat and Ali have been studied on Peristaltic motion of a Jeffrey fluid under the effect of magnetic field in tube. El-dabe and El-Mohandis[23] have studied magneto hydrodynamic flow of second order fluid through a porous medium on an inclined porous plane. Ebaid [24] have studied the effects of magnetic field and wall slip conditions on the peristaltic transport of a Newtonian fluid in an asymmetric channel.

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In the last few decades fractional calculus is increasingly using in the modeling of various physical and dynamical system. Fractional calculus has encountered much success in the description of viscoelastic characteristics. The starting point of the fractional derivative model of non-Newtonian model is usually a classical differential equation which is modified by replacing the time derivative of an integer order by the so-called Riemann–Liouville fractional calculus operators. This generalization allows one to define precisely non-integer order integrals or derivatives. Fractional second grade model is the model of viscoelastic fluid. In general, fractional second grade model is derived from well known second grade model by replacing the ordinary time derivatives to fractional order time derivatives and this plays an important role to study the valuable tool of viscoelastic properties.

Tripathi et.al. [25] have studied the peristaltic flow of a fractional second grade fluid through a cylindrical tube. Some important works [26-31] such as; Numerical and analytical simulation of peristaltic flow of generalized Oldroyd-B fluids, mathematical model for the peristaltic flow of chyme movement in small intestine, peristaltic transport of fractional Maxwell fluids in uniform tubes: applications in endoscopy, peristaltic transport of a viscoelastic fluid in a channel, numerical study on peristaltic transport of fractional bio fluids model, a mathematical model for swallowing of food bolus through the oesophagus under the influence of heat transfer have been studied. Tripathi et.al. [32] have studied the peristaltic transport of a generalized Burgers' fluid: Application to the movement of chyme in small intestine. Peristaltic flow of a Williamson fluid in an inclined asymmetric channel with partial slip and heat transfer has been studied by Akbar et.al. [33]. Rathod and Pallavi [34-36] have studied the peristaltic transport of dusty fluid. Rathod and Mahadev [37-41] have studied the a study of ureteral peristalsis in cylindrical tube through porous medium, the effect of magnetic field on ureteral peristalsis in cylindrical tube, effect of thickness of the porous material on the peristaltic pumping of a Jeffry fluid when the tube wall is provided with non- erodible porous lining, peristaltic flow of Jeffrey fluid with slip effects in an inclined channel and a study of ureteral peristalsis with fluid flow.

Rathod and Laxmi [42-46] have studied the slip effect on peristaltic transport of a conducting fluid through a porous medium in an asymmetric vertical channel by adomian decomposition method, Peristaltic transport of a conducting fluid in an asymmetric vertical channel with heat and mass transfer,effects of heat transfer on the peristaltic MHD flow of a Bingham fluid through a porous medium in an inclined channel, effects of heat transfer on the peristaltic MHD flow of a Bingham fluid through a porous medium in a channel and peristaltic transport in an inclined asymmetric channel with heat and mass transfer by Adomian decomposition method. Rathod and Sridhar[42-45] have studied the peristaltic transport of couple stress fluid in uniform and non-uniform annulus through porous medium, peristaltic pumping of couple stress fluid through non - erodible porous lining tube wall with thickness of porous material and effects of couple stress fluid and an endoscope on peristaltic transport through a porous medium. Rathod and Anita [46-47] have studied the effect of magnetic field on the peristaltic flow of a fractional second grade fluid through a cylindrical tube, peristaltic flow of a fractional second grade fluid through inclined cylindrical tube, and slip effect and magnetic field on the peristaltic flow of a fractional second grade fluid through a cylindrical tube.

In view of this paper, we study the effect of magnetic field on the peristaltic flow of a fractional second grade fluid through inclined cylindrical tube under the assumptions of long wavelength and low Reynolds number has been investigated. Caputo's definition is used to find fractional differentiation and numerical results of problem for different cases are discussed graphically. The effect of fractional parameter is material constant and time on the pressure rise friction force across one wavelength is discussed. This model is applied to study the movement of chyme through small intestine and also applicable in mechanical point of view.

CAPUTO'S DEFINITION

Caputo's definition [23] of the fractional order derivative is defined as

$$D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_b^t \frac{f^n(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau \qquad (n-1 < \mathrm{Re}(\alpha) \le n,\ n \in N),$$

Where, α is the order of derivative and is allowed to be real or even complex, b is the initial value of function f. For the Caputo's derivative we have

$$D^{\alpha} t^{\beta} = \begin{cases} 0 & (\beta \le \alpha - 1) \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha} & (\beta > \alpha - 1) \end{cases}$$

MATHEMATICAL FORMULATION

The constitutive equation for viscoelastic fluid with fractional second grade model is given by

$$s=\mu\left(1+\lambda_1^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\right)\dot{\Upsilon} \tag{1}$$

Where t, s, γ and λ_1, is the time, shear stress, rate of shear strain and material constants respectively, μ is viscosity, and α is the fractional time derivative parameters such that 0<α≤1. This model reduces to second grade model with α=1, and Classical Navier Stokes model is obtained by substituting λ_1 =0.

The governing equations of motion of viscoelastic fluid with fractional second grade model for axisymmetric flow are given by

$$\rho\left(\frac{\partial\bar{u}}{\partial\bar{t}}+\bar{u}\frac{\partial\bar{u}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{u}}{\partial\bar{r}}\right)=-\frac{\partial\bar{p}}{\partial\bar{x}}+\mu\left(1+\bar{\lambda}_1^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\right)\left\{\frac{1}{\bar{r}}\frac{\partial}{\partial\bar{r}}\left(\bar{r}\frac{\partial\bar{u}}{\partial\bar{r}}\right)+\frac{\partial^2\bar{u}}{\partial\bar{x}^2}\right\}-\frac{\sigma B_0^2u}{c}-\eta_1\sin\theta \tag{2}$$

$$\rho\left(\frac{\partial\bar{v}}{\partial\bar{t}}+\bar{u}\frac{\partial\bar{v}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{v}}{\partial\bar{r}}\right)=-\frac{\partial\bar{p}}{\partial\bar{r}}+\mu\left(1+\bar{\lambda}_1^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\right)\left\{\frac{\partial}{\partial\bar{r}}\left(\frac{1}{\bar{r}}\frac{\partial}{\partial\bar{r}}(\bar{r}\bar{v})\right)+\frac{\partial^2\bar{v}}{\partial\bar{x}^2}\right\}-\frac{\sigma B_0^2v}{c}+\eta_2\cos\theta \tag{3}$$

$$\frac{\partial\bar{u}}{\partial\bar{x}}+\frac{1}{r}\frac{\partial(\bar{r}\bar{v})}{\partial\bar{r}}=0 \tag{4}$$

For carrying out further analysis, we introduce the following non –dimensional parameters.

$$\left.\begin{array}{l} x=\dfrac{\bar{x}}{\lambda},\ r=\dfrac{\bar{r}}{\alpha},t=\dfrac{c\bar{t}}{\lambda},\ \lambda_1^\alpha=\dfrac{c\bar{\lambda}_1^\alpha}{\lambda},\ u=\dfrac{\bar{u}}{c},v=\dfrac{\bar{v}}{c\delta} \\ \delta=\dfrac{\alpha}{\lambda},\phi=\dfrac{\bar{\phi}}{\alpha},\ p=\dfrac{\bar{p}\alpha^2}{\mu c\lambda},Q=\dfrac{\bar{Q}}{\pi\alpha^2c}\,\mathrm{Re}=\dfrac{\rho c\alpha\delta}{\mu} \end{array}\right\} \tag{5}$$

Where ρ is fluid density, δ is defined as wave number, λ,r.t,u,v,φ,p and Q stand for wavelength, radial coordinate, time, axial velocities, wave velocity, amplitude, pressure, and volume flow rate respectively in non-dimensional form.

Introducing the non-dimensional parameters and taking long wavelength and low Reynolds number approximations, Eqs. (2) reduce to

$$\frac{\partial p}{\partial x}=\left(1+\lambda_1^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\right)\left\{\frac{\partial^2u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right\}-M^2u-\eta_1\sin\theta \tag{6}$$

$$0=\frac{\partial p}{\partial r}+\eta_2\cos\theta \tag{7}$$

$$\frac{\partial u}{\partial x}+\frac{1}{r}\frac{\partial(rv)}{\partial r}=0 \tag{8}$$

Where $M=B_0\alpha_1\sqrt{\frac{\sigma_e}{\mu}}$, $\eta_1=\frac{pg\alpha^2}{\mu c},\eta_2=\frac{pg\alpha^3}{\mu c\lambda}$ and g is acceleration due to gravity.

Boundary conditions are given by

$$\frac{\partial u}{\partial r}=0\quad at\, r=0 \tag{9}$$

$$u=0\ at\ r=h \tag{10}$$

Integrating eqn (6) with respect to r and using boundary condition of eqn (9) we get

$$\frac{\partial p}{\partial x}=\left(1+\lambda_1^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\right)\left\{\frac{\partial^2u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right\}-M^2u-\eta_1\sin\theta$$

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$$r\frac{\partial p}{\partial x}=A\left\{r\frac{\partial^2 u}{\partial r^2}+\frac{\partial u}{\partial r}\right\}-M^2ur-r\eta_1\sin\theta$$

$$\frac{r}{2}\frac{\partial p}{\partial x}+c_1=A\frac{\partial u}{\partial r}-\frac{M^2ur}{2}-\frac{r}{2}\eta_1\sin\theta \qquad (11)$$

Using boundary condition $\frac{\partial u}{\partial r}=0\ at\ r=0$

We get $c_1=0$ (12)

Substituting eqn (12) in eqn (11)

$$\frac{r}{2}\frac{\partial p}{\partial x}=A\frac{\partial u}{\partial r}-\frac{M^2ur}{2}-\frac{r}{2}\eta_1\sin\theta \qquad (13)$$

Again, integrating eqn (13) with respect to r we get the axial velocity

$$\frac{r^2}{4}\frac{\partial p}{\partial x}+c_2=Au-\frac{M^2ur^2}{4}-\frac{r^2}{4}\eta_1\sin\theta \qquad (14)$$

Again using boundary condition $u=0\ at\ r=h$

$$c_2=-\frac{h^2}{4}\frac{\partial p}{\partial x}-\frac{h^2}{4}\eta_1\sin\theta \qquad (15)$$

Substituting c_2 in eqn(14) we get

$$\frac{1}{4}\frac{\partial p}{\partial x}\left(r^2-h^2\right)-\frac{1}{4}(h^2-r^2)\eta_1\sin\theta=\left(A-\frac{M^2h^2}{2}\right)u$$

$$u=\frac{\frac{1}{4}\frac{\partial p}{\partial x}\left(r^2-h^2\right)-\frac{1}{4}\frac{\partial p}{\partial x}\left(r^2+h^2\right)\eta_1\sin\theta}{\left(A-\frac{M^2h^2}{2}\right)} \qquad (16)$$

The volume flow rate is defined as $Q=\int_0^h 2rudr$ which by virtue of eqn (15) reduces to

$$Q=\int_0^h\frac{\frac{1}{4}\frac{\partial p}{\partial x}\left(r^2-h^2\right)-\frac{1}{4}\left(r^2+h^2\right)\eta_1\sin\theta}{\left(A-\frac{M^2h^2}{2}\right)}rdr$$

$$Q\left(A-\frac{M^2h^2}{2}\right)=-\frac{h^2}{8}\frac{\partial p}{\partial x}-\frac{h^2}{4}\eta_1\sin\theta \qquad (17\}$$

The transformations between the wave and the laboratory frames, in the dimensionless form, are given by

$$X=x-1, R=r, U=u-1, V=v,\ q=Q-h^2 \quad (18)$$

Where the left side parameters are in wave frame & the right side parameters are in the laboratory frame.

We further assume that the wall undergoes contraction & relaxation is mathematically formulated as

$$h=1-\varphi\cos^2(\pi x) \quad (19)$$

The following are the existing relation between the averaged flow rate, the flow rate in the wave frame & that in the laboratory frame.

$$\bar{Q}=q+1-\phi+\frac{3\phi^2}{8}=Q-h^2+1-\phi+\frac{3\phi^2}{8} \qquad (20)$$

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Eqn(17) in view of eqn(19) becomes

$$\left(1+\lambda_1^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}-\frac{M^2h^2}{2}\right)Q=-\frac{h^4}{8}\frac{\partial p}{\partial x}-\frac{h^2}{4}\eta_1\sin\theta$$

$$\frac{\partial p}{\partial x}=-\frac{8\left(\overline{Q}+h^2-1+\phi-3\phi^2/8\right)\left(1+\lambda_1^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}-\frac{M^2h^2}{2}-\frac{h^2}{4}\eta_1\sin\theta\right)}{h^4} \tag{21}$$

Using Caputo's definition in eqn (20) we get

$$\frac{\partial p}{\partial x}=-\frac{8\left(\overline{Q}+h^2-1+\phi-3\phi^2/8\right)\left(1+\lambda_1^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}-\frac{M^2h^2}{2}-\frac{h^2}{4}\eta_1\sin\theta\right)}{h^4} \tag{22}$$

The pressure difference and friction force across one wavelength are given by

$$\Delta p=\int_0^1\frac{\partial p}{\partial x}dx \tag{23}$$

$$F=\int_0^1\left(-h^2\frac{\partial p}{\partial x}\right)dx \tag{24}$$

The above integrals numerically evaluated using the MATHEMATICA software.

NUMERICAL RESULTS AND DISCUSSIONS

In order to study the effect of various parameter(α), material constant(λ),time(t), amplitude(ϕ) ,slip parameter(k) and magnetic field (m) on pressure rire (Δp) and friction force per wavelength(F),the integrals Eqs(23) and (24) are solved numerically. Numerical simulation here is performed using the computational software Mathematica.

Fig.1-6depict the variation of pressure(Δp) with averaged flow rate $\overline{Q}$ for various values of $\alpha,\lambda,t,\phi,m\,and\,\theta$. It is observed that there is a linear relation between pressure and averaged flow rate, also increases in the averaged flow rate reduces the pressure and thus, maximum averaged flow rate is achieved at zero pressure and occurs at zero averaged flow rate.

Fig.1 shows that the pressure rise Δp with averaged flow rate $\overline{Q}$ for various values of α at $\phi=0.4, t=0.5, \eta_1=0.5, \lambda=1, m=2\,and\,\theta=\pi/3$. It is observed that, pressure increases with increases of α for pumping region$(\Delta p>0)$, and as well as free- pumping$(\Delta p=0)$and co-pumping$(\Delta p<0)$regions with an increase in α .Also it can be noted that the fractional behavior of second grade fluid increases, the pressure for flow diminishes. The variation of pressure rise Δp with $\overline{Q}$ for various values of λ at $\alpha=0.2, \phi=0.4, t=0.5, \eta_1=0.5, m=2\,and\,\theta=\pi/3$ is presented in Fig.2. It is revealed that the pressure increases with increasing λ . This means that viscoelastic behavior of fluids increases, the pressure for flow of fluids decreases, i.e. the flow for second grade fluid is required more pressure than that for the flow of Newtonian fluids $(\lambda\to 0)$.

Figs.3 depicts the variation of pressure rise Δp with averaged flow rate $\overline{Q}$ for various values of t at $\alpha=0.6, \phi=0.4, \lambda=1, \eta_1=0.5, m=2\,and\,\theta=\pi/3$.It is found that pressure increases with an increase in the magnitude of the parameter t.

Figs.4 depicts the variation of pressure rise Δp with averaged flow rate $\overline{Q}$ for various values of ϕ at $\alpha=0.2, t=0.5, \lambda=1, \eta_1=0.5, m=2\,and\,\theta=\pi/4$. It is observed that, the pressure increase with increasing amplitude ratioϕ.

Fig. 5 shows the relation between pressure rise Δp and time averaged flux $\overline{Q}$ for different values of m with $\alpha = 0.2, \phi = 0.3, t = 0.5, \lambda = 1, \eta_1 = 0.5$ *and* $\theta = \pi/4$. In the pumping region (Δp >0), the time averaged flow rate $\overline{Q}$ decreases with increase of m. Where as in the free pumping (Δp =0) and co-pumping (Δp <0), region $\overline{Q}$ increases by decreasing the m

Fig.6.depicts the variation of pressure rise Δp with time averaged flow rate $\overline{Q}$ for different values of θ with $\alpha = 0.2, \phi = 0.6, t = 0.5, \lambda = 1, \eta_1 = 0.5$ *and* $m = 3$. It is found that, any two pumping curves intersect in the first quqdrant, to the left of this point of intersection the averaged flow rate increases on decreasing θ and to the right of this point of intersection the $\overline{Q}$ decreases with increasing θ.

Figs.7-12 shows the variations of fractional force (F) with the averaged flow rate $(\overline{Q})$ under the influences of all emerging parameters such as $\alpha, \lambda, t, \phi, m$ *and* θ. From figures, It is observed that the effects of all parameters on friction force are opposite behavior as compared to the pressure rise.

5. CONCLUSIONS

In this article, we have presented a mathematical model that describes a slip effect and magnetic field on peristaltic flow of a fractional second grade fluid through a cylindrical tube is analyzed. The governing equations of the problem were solved analytically under the assumption of long wavelength and low Reynolds number. The Caputo's definition is used for differentiating the fractional derivatives. Closed form solutions are derived for velocity, slip parameter and magnetic field. we conclude with following observations:

- Pressure rise increases with an increase in fractional parameter α.
- The quantitative behavior of λ, t, ϕ on the pressure are similar.
- It is observed that friction forces have an opposite behavior to that of pressure rise.
- The pressure rise first decreases and then increases with increase in m and θ .

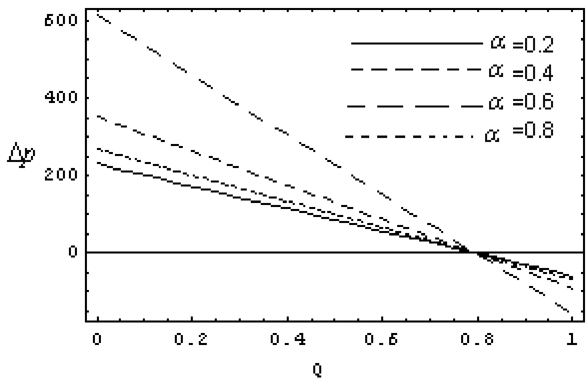

Figure.1.Pressure verses averaged flow rate for various values of α at $\phi = 0.4, t = 0.5, \eta_1 = 0.5, \lambda = 1, m = 2$ *and* $\theta = \pi/3$

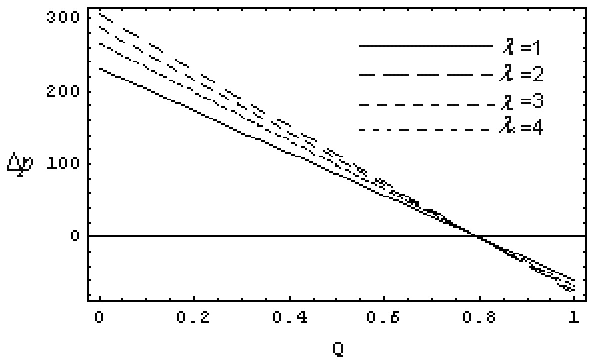

Figure.2.Pressure verses averaged flow rate for various values of λ at $\alpha = 0.2, \phi = 0.4, t = 0.5, \eta_1 = 0.5, m = 2$ *and* $\theta = \pi/3$

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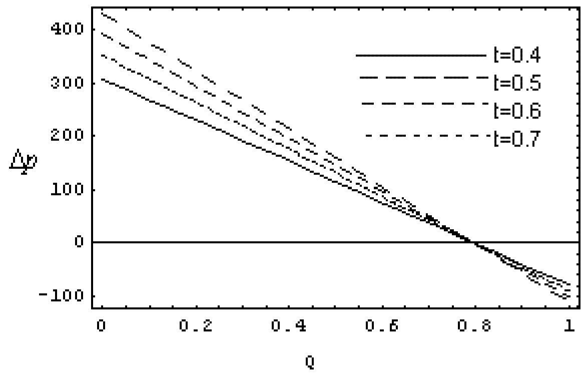

Figure.3.Pressure verses averaged flow rate for various values of t at
$\alpha = 0.6, \phi = 0.4, \lambda = 1, \eta_1 = 0.5, m = 2\, and\, \theta = \pi / 3$

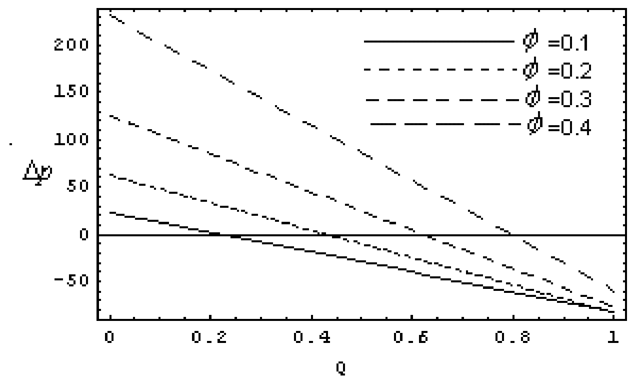

Figure.4.Pressure verses averaged flow rate for various values of ϕ at
$\alpha = 0.2, t = 0.5, \lambda = 1, \eta_1 = 0.5, m = 2\, and\, \theta = \pi / 4$

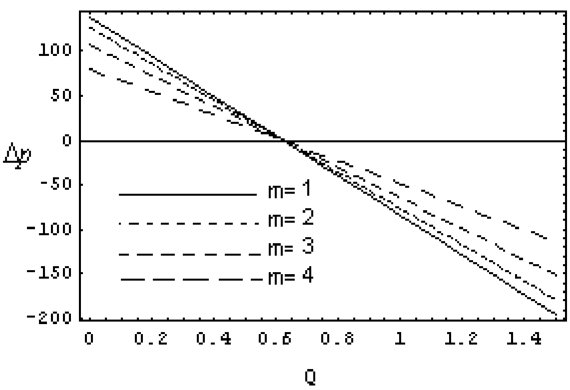

Figure.5.Pressure verses averaged flow rate for various values of m at
$\alpha = 0.2, \phi = 0.3, t = 0.5, \lambda = 1, \eta_1 = 0.5\; and\, \theta = \pi / 4$

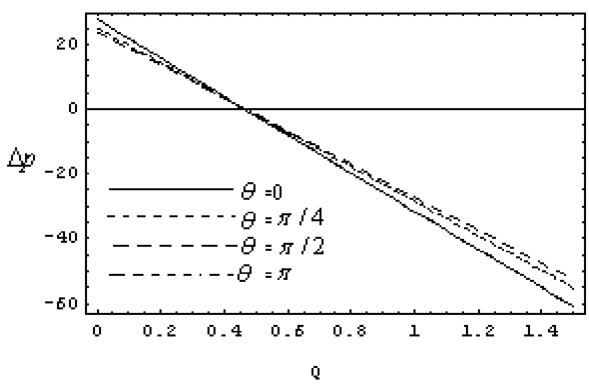

Figure.6.Pressure verses averaged flow rate for various values of Θ at
$\alpha = 0.2, \phi = 0.6, t = 0.5, \lambda = 1, \eta_1 = 0.5\; and\; m = 3$

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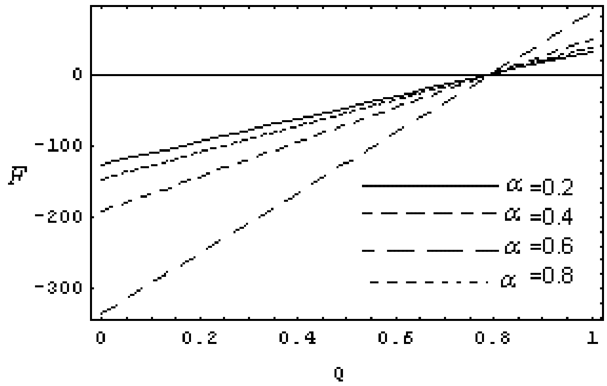

Figue.7.Friction force verses averaged flow rate for various values of α at $\phi = 0.4, t = 0.5, \eta_1 = 0.5, \lambda = 1, m = 2 \, and \, \theta = \pi / 3$

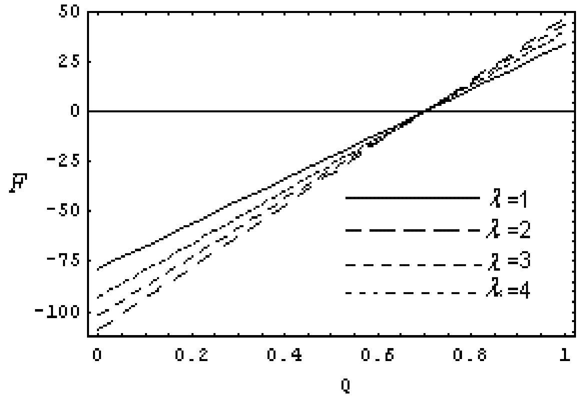

Figure.8.Friction force verses averaged flow rate for various values of λ at $\alpha = 0.2, \phi = 0.4, t = 0.5, \eta_1 = 0.5, m = 3 \, and \, \theta = \pi / 4$

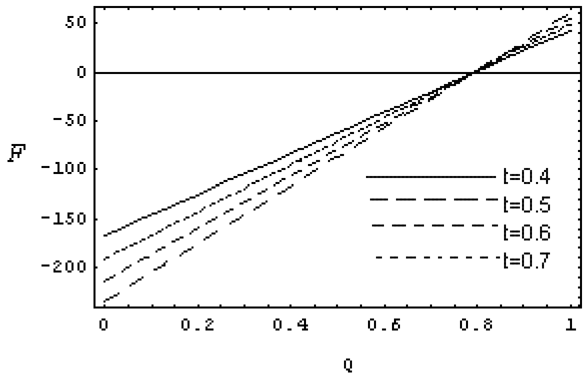

Figure.9.Friction force verses averaged flow rate for various values of t at $\alpha = 0.6, \phi = 0.4, \lambda = 1, \eta_1 = 0.5, m = 2 \, and \, \theta = \pi / 6$

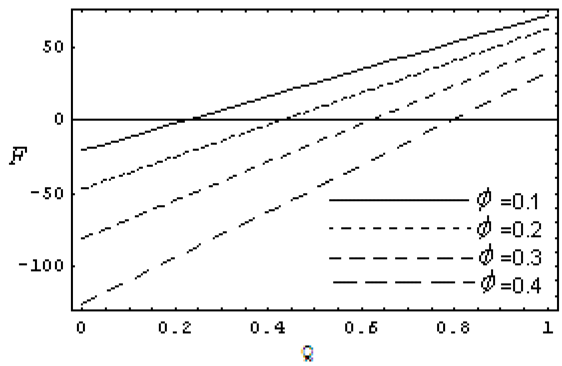

Figure.10.Friction force verses averaged flow rate for various values of ϕ at $\alpha = 0.2, t = 0.5, \lambda = 1, \eta_1 = 0.5, m = 2 \, and \, \theta = \pi / 4$

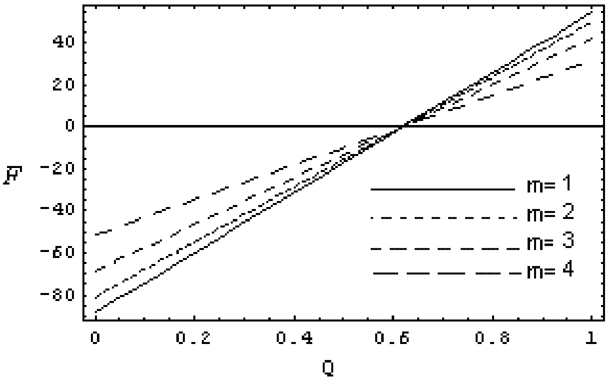

Fig.11.Friction force vs. averaged flow rate for various values of m at
$\alpha = 0.2, \phi = 0.3, t = 0.5, \lambda = 1, \eta_1 = 0.5 \ and \ \theta = \pi / 6$

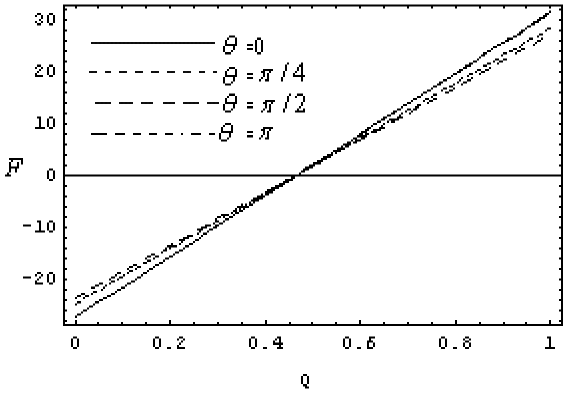

Figure.12.Pressure verses averaged flow rate for various values of Θ at
$x = 0.1, \alpha = 0.2, \phi = 0.4, t = 0.5, \lambda = 1 \ and \ k = 0.5$

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VARIOUS NOISE REDUCTION BASED ON HAAR WAVELETS FOR ECG SIGNALS

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ABSTRACT
In the field of biomedical signal processing, electrocardiogram (ECG) represents a graphical representation of the electrical activity of the heart. ECG signal has been analyzed and utilized for various purposes, such as measuring the heart rate, rhythm of heartbeats and detecting R wave, etc. The removing baseline wanders for the ECG signal contained by various noise. This paper developed a new approach algorithm for various noise reductions using wavelet transformation, and the proposed algorithm gives useful information from the electrocardiogram (ECG) signal based on applications. The threshold value of the ECG signal gives better quality and more information's. The outcoming results are compared with existing results.

Keywords: ECG Signal, Baseline Wander, Wavelet Theory, Noise Reduction, Threshold.

1. INTRODUCTION
The morphology of the ECG signal has been used for recognizing much variability of heart activity, so it is very important to get the feature of the ECG signal. If the ECG signal does not vary, then we do not get any information. The study of various noise reductions for ECG signal to global denoising analysis and can be picked up ECG signal based on the applications. It is simply based on the selection of the part of ECG signal morphology that to be preserved and denoised during the process with minimal loss of information. There is a number of algorithms like digital filters, Finite impulse response (FIR) and Infinite impulse response (IIR) filters, etc. FIR filters required stationary signal with a different window such as Hanning window, Kaiser window, Rectangular window are required window size and IIR filters are of type Chebyshev-I, Chebyshev-II, and Butterworth. These are observed that the choice of the cut-off frequency is required, a lower than required cut-off frequency does not filter the actual ECG signal component [1, 2 and 3]. The one of most popular method is Donaho's denoising algorithm, determination using these algorithms and the idea of not to threshold the approximation coefficients of ECG signal [4]. Wavelet transform makes it very suitable for analysis of non-stationary signals such as the ECG. Wavelet transform does not require cut-off frequency and applied a threshold to the approximation coefficients the reduction of noise after getting features of ECG signals.

In this paper proposed algorithm applies to MIT-BIH database records [5], and considered ECG signal likes normal, Bradycardia and Tachycardia [6]. Before doing any processing subtracts the mean from the ECG signal and removing baseline wander [7]. The record ECG signals are purposefully corrupted by white Gaussian noise or random noise with consequently reduced the noise interfering in ECG signal and decomposed to fourth levels. The second level contains a little noise and fourth levels are loss of information compared to the third level. In this context, we developed an algorithm reduction of various noises and getting better information features.

Now considered time domain ECG signals have 3600 number of samples after adding various noise to the considered ECG signal also has the same number of samples but some important parameters are vanished by psycho visible information. Then analysis by wavelets gives two half of its original signal samples in the first level similarly second, third and fourth are also gives half of these samples, then ECG signal in the wavelet domain and third level approximation coefficients are 450 number of non zeros (nnz) samples and using threshold, after thresholding samples are strictly less than 450 samples and accurate parameters. The 3600 samples in time domain ECG signal does not get clear visible information after adding noise after analyzed by wavelets, and wavelet domain gives more accurate clearly visible information, fewer samples and more features of ECG signals.

2. BASIC CONCEPTS OF HAAR WAVELET THEORY [8]
Wavelets are being beneficial in different fields of science and engineering such as signal and medical imaging processing etc. The wavelets transform gives a small number of large coefficients and a large number of small coefficients. The smaller coefficients values are represented noise components with large coefficients mainly represent the signal values. The wavelets are localized in both time and frequency.

Considered ECG signal $x(t)$ is continuous, generally, we shall express a discrete ECG signal in the form

$$x(n)=(x_1, x_2, ..., x_N),$$

here N is a positive even integer and length of x. The values of x are the N real numbers $x_1, x_2, ..., x_N$. These values are typically measured values of an analog ECG signal.

The wavelets transform, decomposes a discrete ECG signal into two ECG sub-signals of half its length. One ECG sub-signal is an average running and the other one is ECG sub-signal is a running difference. Let us begin with the average running of the Haar wavelet transformation.

The first average running is

$$a^1 = (a_1, a_2, ..., a_{N/2}),$$

for the ECG signal x is computed by taking an average running in the following process. Its first value a_1 is computed by taking the average of the first pair of values of $x = \left(\frac{x_1 + x_2}{2}\right)$ and then multiplying by $\sqrt{2}$. Thus $a_1 = \left(\frac{x_1 + x_2}{2}\right)\sqrt{2}$, similarly, its next value a_2 is computed by taking the average of the next pair of values $x = \left(\frac{x_3 + x_4}{2}\right)$, and then multiplying by $\sqrt{2}$. Hence $a_2 = \left(\frac{x_3 + x_4}{2}\right)\sqrt{2}$, continuing in this processing, all of the values a^1 are produced by taking averages of successive pairs of values of x and then multiplying these averages by $\sqrt{2}$. A formula for the values of a^1 is given

$$a_k = \left(\frac{x_{2k-1} + x_{2k}}{2}\right)\sqrt{2}, \qquad (1)$$

for $k = 1, 2, ..., N/2$.

The other ECG sub-signal is called the first fluctuation. The first running difference of the ECG signal x which is denoted by $d^1 = \left(d_1, d_2, ..., d_{N/2}\right)$, is computed by taking a running difference in the following manner. Its first value, d_1, is found by taking half the difference of the first pair of values of $x = \left(\frac{x_1 - x_2}{2}\right)$, and multiplying by $\sqrt{2}$. That is $d_1 = \left(\frac{x_1 - x_2}{2}\right)\sqrt{2}$. Likewise its next value d_2 is found by taking half the difference of the next pair of values of $x = \left(\frac{x_3 - x_4}{2}\right)$, and multiplying by $\sqrt{2}$. In other words, $d_2 = \left(\frac{x_3 - x_4}{2}\right)\sqrt{2}$. continuing in this processing, all of the values d^1 are produced according to the following formula:

$$d_k = \left(\frac{x_{2k-1} - x_{2k}}{2}\right)\sqrt{2}, \qquad (2)$$

for $k = 1, 2, ..., N/2$.

3. ECG DATABASE [5]

The ECG signal used in this paper is a part of the MIT-BIH arrhythmia database, available online. The database includes 48 annotated records. Each record includes two-channel ECG signals are contains different durations like 10sec, 1 minute, 1 hour, and 12 hours, in these time duration, the considered 10sec and sampled at a frequency 360Hz. The duration of the ECG signal is selected from 24hr recordings of 3 different individuals (i.e., 100, 108 and 215). Header file consists of detailed information such as the number of samples, sampling frequency, the format of ECG signal, type of ECG leads and number of ECG leads, patient's history and the detailed clinical information. ECG signals (.dat files) downloaded from Physionet are first converted into MATLAB readable format (.mat files). The signals from both leads now become readable separately. Then the signals from lead-II are only taken for our analysis.

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4. NOISE REDUCTION BASED ON PROPOSED ALGORITHM

Wavelet analysis decomposes signals into constituent waves of varying time, called wavelets. These wavelets are localized variations of detail in the signal, they can be used for a broad variety of important signal processing job such as compression, reduction noise, enhancing a recording MIT-BIH ECG signals in various ways.

The proposed algorithm is divided into the following steps are given:

1. Remove baseline wanders for the considered MIT-BIH ECG records.

2. Noise Generation and addition: WGN and random noise are generated and added to the signal from step 1.

Mathematically defined:

$$z(t) = x(t) + y(t), \qquad (3)$$

$z(t)$ is corrupted by white Gaussian noise (WGN) or random noise and $y(t)$ is WGN or random noise.

3. The noisy ECG signal (z) z is decomposed into four levels using equation (1) and (2).

4. Choose and apply the threshold value to the third level approximation coefficients [9]

$$thresh = 4\sigma_j \sqrt{\frac{\log n_j}{n_j}} \qquad (4)$$

Here, σ_j is the standard deviation and n_j is the length of the approximation coefficients of the ECG signal and $j = 1, 2, 3, 4.$ levels decomposition.

5. PERFORMANCE OF PSYCHO VISIBILITY

Now, compare the performance of our algorithm results visibility with respect to the various noises added, after removing baseline wanders and reduction of noise using eqn (1). At a particular higher SNR in an ECG signal show better it is desirable and lesser SNR contained an ECG signal is not better visible information [10]. The random noise increasing and SNR decreasing the ECG signal is not a piece of too much better information visible as shown below figs. 1 to 12 is analyzed by Haar wavelets with various noise. Table 1 shows different noise adding to different ECG signals to the MIT-BIH records, threshold values and the number of non-zeros samples.

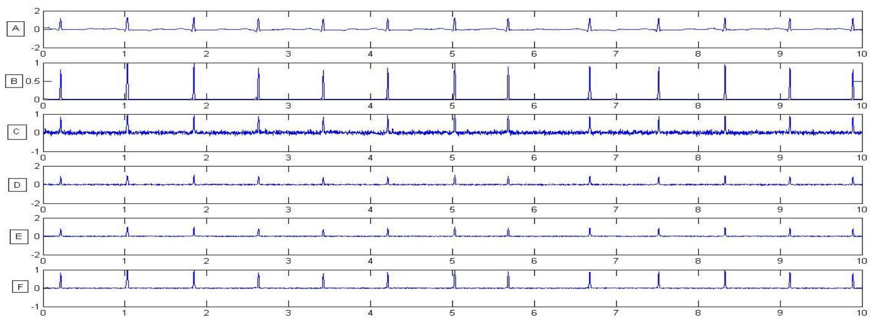

Figure-1: Baseline removed and noise added to MIT-BIH record no 100. (A) Original ECG signal. (B) Baseline removed. (C) - (F) WGN added with 5, 10, 15 and 20 SNR in dB respectively.

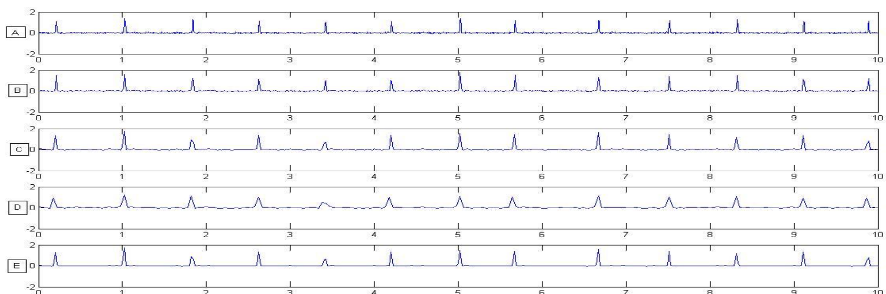

Figure-2: Haar Wavelet transforms to the MIT-BIT record 100 with 10 SNR in dB. (A) - (D) First, Second, Third and Fourth approximation coefficients. (E) The threshold applied to Third approximation coefficients.

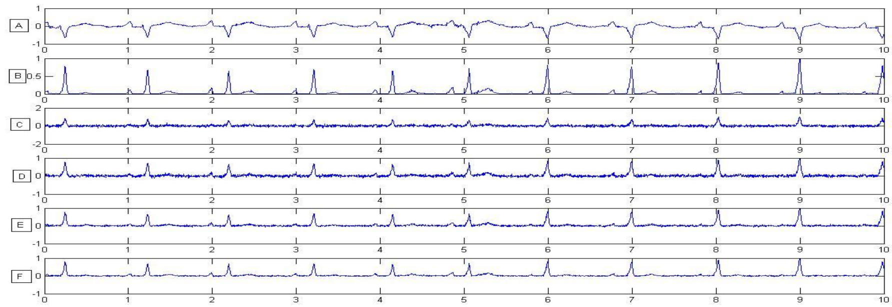

Figure-3: Baseline removed and noise added to MIT-BIH record no 108. (A) Original ECG signal. (B) Baseline removed. (C) - (F) WGN added with 5, 10, 15 and 20 SNR in dB respectively.

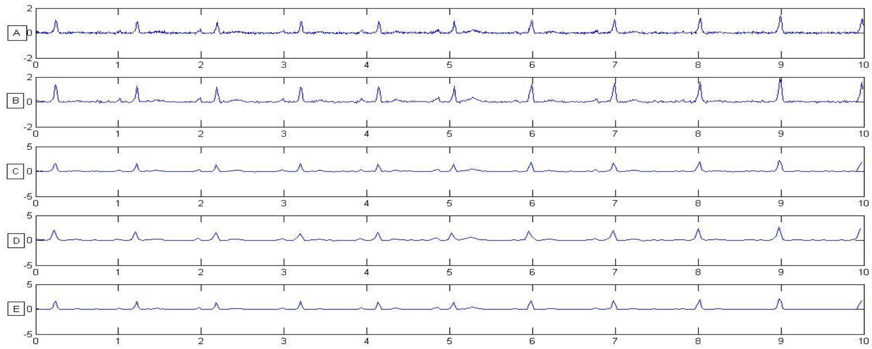

Figure-4: Haar Wavelet transforms to the MIT-BIT record 108 with 10 SNR in dB. (A) - (D) First, Second, Third and Fourth approximation coefficients. (E) The threshold applied to Third approximation coefficients.

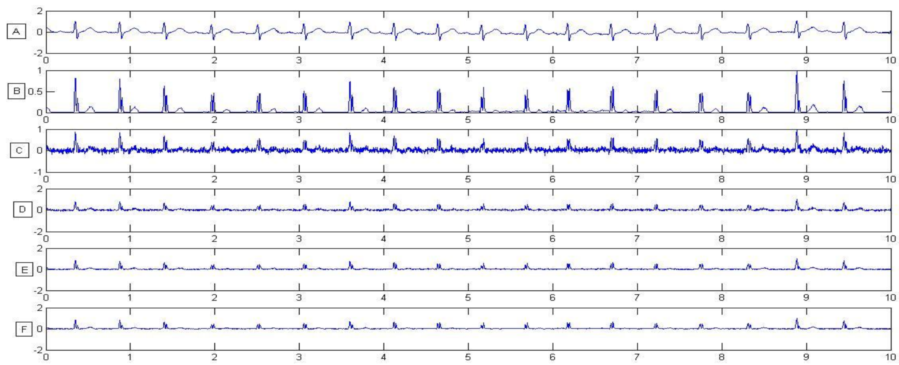

Figure-5: Baseline removed and noise added to MIT-BIH record no 215. (A) Original ECG signal. (B) Baseline removed. (C) - (F) WGN added with 5, 10, 15 and 20 SNR in dB respectively.

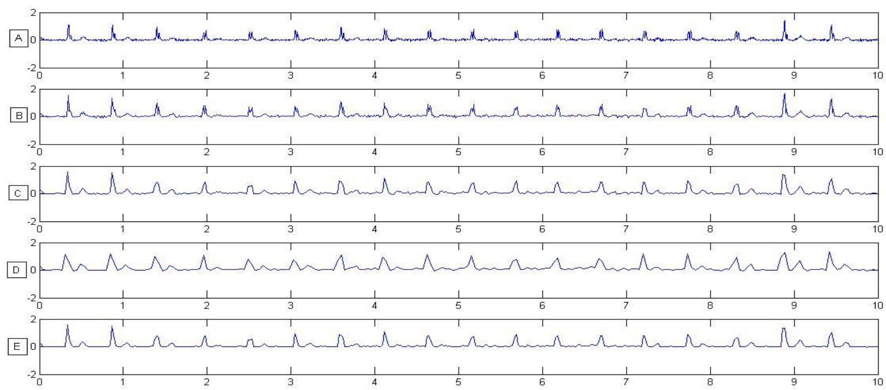

Figure-6: Haar Wavelet transforms to the MIT-BIT record 215 with 10 SNR in dB. (A) - (D) First, Second, Third and Fourth approximation coefficients. (E) The threshold applied to Third approximation coefficients.

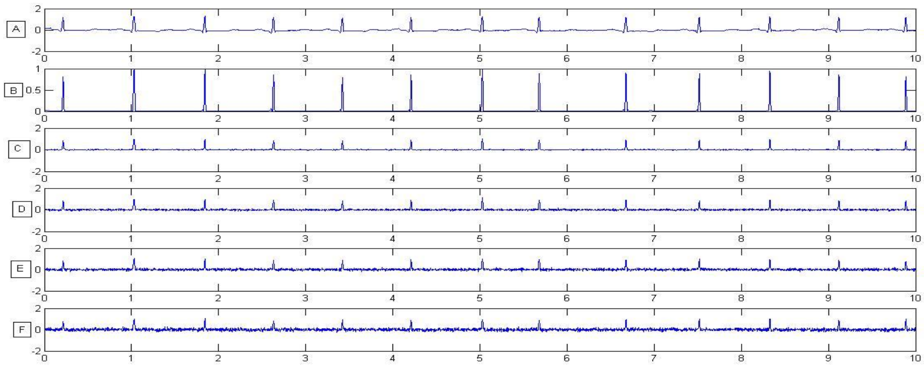

Figure-7: Baseline removed and noise added to MIT-BIH record no 100. (A) Original ECG signal. (B) Baseline removed. (C) - (F) Random noise added with 0.02, 0.04, 0.06, and 0.08 respectively.

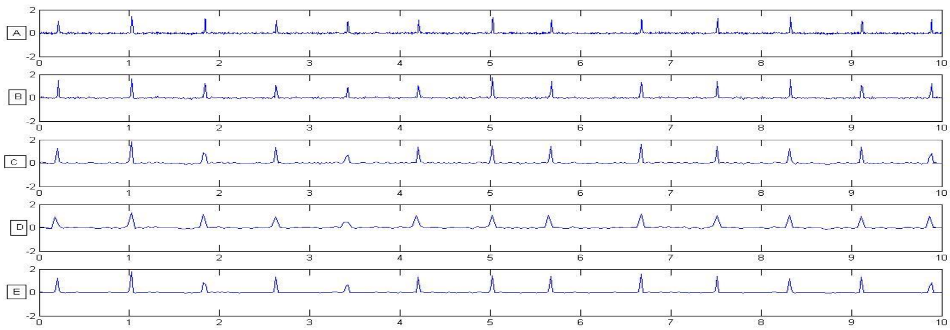

Figure-8: Haar Wavelet transforms to the MIT-BIT record 100 with 0.04 random noises. (A) - (D) First, Second, Third and Fourth approximation coefficients. (E) The threshold applied to Third approximation coefficients.

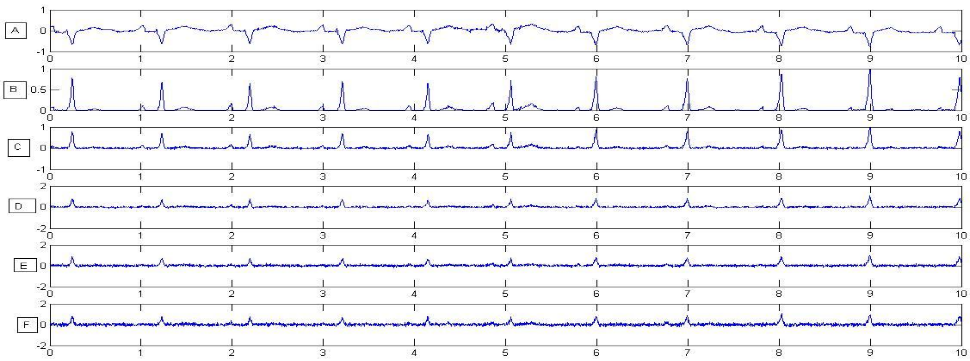

Figure-9: Baseline removed and noise added to MIT-BIH record no 108. (A) Original ECG signal. (B) Baseline removed. (C) - (F) Random noise added with 0.02, 0.04, 0.06, and 0.08 respectively.

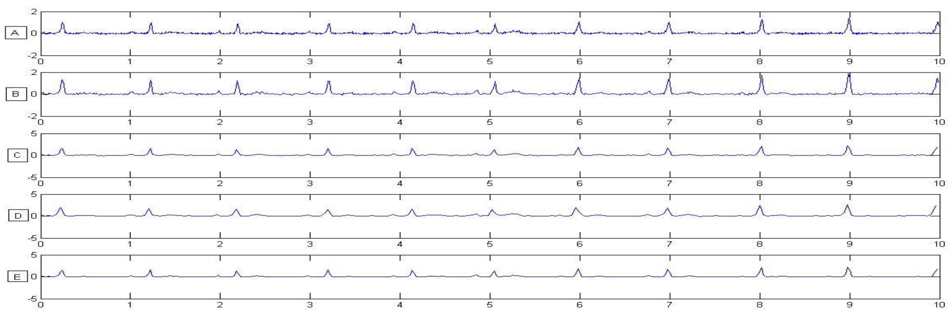

Figure-10: Haar Wavelet transforms to the MIT-BIT record 108 with 0.04 random noises. (A) - (D) First, Second, Third and Fourth approximation coefficients. (E) The threshold applied to Third approximation coefficients.

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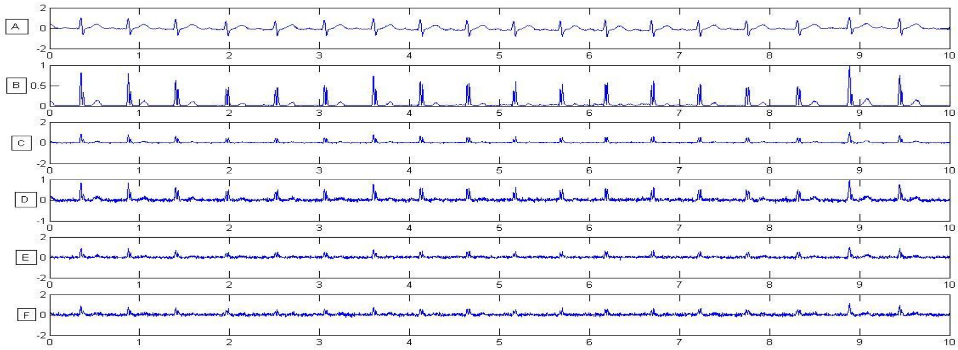

Figure-11: Baseline removed and noise added to MIT-BIH record no 215. (A) Original ECG signal. (B) Baseline removed. (C) - (F) Random noise added with 0.02, 0.04, 0.06, and 0.08 respectively.

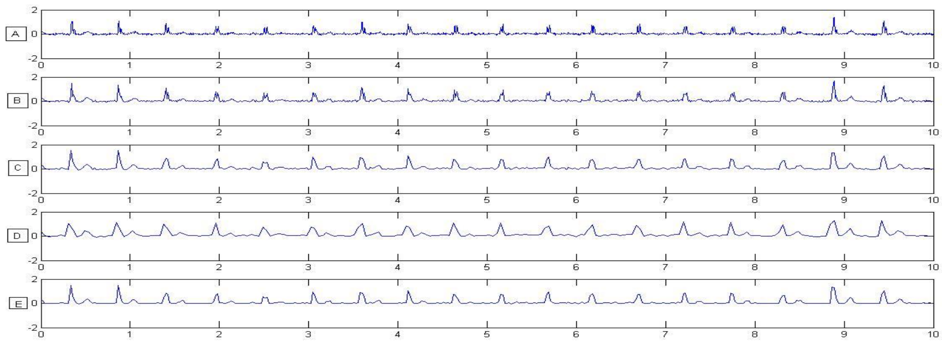

Figure-12: Haar Wavelet transforms to the MIT-BIT record 215 with 0.04 random noises. (A) - (D) First, Second, Third and Fourth approximation coefficients. (E) The threshold applied to Third approximation coefficients.

Table 1.

MIT-BIH Record No 100 is Normal[6]					
Random Noise	Threshold Values	After Threshold nnz samples	SNR in dB	Threshold Values	After Threshold nnz samples
0.02	0.0378	82	5	0.0387	227
0.04	0.0381	169	10	0.078	144
0.06	0.0387	250	15	0.0377	62
0.08	0.0388	262	20	0.0377	43
MIT-BIH Record No 108 is Normal[6]					
Random Noise	Threshold Values	After Threshold nnz samples	SNR in dB	Threshold Values	After Threshold nnz samples
0.02	0.0528	166	5	0.0552	279
0.04	0.0532	220	10	0.0533	198
0.06	0.0538	252	15	0.0527	172
0.08	0.0547	271	20	0.0526	162
MIT-BIH Record No 215 is Normal[6]					
Random Noise	Threshold Values	After Threshold nnz samples	SNR in dB	Threshold Values	After Threshold nnz samples
0.02	0.0406	228	5	0.0428	317
0.04	0.0405	288	10	0.0408	226
0.06	0.0418	295	15	0.0404	240
0.08	0.0425	335	20	0.0402	232

Note: Original ECG signal has 3600 samples. Third level approximation coefficients have the number of non zero (nnz) samples 450.

6. RESULTS AND DISCUSSIONS

Considered an ECG signal, before doing any processing must be subtracted the mean of its original ECG signal itself. Then successfully removed baseline wanders using squaring and moving window algorithm with order 2. Then added WGN of 5, 10, 15 and 20 SNR in dB to the MIT-BIH records 100, 108 and 215 respective then decomposed the signal using equations (1) and (2), up to 4 levels; the decomposed ECG signal only considered approximation coefficient and details coefficients are neglected. Then comparisons between SNR of 5, 10, 15 and 20 dB added in the ECG signal, lesser SNR in dB high decomposing levels required to better information visible and high SNR in dB lesser decomposition required to better information visible.

The comparisons between of MIT-BIH records 100, 108 and 215 corrupted by random noise 0.02, 0.04, 0.06 and 0.08 and decomposed using equations (1) and (2), the lesser random noise fewer decomposition levels required and higher random noise high decomposition levels required are better visible information of ECG signal, but lesser random noise high decomposition loss of information and high random noise less decomposition its contains a little noise and ECG signal depends upon both WGN or random noise quantity.

7. CONCLUSIONS

Diagnosing of ECG signal from obtained noisy ECG signal is a very difficult task requiring observation of the various situations in real life. We presented our work on Haar wavelets based denoising algorithm applied to the ECG signal. Our algorithm is to the reduction of noise the white Gaussian noise as well as random noise that is corrupted to the original ECG signal. Then, by an able to processing and different level discrete wavelet transform the detail and approximation sequences of the noisy; finally, ECG signal can be extracted from the third level discrete wavelet transform is better for various noise reduction. The detail coefficients are not considered for analysis in this paper with features extract aid of approximation coefficients with different threshold values can be obtained from different ECG signals. The denoising algorithm has been applied on the MIT-BIH Arrhythmia database ECG records corrupted with a WGN of an SNR of 5, 10, 15, 20 dB and random noise 0.02, 0.04, 0.06 and 0.08. In our algorithm, the random noise was also highly and efficiently suppressed. The corrupted noise and highly recovering the ECG signal components exist obtained as features. The original ECG signal and after adding various noise has 3600 samples then analyzed ECG signal recovering in less than 450 samples and important parameters are recognized in fewer samples. An important task in this paper successfully removed various noise and accurate features within fewer samples and 25200 Bytes memories saved.

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IMPACT OF THERMAL RADIATION ON THE 2D MAGNETOHYDRODYNAMIC FLOW OF NANOFLUID PAST A VERTICALLY ELONGATED SHEET IN THE PRESENCE OF ALLOY NANOPARTICLES

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ABSTRACT
Present study deals with the boundary layer analysis of a 2D MHD flow of methanol based $Al_{50}Cu_{50}$ nanofluid past semi-infinite elongated porous plate. The energy equation is incorporated with heat source/sink and thermal radiation effects. For making the analysis more attractive we pondered alloy nanoparticles of aluminium and copper in the ratio of 50% in methanol based nanofluid. The transmuted governing PDEs are resolved analytically by employing regular perturbation method. The effect of physical parameters on velocity and temperature profiles along with the skin friction factor and local Nusselt number are examined and presented with the help of graphical and tabular forms. Results describes that, thermal radiation and heat source parameters have tendency to improve the heat transfer rate. And also, use of alloy nanoparticles improves the local Nusselt number.

Keywords: Alloy nanoparticles, MHD, Thermal Radiation, Heat source/sink, Grashof number.

I. INTRODUCTION

Extremely high-performance cooling process is one of the most vital needs of many industries like microelectronics, transportation, manufacturing and metallurgy. Conventional methods leading to enhanced heat transfer rates, such as extended surfaces and micro-channels, have the disadvantage to increase the required pumping power of the cooling fluid, or the use of microfluids pose problems in terms of gravity settling, clogging, etc. However, essentially low thermal conductivity is a primary constraint in modelling energy-efficient heat transfer fluids that are essential for extremely high-performing cooling process. The development of advanced fluids with improved flow and thermal characteristics are of paramount importance to achieve higher heat flux densities. Thermal conductivities of solids are of greater magnitude than that of conventional fluids, hence it is expected that dispersion of solid particles will significantly improve the thermal behavior of base fluids. In order to achieve these major requirements of industries, Choi [1] introduced new class of fluids termed as Nanofluids (nanoparticle fluid suspensions). Sandeep and Sulochana [2] addressed the impact of induced magnetic field effect stagnation point flow of a nanofluid along a stretchable cylinder. Sulochana et al. [3] extended this concept by the usage of transpiration effects. Reddy et al. [4] explored the impact of radiation and chemical reaction on MHD flow of nanofluid passing over a moving porous plate. They conveyed that, water based Silver nanoparticles are more effective in magnifying the local Nusselt number compared with Titania nanoparticles. Later on several authors [5-9] discussed the use of nanofluids in their analysis.

The studies of flow through a porous medium has achieved abundant prominence owing to enormous practical applications in diverse sections of science and engineering, few of them are polymer extrusion, heat exchangers, in geophysical and geothermal processes etc. and also in ground water purification, petroleum reservoirs, in grain storage devices etc. In recent days, the thermal and mass transport in MHD flow is a focusing area by the researchers, due to prominent applications in engineering and industries. Particularly, in magnetic material processing, in geophysics and control of cooling rate, etc. in view of these applications, Sandeep et al. [10] elaborated the impact aligned magnetic field on the natural convection flow on a vertical plate engrossed in porous medium. The literatures focusing on this topic are reviled by few authors (ref. [11-14]). Very recently, Ahmed et al. [15] depicted the impact of frictional heating and chemical reaction on a flow of an electrically conducting fluid along a vertical porous plate in the presence of constant magnetic field. The impact cross diffusion on 2D flow of power-law fluid past porous plate were examined by Pal and Chatterjee [16].

Present literature addresses, the momentum and heat transfer in MHD flow of nanofluid over an inclined semi-infinite moving porous plate in presence of thermal radiation, heat source/sink, thermo diffusion effects. The transmuted governing PDEs are resolved analytically employing regular perturbation method. The effects of various non-dimensional pertinent parameters on flow and thermal distributions along with friction factor and Nusselt numbers are examined and produced with the succour of graphical and tabular values.

II. FORMULATION OF THE PROBLEM:

We presume, the 2D boundary layer flow of electrically conducting nanofluid over a permeable semi-infinite plate immersed in porous medium under the consideration of thermal and concentration buoyancy effects as shown in fig. 1.

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The plate is placed along the x-axis and it is perpendicular to the y-axis. Here T_w and C_w are the constant temperature and concentration applied at the wall which are higher than T_∞ and C_∞. Where T_∞ and C_∞ are the ambient temperature and concentration, respectively. And also for making the analysis more attractive we pondered alloy nanoparticles of aluminium and copper in the ratio of 50% in methanol based nanofluid. Under the presence of above physical assumptions, the flow governing equations are described as below:

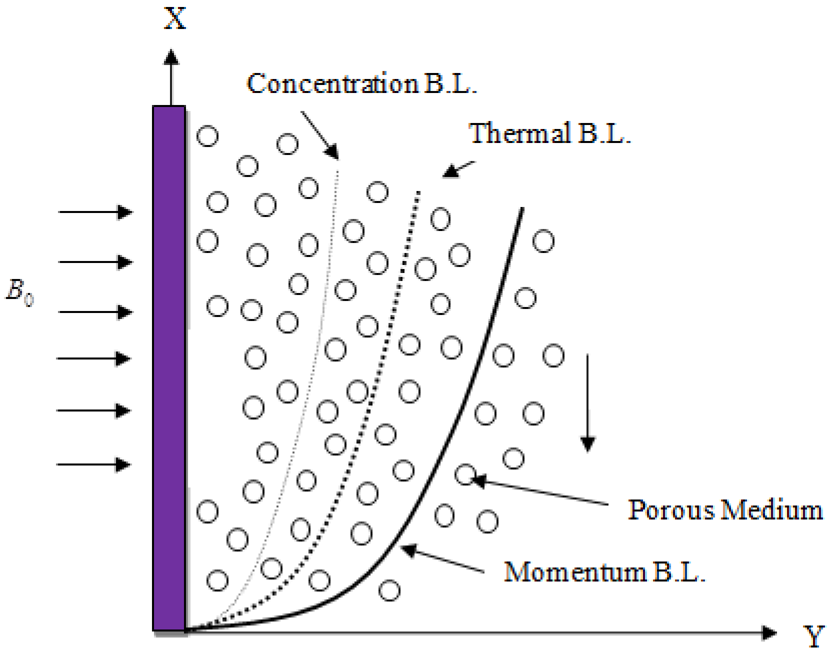

Fig-1: Flow geometry of the problem.

$$\frac{\partial v}{\partial y} = 0 \Rightarrow v' = -v_0 \tag{1}$$

$$v'\frac{\partial u'}{\partial y'} = \frac{1}{\rho_{nf}}\left\{\frac{\partial^2 u'}{\partial y'^2} - \frac{\mu_{nf}}{K'} - \sigma B_0^2 u' + (\rho\beta)_{nf}\, g(T' - T_\infty) + (\rho\beta)_{nf}\, g(C' - C_\infty)\right\} \tag{2}$$

$$v'\frac{\partial T'}{\partial y'} = \frac{1}{(\rho c_p)_{nf}}\left\{k_{nf}\frac{\partial^2 T'}{\partial y'^2} + \mu_{nf}\left(\frac{\partial u'}{\partial y'}\right)^2 - \frac{\partial q_r}{\partial y'} + \sigma B_0^2 u'^2 - Q_0(T' - T_\infty)\right\} \tag{3}$$

where u and v are the velocity towards the $x-y$ axis, respectively. And ϕ, ρ_{nf}, μ_{nf}, $(\rho c_p)_{nf}$ k_{nf} are the volume fraction, effective density, effective dynamic viscosity, heat capacitance and thermal conductivity of the nanofluid which are stated as follows:

$$k_{nf} = K_f\left(\frac{K_s + 2K_f - 2\phi(K_f - K_s)}{K_s + 2K_f + 2\phi(K_f - K_s)}\right) \tag{4}$$

$$\mu_{nf} = \mu_f(1-\phi)^{-5/2}, \tag{5}$$

$$(\rho c_p)_{nf} = (\rho c_p)_f\{(1-\phi) + \phi(\rho c_p)_s/(\rho c_p)_f\}, \tag{6}$$

$$\rho_{nf} = \rho_f\{(1-\phi) + \phi\rho_s/\rho_f\} \tag{7}$$

The radiative heat flux is given by

$$\frac{\partial q_r}{\partial y'} = 4(T' - T_\infty)L', \tag{8}$$

The related boundary conditions are:

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$$\left.\begin{array}{l} u'=0,\quad T'=T_\infty,\quad C'=C_\infty,\quad \text{at } y=0 \\ u'\to 0,\quad T'\to T_\infty,\quad C'\to C_\infty,\quad \text{as } y\to\infty \end{array}\right\}, \tag{9}$$

Now utilizing the suitable transmutations:

$$u=\frac{u'}{v_0},\, y=\frac{v_0 y'}{\upsilon_f},\, \theta=\frac{(T'-T_\infty)}{(T_w-T_\infty)},\, K=\frac{K'V_0^2}{\upsilon_f^2}, \tag{10}$$

Using eqn. (5)-(12) in basic eqn. (2)-(4) reduces to the dimensionless form:

$$\left((1-\phi)^{-5/2}\right)\frac{\partial^2 u}{\partial y^2}+\left\{1-\phi+\phi\left(\frac{\rho_s}{\rho_f}\right)\right\}\frac{\partial u}{\partial y}-\left\{M^2+\frac{(1-\phi)^{-5/2}}{K}\right\}A_3=-A_{44}Gr\theta-A_{44}Gm\psi, \tag{11}$$

$$\frac{k_{nf}}{k_f}\frac{\partial^2\theta}{\partial y^2}+\Pr\left\{1-\phi+\phi\left(\frac{(\rho c_p)_s}{(\rho c_p)_f}\right)\right\}\frac{\partial\theta}{\partial y}-\Pr(R+Q_H)\theta=-\left((1-\phi)^{-5/2}\right)\Pr Ec\left(\frac{\partial u}{\partial y}\right)^2-\Pr M^2 Ecu^2, \tag{12}$$

where

$$\left.\begin{array}{l} M=\dfrac{\sigma B_0^2\upsilon_f^2}{v_0\mu_f},Gr=\dfrac{(\rho\beta)_f g(T_W-T_\infty)\upsilon_f^2}{v_0^3\mu_f},Gm=\dfrac{(\rho\beta)_f g(C_W-C_\infty)\upsilon_f^2}{v_0^3\mu_f}, \\ Ec=\dfrac{v_0^2}{c_p(T_w-T_\infty)},R=\dfrac{4\upsilon_f L'}{(\rho c_p)_f v_0^2},Q_H=\dfrac{Q_0\upsilon_f}{(\rho c_p)_f v_0^2},\Pr=\dfrac{\mu_f c_{p_f}}{k_f}, \end{array}\right\}, \tag{13}$$

The corresponding transformed boundary conditions are given by:

$$\left.\begin{array}{l} u=0,\quad \theta=1,\quad \psi=1 \quad \text{at} \quad y=0 \\ u\to 0,\quad \theta\to 1,\quad \psi\to 1 \quad \text{as} \quad y\to\infty \end{array}\right\}, \tag{14}$$

III. PERTURBATION SOLUTION OF THE PROBLEM:

The set of PDE (11) - (12) cannot be solved in closed form, however they can be solved analytically after reducing them into ODE by taking the expressions for velocity as $U(y)$ and temperature as $\theta(y)$ in dimensionless form as follows:

$$U(y)=u_0(y)+Ecu_1(y)+O(Ec^2), \tag{15}$$

$$\theta(y)=\theta_0(y)+Ec\theta_1(y)+O(Ec^2), \tag{16}$$

We get the final form of solutions for velocity and temperature as given below:

$$\begin{aligned} U(y)=&(D_4e^{-B3y}-D_2e^{-B1y}-D3e^{-B2y})+Ec(D_{36}e^{-B3y}-D_{28}e^{-B1y}-D_{29}e^{-B2y} \\ &+D_{30}e^{-2B3y}+D_{31}e^{-2B1y}+D_{32}e^{-2B2y}-D_{33}e^{-B4y}+D_{34}e^{-B5y}-D_{35}e^{-B6y}) \end{aligned} \tag{17}$$

$$\theta(y)=e^{-B1y}+Ec(D_{11}e^{-B1y}-D_5e^{-2B3y}-D_6e^{-2B1y}-D_7e^{-2B2y}+D_8e^{-B4y}-D_9e^{-B5y}+D_{10}e^{-B6y}) \tag{18}$$

The physical quantities of practical interest are local skin friction factor and local Nusselt number are given as follows:

$$\begin{aligned} C_{fx}=\frac{\tau_w}{\rho_f v_0^2}=u'(0)=&(-B_3D_4+B_1D_2+B_2D_3)+Ec(-B_3D_{36}+B_1D_{28}+B_2D_{29}-2B_3D_{30} \\ &-2B_1D_{31}-2B_2D_{32}+B_4D_{33}-B_5D_{34}+B_6D_{35}) \end{aligned} \tag{19}$$

$$\frac{Nu_x}{\mathrm{Re}_x}=-\left.\frac{\partial\theta}{\partial y}\right|_{y=0}=\theta'(0)=B_1+Ec(B_1D_{11}-2B_3D_5-2B_1D_6-2B_2D_7+B_4D_8-B_5D_9+B_6D_{10}) \tag{20}$$

IV. RESULTS AND DISCUSSION

In this section we presented the impacts of physical parameters namely, magnetic field M , thermal Grashof number Gt , concentration Grashof number Gc , volume fraction parameter ϕ, Radiation R , heat source/sink parameter Q_H , on the flow and temperature profiles are discussed with the help of graphical illustrations. Also skin friction factor and heat transport rates are displayed in the form of tabular values. We pondered alloy nanoparticles of aluminium and copper in the ratio of 50% in methanol based nanofluid. In current analysis, we presumed the parametric values as constant in our complete analysis $\Pr = 0.7; R = 0.5; K = 0.5;$ $Gt = 4.0; Gc = 2.0; Q_H = 2.0;$ $Ec = 0.01;$ $\phi = 0.05; M = 2.0;$ except the varied parameter values as shown in figures.

Figs. 2 - 3 are depicted to analyse the impact of thermal Grashof number and concentration Grashof number on the velocity profiles of the nanofluid flow. It is observed that, the rise in the value of thermal Grashof number Gt and concentration Grashof number and Gc , we witnessed improvement in the velocity profiles of the flow. Generally, the ratio of buoyancy force to the viscous force is known as Grashof number, Disparity in the buoyancy forces regulates the momentum boundary layer.

The nature of velocity and thermal distributions for rising values of volume fraction parameter ϕ are shown in Figs. 4 - 5. It shows that, hike in the value of volume fraction parameter ϕ decreases the velocity fields and improves the thermal fields. Physically, improved value of ϕ enlarges the viscosity of the nanofluid as a result it slows down the velocity of the fluid. Hence, we seen declination in velocity fields, and as fluid moves with less velocity correspondingly temperature of the fluid increases. Hence we observed enhancement in the temperature profiles.

Figs. 6 - 7 illustrates the impact of thermal radiation parameter R and heat source parameter on the thermal distributions. It is evident from figures that, escalating values of thermal radiation parameter and heat source parameter enhances the temperature profiles. Generally, radiation is transmission of thermal energy. Rise in R and Q_H contributes heat energy to the flow.

Table 1 and 2 contrast the thermo-physical properties of alloy nanoparticles $Al_{50}Cu_{50}$ nanoparticles and methanol as a base liquid. Numerical values for in skin friction factor and heat transfer rates for diverse values pertinent flow parameters are depicted in Table 2. From table 2 it is observed that that, M has tendency to enhance thermal transfer rate. The ascending Gt and Gc leads to boost the skin friction factor. A raise in R and Q_H enhances the heat transport rates.

Table-1: Thermophysical properties of base fluid (methanol) and nanoparticles.

Physical properties	Base Fluid (Methanol)	Nanoparticle $Al_{50}Cu_{50}$
$C_p\ (J/Kg\ K)$	792	747.90
$\rho(Kg/m^3)$	2545	4133.68
$\kappa(W/mK)$	0.2035	112
$\beta \times 10^5 (K^{-1})$	$0.5x10^{-6}$	$1.4993x10^7$

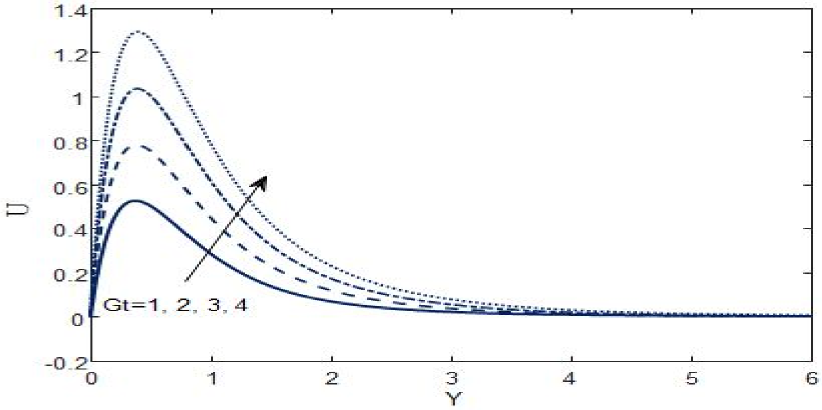

Fig-2: Impact of Gt on velocity profiles.

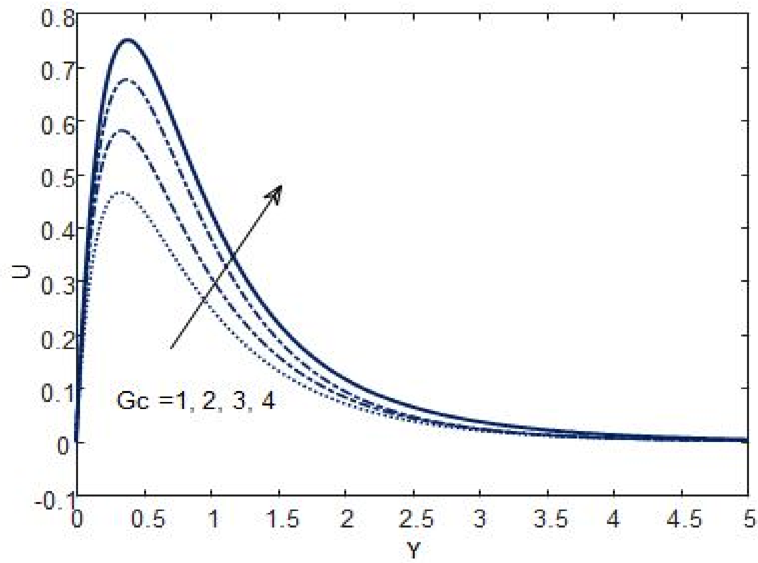

Fig-3: Impact of Gc on velocity profiles.

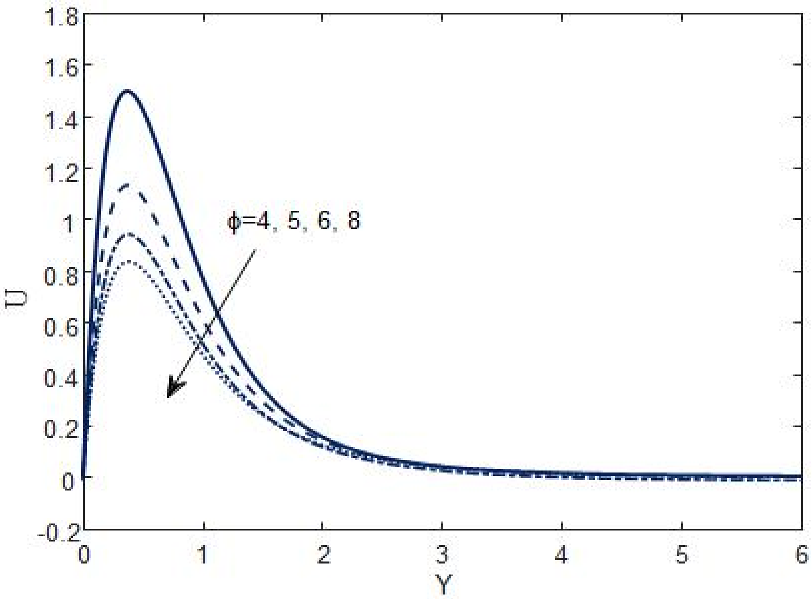

Fig-4: Impact of ϕ on velocity profiles.

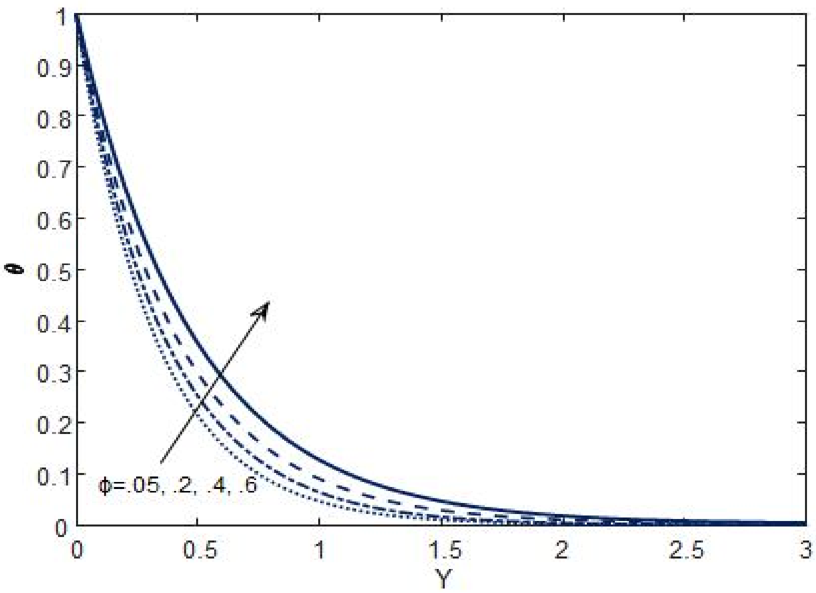

Fig-5: Impact of ϕ on Temperature profiles.

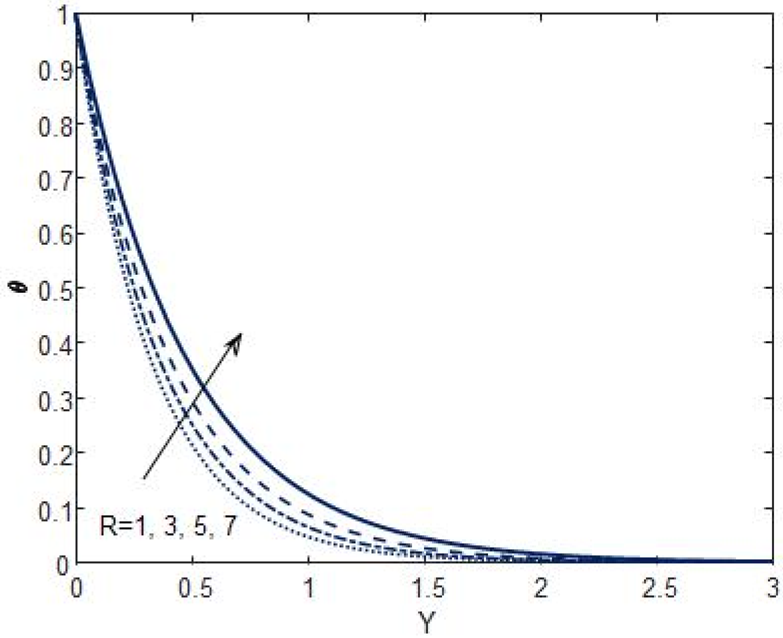

Fig-6: Impact of R on Temperature profiles.

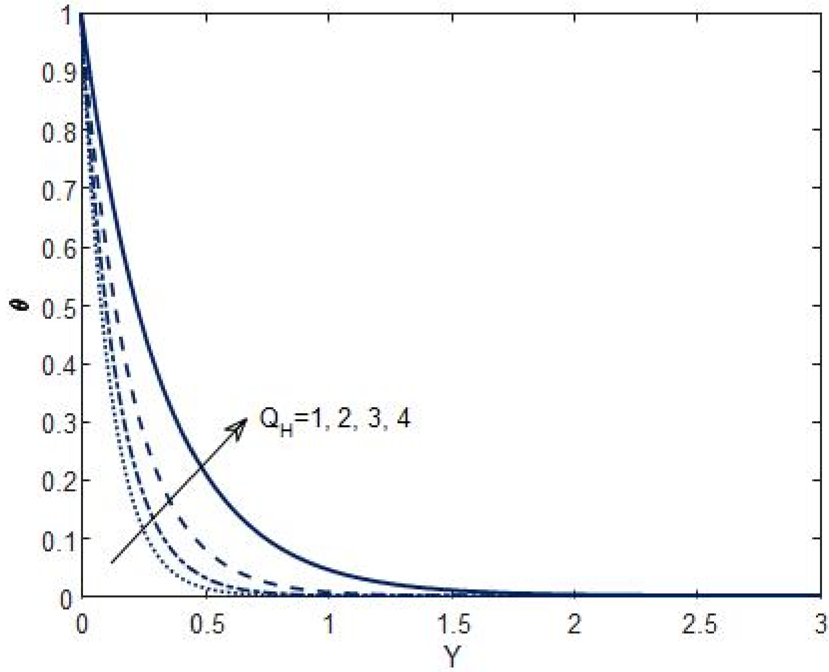

Fig-7: Impact of Q_H on Temperature profiles.

Table-2: Values of friction factor and Nusselt number for different values of non-dimensional parameters for $Al_{50}Cu_{50}$-Methanol nanofluid.

ϕ	Gr	Gm	R	Q_H	$Al_{50}Cu_{50}$-Methanol	
					C_{fx}	Nu_x / R_{ex}
0.1					3.99325	0.85239
0.2					1.42587	1.04335
0.3					0.64984	1.59662
	1				2.44173	2.43236
	2				2.95142	2.43236
	3				3.46807	2.43236
		1			2.27394	2.44872
		2			3.99329	2.42581
		3			5.70705	2.38957

			1		3.27512	0.85236
			2		3.42043	1.81958
			3		3.99328	2.79475
				1	3.42044	0.95587
				2	3.99322	1.21872
				3	5.96967	1.98303

V. CONCLUSIONS

The present study describes the 2D MHD flow of $Al_{50}Cu_{50}$-methanol nanofluid over infinitely moving porous plate in the presence of thermal radiation and heat source effects. The major findings of the present study are summarized as follows:

- Heat transfer rate in increased due to the enhancement in ϕ.
- Higher the values of M inflates the thermal profiles.
- The improvement in Gr and Gm leads to regulate the skin friction factor.
- Raise R and Q_H has tendency to upturn the thermal transport rates.

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A FUZZY INVENTORY MODEL FOR DETERIORATING ITEMS WITH PRICE-DEPENDENT DEMAND AND SHORTAGES

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ABSTRACT
A fuzzy inventory model has been developed here for deteriorating items in which the deteriorating rate has been considered as Weibull distribution with two parameters and the Demand rate has been considered as a function of selling price in which completely backlogged shortages are allowed. Most of the parameters are taken to be triangular fuzzy numbers and for defuzzification, Graded mean integration representation method has been used. The aim of this work is to minimize the inventory cost. Finally, some numerical examples have been illustrated for the result analysis and sensitive analysis have been carried out to check the behaviour of different parameters.

1. INTRODUCTION
Inventory is a stock of goods or other economic valued resources used in various organisations. The retailer continues his business in a smooth and efficient manner to maintain stock for their future affairs of an organisation. Inventory system is affected by numerous factors like demand cost, carrying cost, deterioration cost, shortage cost, pilferage cost etc. in the real world. Deterioration has a very significant role in the inventory system among all of the above given factors, which has been used many times by various inventory researchers. Deterioration refers the damage, decay, dryness, pilferage, expired, devaluation and loss marginal value of the product, which in turn, cannot be used or decrease the worth of the original one. For instance, fruits, vegetables, meat, alcohols, medicine, flowers, sea-foods undergo deterioration over time. During last few years, many researchers reported about deterioration items of inventory models [1–2]. The market price is always affected by the inventory of goods held by the manufactures rather than the rate at which manufacturers are supplying goods. It is believed that if the manufacturers are supplying goods at a rate equivalent to consumer demand, the static classical theory would propose that the market is in equilibrium. Always demand is the rate at which consumers want to buy a product. Economic theory embraces that demand consists of two factors: taste and ability to buy. Taste, which is the desire for a good, determines the readiness to buy the good at a specific price. The factors of demand depend on the market price and we know that when the market price for a product is high, the demand will be low. When the price is low, demand is high.

The main purpose of this work is to study the optimal retailer's restoration decisions for deteriorating items including price dependent demand. Here we have considered the selling price as time dependent. The deterioration rate is kept constant for the crisp inventory model. Hence we have to consider better approach of Fuzzy inventory model for deteriorating items for the slight difference of the original crisp value. The theoretical derivation of deteriorating inventory models began with Ghare and Schrader [3] who developed deterioration rate was exponential in nature. Weibull distribution, one of the continuous probability distributions is first described by a famous Swedish mathematician Waloddi Weibull in 1951 and it is first applied by Rosin & Rammler in the year 1933. Earlier price dependent demand has been reported by most of the researchers, namely Burwell et al[4] , Chakrabarty et al.[5] Mandal. B et al.[6] P.N.samanta et al.[7] Singh et al.[8] and recently Sumit Saha et al.[9] Garai et al.[10] reported about fully fuzzy inventory model for deteriorating items with price-dependent demand. The concept of Fuzzy sets is introduced by Lafti A Zadeh in 1965[11]. In 1998, Lee and Yao.[12] incorporated the fuzziness in the inventory model., Indrajit Singha et al.[13]developed fuzzy inventory model with shortages under fully backlogged using signed distance method. D.Datta et al[14]. developed fuzzy inventory model for deteriorating items with shortages under fully backlogged condition.

Here we have considered an EOQ model for deteriorating items having demand rate is a function of selling price. Shortages are also allowed during lead time which are entirely backlogged. All fuzzy parameters are considered as triangular fuzzy numbers. The main purpose of our work is to defuzzify the fuzzy model by using Graded mean integration representation method which are used to obtain Optimal EOQ to minimize the total inventory cost.

2. ASSUMPTIONS AND NOTATIONS
2.1 Assumptions
Here, we have assumed that

(a) The demand rate is function of selling price i.e. $U(\check{\lambda}) = m\check{\lambda}^{-n}$, $m, n > 0$, $\check{\lambda}$ is a Fuzzy selling price.

(b) Shortages are allowed and it is fully backlogged.

(c) The replenishment rate is unbounded and constant lead time.

(d) All items are deteriorating items but deterioration is not spontaneous.

(e) The deterioration rate follows Weibull Distribution i.e. $\check{\theta}(t) = \alpha\beta t^{\beta-1}$, $t > 0$,

Where α $(0 < \alpha < 1)$is a scale parameter and $\beta > 0$ is a shape parameter.

(f) The inventory cost of each item remains unchanged and order size is irrespective in nature.

(g) Each cycle period the inventory is replenished.

(h) C is the shortage cost per unit per unit time.

(i) The initial inventory cost is zero.

(j) When lead time L= t, ordering quantity is Q+L $U(\check{\lambda})$.

2.2. Notations

There are some notations are used throughout this paper:

y (t): Inventory level at any time t $(L \leq t \leq T)$.

O_C: Ordering cost per order at time zero.

O_D :Deterioration amount of the material per each cycle per unit time.

$\check{\theta}(t)$: Time dependant deterioration rate.

$U(\lambda)$: Unit item demand rate per unit time.

L: Lead time.

$\check{U}_c$: Fuzzy purchase cost per unit time.

U_s: Unit item shortage cost per unit time.

H_c: Total inventory holding cost per cycle.

D_c: Total inventory deterioration cost per cycle.

Q: Maximum inventory level.

$\check{h}$: Fuzzy holding cost per item per unit time.

$F_{tc}(t)$: Fuzzy total cost of the inventory.

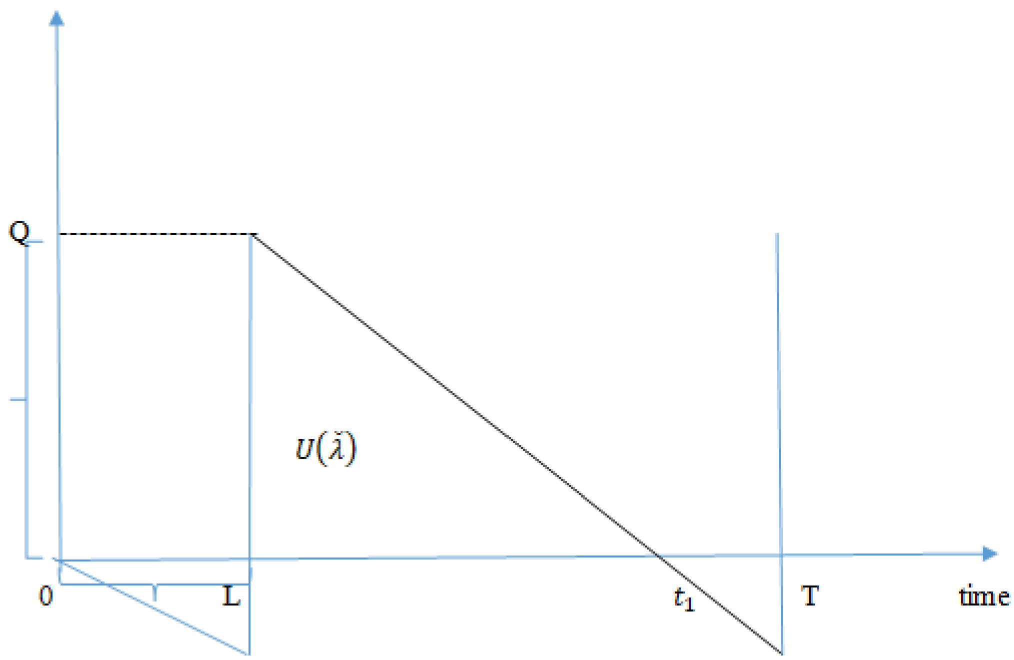
$G_{tc}(t)$: Defuzzified total cost of the inventory per cycle.

$G_{tc}(t^*)$: Optimal defuzzified total cost of the inventory per cycle

3. MATHEMATICAL MODEL

Let Q be the maximum inventory level at time t, $U(\check{\lambda})$ be the demand rate per unit and L is the lead time. Since the price dependant demand is deterministic in nature, hence the depletion of inventory occurs due to deterioration as well as demand in each cycle. The main purpose of inventory model is to determine optimal order quantity for a particular order cycle that keep the total relevant cost as low as possible .in this model holding cost is a function of time and shortages are allowed. Hence at time T shortages occurred and assemble to the level s_1. In the first interval [0, L] inventory level is zero.

Figure-1: Inventory Level at any time t



The differential equation which described the instantaneous state of y(t) over the period [L,T] is given by

$$\frac{dy(t)}{dx} + \alpha\beta t^{\beta-1}y(t) = -m\check{\lambda}^{-n} \quad , L \le t \le t_1 \tag{1}$$

$$\frac{dy(t)}{dx} = -m\check{\lambda}^{-n} \quad , t_1 \le t \le T \tag{2}$$

With the boundary conditions y (t_1) =0 and y(T)= -s_1

Solving the above linear differential equation (1) we get

$$y(t).e^{\alpha t^{\beta}} = -m\check{\lambda}^{-n}\left[t + \alpha\frac{t^{\beta+1}}{\beta+1}\right] + K_1 \tag{3}$$

Where K_1 is a constant of integration. Now using the boundary condition y (t_1)= 0 we get

$$y(t) = m\check{\lambda}^{-n}\left[(t_1 - t) + \frac{\alpha}{\beta+1}({t_1}^{\beta+1} - t^{\beta+1})\right][1 - \alpha t^{\beta}] \quad , L \le t \le t_1 \tag{4}$$

From the figure-1, initially time t=0, now at time t = L then the organisation reach maximum inventory. Hence equation (4) represented by Q

$$Q = m\check{\lambda}^{-n}\left[(t_1 - L) + \frac{\alpha}{\beta+1}({t_1}^{\beta+1} - L^{\beta+1})\right][1 - \alpha L^{\beta}] \tag{5}$$

Again solving the differential equation (2) we get

$$y(t) = -m\check{\lambda}^{-n}t + K_2 \tag{6}$$

Using the second boundary condition in equation (6) we get

$$y(t) = m\check{\lambda}^{-n}(T - t_1) - s_1 \tag{7}$$

If t=T then $y(t) = -s_1$.Therefore equation (7) becomes $s_1 = m\check{\lambda}^{-n}(T - t_1)$ (8)

Hence the total amount of materials deteriorates during one cycle is given by

$$O_D(t) = m\check{\lambda}^{-n}\left[(t_1 - L) + \frac{\alpha}{\beta+1}({t_1}^{\beta+1} - L^{\beta+1})\right][1 - \alpha L^{\beta}] - D(t_1 - L) \tag{9}$$

The fuzzy average total cost of inventory includes the following costs:

(I) Ordering cost: OC = O_C (10)

(II) Carrying cost: Holding cost is a function of average inventory cost and it is given by the

Formula $\quad H_c = \int_L^{t_1} \check{h}\, y(t)dt$

$H_c \quad =\check{h}\, m\check{\lambda}^{-n} \left\{\left[\left(\frac{t_1^{\,2}}{2} - \alpha\frac{t_1^{\,\beta+2}}{\beta+1} + \alpha\frac{t_1^{\,\beta+2}}{\beta+2}\right) - \left(Lt_1 - \frac{L^2}{2} - \alpha\frac{t_1L^{\beta+1}}{\beta+1} + \alpha\frac{L^{\beta+2}}{\beta+2}\right)\right] + \frac{\alpha}{\beta+1}\left[\left(t_1^{\,\beta+2} - \frac{t_1^{\,\beta+2}}{\beta+2} -\right.\right.\right.$
$\alpha t12\beta+2\beta+1+\alpha t12\beta+22\beta+2-L\beta+1-L\beta+2\beta+2-\alpha t1\beta+1L\beta+1\beta+1+\alpha L2\beta+22\beta+2$
(11)

(III) Deterioration Cost: $D_c = \check{c} \times O_D(t)$

$$D_c = \check{c}\left\{m\check{\lambda}^{-n}\left[(t_1 - L) + \frac{\alpha}{\beta+1}\left(t_1^{\,\beta+1} - L^{\beta+1}\right)\right]\left[1 - \alpha L^{\beta}\right] - D(t_1 - L)\right\} \tag{12}$$

(IV) Purchase Cost: $\breve{U}_c = \check{c}\left[Q \;+ \text{L}\; U(\check{\lambda})\right]$

$$U_c = \check{c}\left\{m\check{\lambda}^{-n}\left[(t_1 - L) + \frac{\alpha}{\beta+1}\left(t_1^{\,\beta+1} - L^{\beta+1}\right)\right]\left[1 - \alpha L^{\beta}\right] + Lm\check{\lambda}^{-n}\right\} \tag{13}$$

(V) Shortage Cast: $U_s = S\left[-\int_{t_1}^{T} y(t)dt\right]$

$$U_s = S\left[m\check{\lambda}^{-n}\left(\frac{T^2 + t_1^{\,2}}{2} - Tt_1\right)\right] \tag{14}$$

Hence, fuzzy total variable cost for one cycle is given by

$$F_{tc}(t_1) = O_C + H_c + D_c + U_c + U_s$$

$F_{tc}(t) = O_C + \check{h}\, m\check{\lambda}^{-n} \left\{\left[\left(\frac{t_1^{\,2}}{2} - \alpha\frac{t_1^{\,\beta+2}}{\beta+1} + \alpha\frac{t_1^{\,\beta+2}}{\beta+2}\right) - \left(Lt_1 - \frac{L^2}{2} - \alpha\frac{t_1L^{\beta+1}}{\beta+1} + \alpha\frac{L^{\beta+2}}{\beta+2}\right)\right]\frac{\alpha}{\beta+1}\left[\left(t_1^{\,\beta+2} - \frac{t_1^{\,\beta+2}}{\beta+2} -\right.\right.\right.$
$\alpha t12\beta+2\beta+1+\alpha t12\beta+22\beta+2-L\beta+1-L\beta+2\beta+2-\alpha t1\beta+1L\beta+1\beta+1+\alpha L2\beta+22\beta+2+cm\lambda-n$
$(t1-L)+\alpha\beta+1(t1\beta+1-L\beta+1)1-\alpha L\beta-D(t1-L)+cm\lambda-n\;\;(t1-L)+\alpha\beta+1(t1\beta+1-L\beta+1)1-\alpha L\beta+Lm\lambda-n$
$+\,Sm\lambda-n\,T2+t122-Tt1$ (15)

4. ALGORITHM FOR SOLUTION

In order to find optimal cycle length to minimize $F_{tc}(t)$ by using following steps.

Step-1: Let us consider the fuzzy parameters as triangular fuzzy numbers i.e. $\check{\lambda}, \check{c}, \check{h}$

As $(\check{\lambda}_1, \check{\lambda}_2, \check{\lambda}_3), (\check{c}_1, \check{c}_2, \check{c}_3) and (\check{h}_1, \check{h}_2, \check{h}_3)$ respectively.

Step-2: Find out $\widetilde{F_{tc_1}}(t_1) = F_{tc}(\check{p}_1, \check{c}_1, \check{h}_1)$, $\widetilde{F_{tc_2}}(t_1) = F_{tc}(\check{p}_2, \check{c}_2, \check{h}_2)$, $\widetilde{F_{tc_3}}(t_1) = F_{tc}(\check{p}_3, \check{c}_3, \check{h}_3)$,

Step-3: Defuzzified the fuzzy total cost by Graded mean integration representation method

i.e. $GF_{tc}(t_1) = \frac{1}{6}\left[\widetilde{F_{tc_1}} + 4\widetilde{F_{tc_2}} + \widetilde{F_{tc_3}}\right]$ (16)

Here $GF_{tc}(t)$ is the defuzzified total average cost, and $\widetilde{F_{tc_k}}(t)$ (k=1,2,3) are the fuzzy costs.

Step-4: Find $t_1^{\,*}$ such that $\frac{dGF_{tc}(t_1^{\,*})}{dt_1} = 0$ and $\frac{d^2GF_{tc}(t_1^{\,*})}{dt_1^{\,2}} > 0$. Where $t_1^{\,*}$ is the optimal cycle length.

Step-5: Hence $GF_{tc}(t_1^{\,*})$ is called optimal cost of the material.

5. NUMERICAL EXAMPLES

The following examples are presented to validate the proposed model.

Example-1: $\boldsymbol{O_c = 300, m = 12, n = 1, \check{c} = 9, \alpha = 0.004, \beta = 0.03, L = 1, \check{h} = 3, s = 10, \check{\lambda} = 9}$,

$\boldsymbol{T = 1}, \check{\lambda}_1 = 7, \check{\lambda}_2 = 9, \check{\lambda}_3 = 11, \check{c}_1 = 6, \check{c}_2 = 9, \check{c}_3 = 12, \check{h}_1 = 2, \check{h}_2 = 3, \check{h}_3 = 4$

Solution:

The optimal solution is $\quad t_1^{\,*}$ =0.54854514, $\quad GF_{tc}(t_1^{\,*})$=308.131

Example-2: $\boldsymbol{O_c = 500, m = 12, n = 1, \check{c} = 9, \alpha = 0.004, \beta = 0.03, L = 1, \check{h} = 3, s = 10, \check{\lambda} = 9}$,

$\boldsymbol{T = 20}, \check{\lambda}_1 = 7, \check{\lambda}_2 = 12, \check{\lambda}_3 = 15, \check{c}_1 = 4, \check{c}_2 = 9, \check{c}_3 = 10, \check{h}_1 = 4, \check{h}_2 = 5, \check{h}_3 = 3$

Solution:

The optimal solution is $t_1{}^* = 13.91475$, $GF_{tc}(t_1{}^*)$=1155.21755

Example-3: $\boldsymbol{O_c = 400, m = 12, n = 1, \check{c} = 9, \alpha = 0.004, \beta = 0.03, L = 1, \breve{h} = 3, s = 10, \check{\lambda} = 9,}$

$\boldsymbol{T = 10}, \check{\lambda}_1 = 6, \check{\lambda}_2 = 7, \check{\lambda}_3 = 10, \check{c}_1 = 5, \check{c}_2 = 8, \check{c}_3 = 11, \breve{h}_1 = 3, \breve{h}_2 = 5, \breve{h}_3 = 4$

Solution:

The optimal solution is $t_1{}^* = 7.01332$, $GF_{tc}(t_1{}^*)$ = 650.1204

Example-4: $\boldsymbol{O_c = 1000, m = 12, n = 1, \check{c} = 9, \alpha = 0.004, \beta = 0.03, L = 1, \breve{h} = 3, s = 10, \check{\lambda} = 9,}$

$\boldsymbol{T = 40}, \check{\lambda}_1 = 7, \check{\lambda}_2 = 9, \check{\lambda}_3 = 11, \check{c}_1 = 6, \check{c}_2 = 9, \check{c}_3 = 12, \breve{h}_1 = 2, \breve{h}_2 = 3, \breve{h}_3 = 4$

Solution:

The optimal solution is $t_1{}^* = 30.7541$, $GF_{tc}(t_1{}^*)$=3569.117

Example-5: $\boldsymbol{O_c = 1000, m = 12, n = 1, \check{c} = 9, \alpha = 0.004, \beta = 0.03, L = 1, \breve{h} = 3, s = 10, \check{\lambda} = 9,}$

$\boldsymbol{T = 30}, \check{\lambda}_1 = 10, \check{\lambda}_2 = 15, \check{\lambda}_3 = 20, \check{c}_1 = 11, \check{c}_2 = 12, \check{c}_3 = 13, \breve{h}_1 = 7, \breve{h}_2 = 8, \breve{h}_3 = 9$

Solution:

The optimal solution is $t_1{}^* = 17.04416$, $GF_{tc}(t_1{}^*)$=2595.7387.

6. SENSITIVITY ANALYSIS

Table-1

O_c	$t_1{}^*$	$GF_{tc}(t_1{}^*)$
300	19.2946	839.409
400	19.2946	939.409
500	19.2946	1039.409
600	19.2946	1139.409

Table-2

T	$t_1{}^*$	$GF_{tc}(t_1{}^*)$
25	19.2946	839.409
30	22.8306	1145.02
40	29.8991	1964.88
50	36.9646	3063.25

Table-3

α	$t_1{}^*$	$GF_{tc}(t_1{}^*)$
0.006	19.2917	840.036
0.007	19.2889	840.66
0.008	19.2860	841.281
0.009	19.2832	841.901

Table-4

β	$t_1{}^*$	$GF_{tc}(t_1{}^*)$
0.05	19.3491	829.461
0.06	19.4057	819.167
0.07	19.4645	808.513
0.08	19.5257	797.489

Important observation from Table (1 to 4):

1. If ordering cost O_c increase then optimal total cast is increase but the optimal cycle length remain unchanged (from table-1).

2. If order of the cycle length T increase then optimal cycle length as well as total cost increase.

3. If scale parameter (α) increase then $t_1{}^*$ and $GF_{tc}(t_1{}^*)$ are increase.

4. If shape parameter (β) value is increased, then $t_1{}^*$ increase but $GF_{tc}(t_1{}^*)$ decrease.

7. CONCLUSION

This work reports an inventory model for deterioration items in which price dependent demand model is developed in the fuzzy environment. The mathematical approach considered as triangular fuzzy numbers and Graded mean integration representation method. Numerical examples are provided to justify in this work successfully. The sensitivity analysis is carried out to get the behaviour of the different parameters.

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LAGUERRE WAVELET BASED SERIES SOLUTION FOR SOLVING NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT
In this paper, we present a new simple and effective method for solving neutral delay differential equations (NDDE). This series solution numerical method is based on Laguerre wavelets namely Laguerre wavelet polynomials. We first introduce the properties of Laguerre wavelets and then employed them to solve NDDE. Only few terms of Laguerre wavelets are needed to obtain very good results. Numerical examples with comparison show the simplicity, efficiency and accuracy of the method.

Keywords: Laguerre wavelets; Neutral Delay Differential Equations (NDDE); convergence.

1. INTRODUCTION

Let us consider the Neutral Delay Differential Equation (NDDE) of the form

$$\frac{dy}{dt} = f(t, y(t), y(t-\mu(t, y(t))), y'(t-\lambda(t, y(t)))), \quad t_1 \le t \le t_f,$$

With $y(t) = \phi(t) \;\; t \le t_1$ where $f : [t_1, t_f] \times \Re \times \Re \times \Re \to \Re$ is a differentiable function, $\mu(t, y(t))$ and $\lambda(t, y(t))$ $\mu(t, y(t))$ and $\lambda(t, y(t))$ are continuous functions on $[t_1, t_f] \times \Re$ such that $t - \mu(t, y(t)) < t_f$ and $t - \lambda(t, y(t)) < t_f$. Also $\phi(t)$ represents the initial function (Karimi & Aminataei, 2008).

Neutral Delay Differential Equations represent particular class of Delay Differential Equations and they possess abundant applications in science & engineering like in mathematical modelling of biological, physiological, chemical process, electronic transportation system (control of ships and aircraft), neural networks and economic growth etc. Evidently most of these equations cannot be solved exactly. Therefore it is necessary to design efficient numerical methods to approximate solutions.

Some Researchers replaced the delay differential equations by a system of ordinary differential equations to find the solution which inherit the partial differences among the delay differential equations and system of ordinary differential equation [Kuang, 1993]. So it is better to solve delay differential equation independently. Many researchers try to solve delay differential equation using Runge-Kutta method (Bellen and Zennaro, 2004), Radau, waveform relaxation (Lelarsmee et al.,) Bellman's method (Bellen and Zennaro,2004), variational iteration method (Wang & Chen,2010), reproducing kernel Hilbert space method(RKHSM)(Lv and Gao,2012) etc. In this paper we solved NDDE using Laguerre Wavelets.

This paper is organized as follows: In section 2, basic idea about Wavelets and Laguerre wavelets presented. The method of solution to solve NDDE is discussed in Section 3. Section 4 is devoted to numerical comparisons between the results obtained by LWM in this work and some existing methods. Finally Conclusions are stated in the last section 5.

2. PROPERTIES OF LAGUERRE WAVELETS

In recent years, wavelets have found their way into many different fields of science and technology. Wavelets constitute a family of functions constructed from dilation and translation of a single function called mother wavelet. When dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets [19]:

$$\psi_{a,b}(x) = |a|^{-1/2} \psi\left(\frac{x-b}{a}\right), \forall a, b \in R, a \ne 0. \qquad (1)$$

If we limit the parameters a and b to discrete values as $a = a_0^{-k}, b = nb_0 a_0^{-k}, a_0 > 1, b_0 > 0.$ We have the following family of discrete wavelets,

$$\psi_{k,n}(x) = |a|^{1/2} \psi(a_0^k x - nb_0), \forall a, b \in R, a \ne 0, \qquad (2)$$

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where $\psi_{k,n}$ form a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$ and $b_0 = 1$,then $\psi_{k,n}(x)$ forms an orthonormal basis.

The Laguerre wavelets $\psi_{k,n}(x) = \psi(k, n, m, x)$ involve four arguments n =1, 2, 3...2^{k-1}, k is assumed any positive integer, m is the degree of the Laguerre polynomials and t is the normalized time. They are defined on the interval [0, 1) as [16]

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}} L_m\left(2^k t - 2n + 1\right) & , \frac{n-1}{2^{k-1}} \le t < \frac{n}{2^{k-1}} \\ 0, \quad \text{otherwise} & \end{cases} \tag{3}$$

where $L_m(t) = \frac{1}{m!} L_m\left(t\right)$

Here $m = 0,1,2,3,...,M-1$ & L_m are the Laguerre polynomials of degree m with respect to the weight function $w(t) = 1$ on the interval [0,∞], and satisfy the following recursive formula:

$$L_0(t) = 1, L_1(t) = 1 - t \quad \text{and} \quad L_{m+2}(t) = \frac{\left(\left(2m+3-t\right)L_{m+1}(t) - \left(m+1\right)L_m(t)\right)}{m+2}, m = 0,1,2,3... \tag{4}$$

3. LAGUERRE WAVELETS METHOD OF SOLUTION

The solution of Neutral Delay Differential Equation can be stretched as a Laguerre wavelet series as:

$$y(t) \cong \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} k_{n,m}\psi_{n,m}(t)$$

where $\psi_{n,m}(x)$ is given by the equation (3). We estimate y(x) by reduced series

$$y_{K,M}(t) \cong \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} K_{n,m}\psi_{n,m}(t) = K^T\psi(t) \tag{5}$$

Here, $$K^T = \left[K_{1,0},...,K_{1,M-1},K_{2,0},...,K_{2,M-1},...,K_{2^{k-1},0},...,K_{2^{k-1},M-1}\right]$$

$$\psi(t) = \left[\psi_{1,0},...,\psi_{1,M-1},\psi_{2,0},...,\psi_{2,M-1},...,\psi_{2^{k-1},0},...,\psi_{2^{k-1},M-1}\right]$$

Then a total number of $2^{k-1}M$ conditions should exist to determine the $2^{k-1}M$ coefficients

$$K_{10}, K_{11}, , \ldots, K_{1M-1,} K_{20}, K_{21}, \ldots, K_{2M-1}, \ldots, K_{2^{k-1}0}, K_{2^{k-1}1}, \ldots, K_{2^{k-1}M-1}.$$

When given equation is of order n then there are n initial conditions and namely:

$$\sum_{k=0}^{m_1-1} {}_{i,k}\, y^k(0) = \lambda_i \quad i = 0,1,...m_1-1,\ m_1 = 2\ where\ K_{i,k} \neq 0. \tag{6}$$

Now, we observe that there should be $2^{k-1}M - 2$ extra conditions to recuperate the unknown coefficients $K_{n,m}$. This we are carrying by putting Equation (5) in Equation (1):

$$\frac{d^{m_1}}{dx^{m_1}} \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} K_{n,m}\psi_{n,m}(x) = \sum_{i=1}^{J}\sum_{k=0}^{m_1-1} p_{i,k}(x)\frac{d^k}{dx^k}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} K_{n,m}\psi_{n,m}(\alpha_i x) + f(x) \tag{7}$$

Suppose, equation (7) is exact at $2^{k-1}M - 3$ points x_{i_1} as follows:

$$\frac{d^{m_1}}{dx^{m_1}} \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} K_{n,m}\psi_{n,m}(x_{i_1}) = \sum_{i=1}^{J}\sum_{k=0}^{m_1-1} p_{i,k}(x_{i_1})\frac{d^k}{dx^k}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} K_{n,m}\psi_{n,m}(\alpha_i x_{i_1}) + f(x_{i_1}) \tag{8}$$

$x_{i_1}'s$ are limit points of the following sequence for different values of i:

$$\{t_{i_1}\} = \frac{s_i + 1}{2} \text{ where } s_i = \cos\left(\frac{(i_1 - 1)\pi}{2^{k-1}M - 1}\right) \qquad i=2,3,\ldots,\ 2^{k-1}M - 2. \tag{9}$$

Combine Equation (8) and (7) to obtain $2^{k-1}M$ linear equations from which we can compute values for the unknown coefficients $K_{n,m}$ by solving Ax=B form.

Same procedure is repeated for equations of other and higher order also.

Note that here *k =1* fixed and *M* should be greater than or equal to the order of given equations.

4. NUMERICAL ILLUSTRATIONS

In this part, some test problems are solved to illustrate the accuracy and performance of proposed method. We consider some examples from Raza A and Khan A (2018) [1]:

Example-1. Consider the linear Neutral Delay Differential Equation [NDDE]

$$y'(t) = 0.50y'(0.50t) + 0.50y(0.50t) - y(t) \quad t \in [0,1] \tag{10}$$

with initial condition $y(0) = 1 \quad t \le 0$.

The exact solution of (10) is $y(t) = e^{-t} \quad t \le 0$.We solved this problem by using Laguerre Wavelet Method (LWM) for M=5 & K=1 with the procedure as described in section 3. The obtained results with errors are compared with the exact solution & existing methods from [1] in the tables 1, 2 and graph is given in Fig.1

Table-1: Maximum absolute errors of example 1 with existing methods [1]

Wang et al.(2009a)	Wang et al.(2009b)	Wang and Chem (2010)	Lv and Gao(2012)	Haar Wavelet Method(2018)	Our Method[LWM]
1.85E-003	7.66E-002	4.94E-002	4.01E-004	8.9532E-005	3.5497E-010

Table 2 Comparison between analytical (Exact) solution and Laguerre Wavelet Method (LWM) of Example1

t	Exact Solution	LWM Solution	Error
0	1.000000000000000E+000	9.999999996450331E-00	3.5496694472E-010
0.090909090909091	9.131007162822623E-001	9.130955655690972E-001	5.6409036519E-006
0.181818181818182	8.337529180751806E-001	8.337442469483134E-001	1.0400115764E-005
0.272727272727273	7.613003866968737E-001	7.612971107601666E-001	4.3030803141E-006
0.363636363636364	6.951439283988787E-001	6.951523762453569E-001	-1.2152658079E-005
0.454545454545455	6.347364189402818E-001	6.347554149078020E-001	-2.9927331965E-005
0.545454545454545	5.795782787848095E-001	5.795987505146356E-001	-3.5321768560E-005
0.636363636363636	5.292133415000503E-001	5.292220590962072E-001	-1.6472744493E-005
0.727272727272727	4.832250811898254E-001	4.832121689460841E-001	2.6720971745E-005
0.818181818181818	4.412331677599840E-001	4.412030606210483E-001	6.8234079247E-005
0.909090909090909	4.028903215291330E-001	4.028758669410991E-001	3.5877228271E-005
1.000000000000000	3.678794411714423E-001	3.679588729894524E-001	-2.1591806749E-004

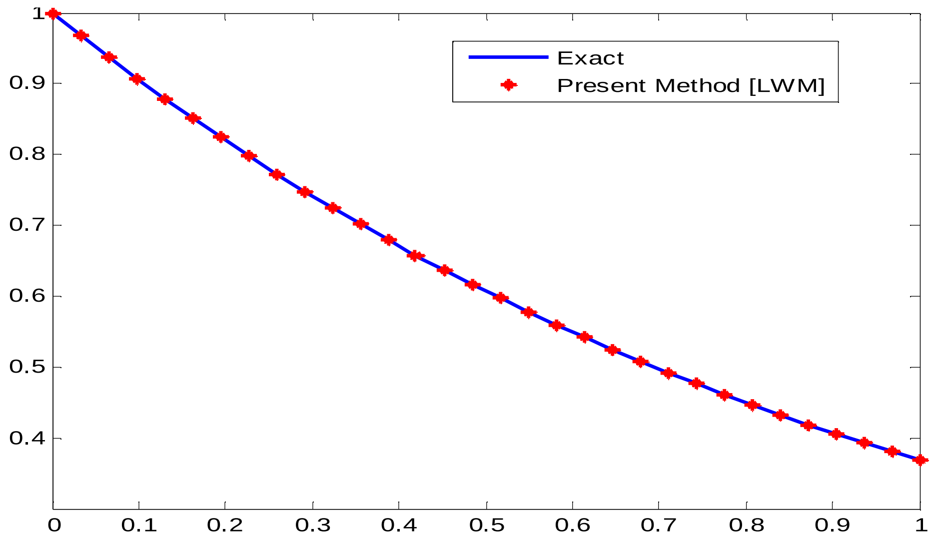

Fig-1: Graph of example 1 showing the Laguerre Wavelet Method [LWM] solution and exact solution.

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Example 2. Let us take the following Non-linear Neutral Delay Differential Equation [NDDE] .

$$y'(t)y(t)+\sqrt{\cos t}\ \ y'(\sqrt{t})+(\sin(\sqrt{t})+e^{t})y(\sin t)=e^{2t}+\sqrt{\cos t}\ \ e^{\sqrt{t}}+(\sin(\sqrt{t})+e^{t})e^{\sin t} \qquad (11)$$

where $t\in[0,1]$ and with the initial condition with $y(0)=1\ \ t\le 0.$

The analytical solution of (11) is $y(t)=e^{t}\ \ t\le 0.$

This non-linear equation solved by using the Laguerre Wavelet Method as described in section 3 and the comparison between analytical solution and exact solution with error is given in table 3 and solution graph in Fig.2.

Table-3: Comparison between Exact solution and Laguerre Wavelet Method (LWM) of the Example 2

t	Exact Solution	LWM Solution	Error
0	1.000000000000000E+000	9.999999999999586E-001	4.141131881851834E-014
0.090909090909091	1.095169439874664E+000	1.095186611758900E+000	-1.567965979509829E-005
0.181818181818182	1.199396102035386E+000	1.199434187729779E+000	-3.175405883736814E-005
0.272727272727273	1.313541957253949E+000	1.313580245661489E+000	-2.914897946625113E-005
0.363636363636364	1.438551009577678E+000	1.438567148610190E+000	-1.121895046092366E-005
0.454545454545455	1.575457103390318E+000	1.575442104939316E+000	9.520063079119966E-006
0.545454545454545	1.725392473466536E+000	1.725357168319567E+000	2.046209631229981E-005
0.636363636363636	1.889597108730308E+000	1.889569237728923E+000	1.474970577379926E-005
0.727272727272727	2.069429007156956E+000	2.069440057452628E+000	-5.339780022978386E-006
0.818181818181818	2.266375406628466E+000	2.266436217083194E+000	-2.683158957243795E-005
0.909090909090909	2.482065084623012E+000	2.482129151520412E+000	-2.581193289273260E-005
1.000000000000000	2.718281828459046E+000	2.718195140971337E+000	3.189054453462161E-005

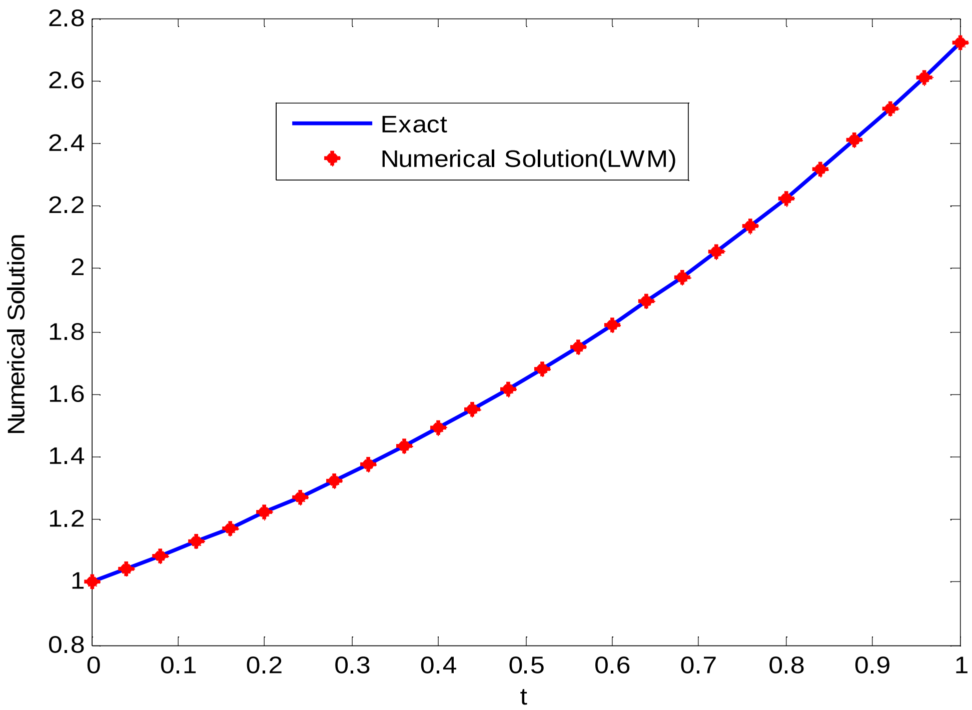

Fig-2: Comparison of the numerical solution with exact solution of example 2.

5. CONCLUSION

The Laguerre Wavelet Method (LWM) has been applied successfully to solve linear and non linear Neutral Delay Differential Equations (NDDE) which is easy to apply to the problems of complex nature. We obtained a very high accuracy numerical solution. The efficiency and accuracy of the proposed method were demonstrated by some test problems. This method has less calculation and takes less time to solve the problems. It is concluded from the above mentioned tables and figures that LWM is an accurate and efficient method to solve Neutral Delay Differential Equations.

ACKNOWLEDGMENT
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A STUDY ON PLANAR GRAPHS: EMBEDDINGS, DETECTION OF PLANARITY AND EULER'S FORMULA

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ABSTRACT

An abstract graph G is defined as a set of three sets, namely V, E and R ,that is G= (V, E, R)where V = set of p objects(vertices), E= set of q objects(edges) which are different from the objects of V and R is the set of relations from the set of objects of E to the unordered pairs of objects belongings to V. A graph is said to be a planar graph if there exists a geometric representation of G drawn on a plane such that no two edges intersect or cross each other. In this paper Detection of Planarity by Elementary Reduction, Kurotowski's graphs, Euler's Formula ,theorems-lemmas related to them are discussed.

Keywords: Planar Graphs, Embeddings, Detection of planarity, Kurotowski's graphs, Euler's Formula.

INTRODUCTION

COMBINATORIAL REPRESENTATION

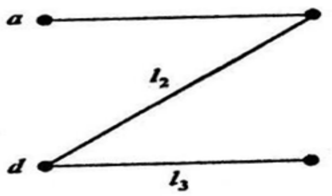
An abstract graph G is defined as a set of three sets, namely *V, E* and *R* ,That is G= (*V, E, R*) where *V* = set of p objects, *E* = set of q objects which are different from the objects of *V* and *R* is the set of relations from the set of objects of *E* to the unordered pairs of objects belongings to *V.*

Let V = { a, b, c, d} , E = {l_1 , l_2, l_3 }

R = { $l_1 \rightarrow$(a , b) , $l_2 \rightarrow$(b , d) , $l_3 \rightarrow$(d , c)}

By $l_1 \rightarrow$(a , b) we understand that l_1 is mapped to the unordered pair(a , b)

The above abstract G can be represented by a geometric figure on a plane.



Topological graph theory, broadly conceived, is the study of graph layouts. Initial motivation for this involved the famous Four Color Problem: can the regions of any map on the globe can be colored with four colors so that regions sharing a nontrivial boundary have different colors? More recent motivation comes from the study of circuit layouts on silicon chips. Crossings cause problems in a layout, so we want to know which circuits have layouts without crossings.

PLANAR GRAPHS

A graph is said to be a **planar graph** if there exists a geometric representation of G drawn on a plane such that no two edges intersect or cross each other.

A graph that cannot be drawn on a plane without a cross over or intersection between the edges, is called a **nonplanar graph**.

Faces or Regions

A planar graph G, partitions the plane into many regions. These regions are also called faces. The bounded regions are called interior faces and unbounded region is called the exterior face.

Degree of a Region

A face is said to be incident with vertices and edges forming the boundary of the face. The degree of the face is the number of edges incident on the face.

Exampls: (1) K_5 is a nonplanar graph.

We can prove this by drawing K_5 graph which has 5 vertices and 10 edges. Plot the vertices a, b, c, d, e. Draw the edges x_1, x_2 , x_3 , x_4 , x_5 , x_6 , x_7 , x_8 and x_9 so that none of then cross each other. Now we are left with the pair of points b , d. The joining b and d cannot be drawn without crossing one of the edges already drawn.

Therefore K_5 is non planar.(see the dia below)

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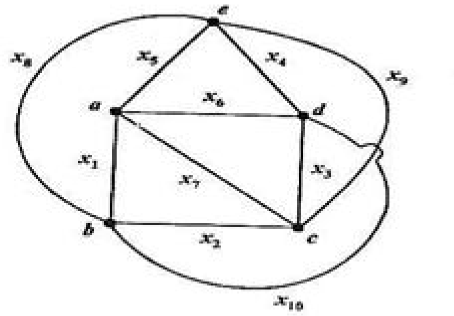
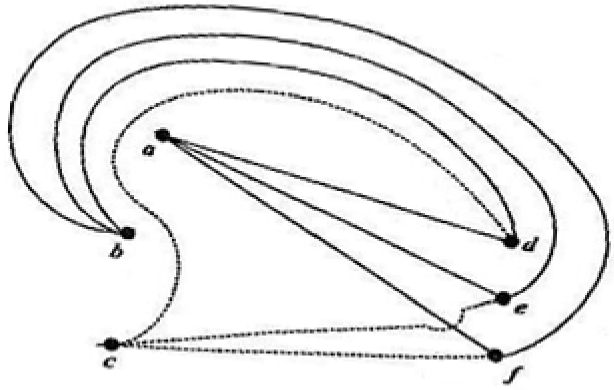


(2) $K_{3,3}$ is nonplanar graph.

Let points be x = (a, b, c) and y = (d, e, f). Draw 3 edges joining (a, d), (a, e) and (a, f). Again draw 3 edges joining (b, d), (b, e) and Draw 2 edges joining (c, d) and (c, f). But we can draw the remaining edge (c, e) only by crossing one of the edges already drawn.



Therefore by the actual drawing we observe that $K_{3,3}$ is nonplanar graph.

The above two graphs K_5 and $K_{3,3}$ are called **Kurotowski's graphs**.

PROPERTIES OF KUROTOWSKI'S GRAPHS

1. Both are regular graphs
2. Both are non planar. But removal of one edge converts them to planar graphs
3. The removal of one vertex converts them into planar graphs
4. K_5 is nonplanar graph with smallest number of vertices
5. $K_{3,3}$ is nonplanar graph with smallest number of edges.

Definition: A *polygonial path* or *polygonal curve* in the plane is the union of finitely many line segments such that each segment starts at the end of the previous one and no point appears in more than one segment except for common endpoints of consecutive segments. In a *polygonal u , v*-path, the beginning of the first segment is u and the end of the last segment is v.

A *drawing* of a graph G is a function that maps each vertex $v \in V(G)$ to a point *f(v)* in the plane and each edge *uv* to a polygonal *f(u), f(v)*-path in the plane. The images of vertices are distinct. A point in $f(e) \cap f(e')$ other than a common end is a crossing. A graph is planar if it has a drawing without crossings. Such a drawing is a *planar embedding* of G. A *plane graph* is a particular drawing of a planar graph with no crossings.

Theorem: (Restricted Jordan Curve Theorem). A simple closed polygonal curve C consisting of finitely many segmemts partitions the plane into exactly two faces, each having C as boundary.

Proof: Because we have finitely many segments, nonintersecting segments cannot be arbitrarily close together, and hence we can leave a face only by crossing C. As we traverse C in one direction, the nearby points on our right are all in the same face, and similarly for the points on the left. (There is a precise algebraic notion of the meaning of left and right here). If $x \notin C$ and $y \in C$, the segment xy first meets C somewhere, approaching it from the right or the left. Hence every point not in the plane lies in the same face with at least one of the two sets we have described.

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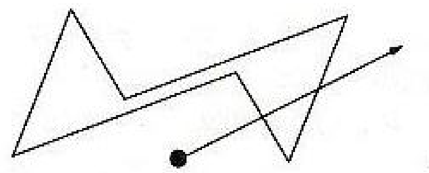
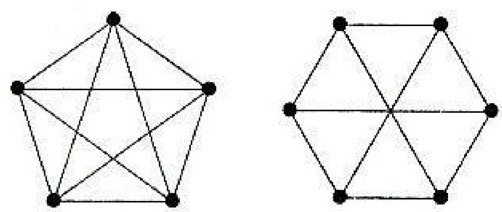
Intuitively, the sets are distinct because in the plane we cannot travers a closed curve and switch the meaning of left and right. We can also distinguish the inside from the outside without defining an orientation for the curve. Consider a ray that starts at a given point p. A direction for the ray is "bad" if the resulting ray contains an endpoint of a segment of C. Since C has finitely many segments, there are finitely many bad directions, and in each good direction the ray crosses C finitely often. As the ray rotates, the number of crossings can change only when the ray passes through a bad direction, but before and after those directions the parity of the number of crossings is the same. Call this the *parity* of *p*.



Suppose x, y are points in the same face of C, and let P be a polygonal x, y - path that avoids C. Because C has finitely, many segments, the ends of the segments of P can be adjusted slightly (if necessary) so that the rays along segments of P are in good directions for their endpoints. A segment of P belongs to a ray from one end that contains the other. Since the segment does not intersect C, the parity of the two points is the same. Hence any pair of points in the same face have the same parity. Because the endpoints of a short segment intersecting C exactly once have opposite parity, there must be two distinct faces. The even points and odd points comprise the outside face inside face, respectively.

Example: Consider a drawing of K_5 or $K_{3,3}$ in the plane using polygonal curves. Let *C* be a spanning cycle in the graph; *C* also is drawn as a closed polygonal curve. If the drawing is an embedding, chords of C must be drawn inside or outside this curve. Two chords conflict if their endpoints on C occur in alternating order. Conflicting chords must embed in opposite faces of C. Considering a 6-cycle from $K_{3,3}$, we are left with three pairwise conflicting chords and can put at most one inside and one outside. Considering a 5-cycle from K_5 , at most two chords can go inside or outside. Hence neither of these graphs is planar.



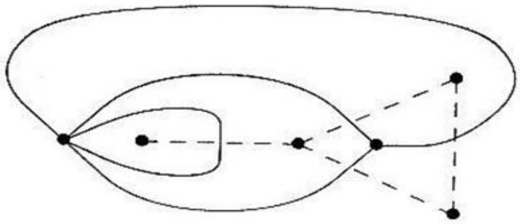
DUAL GRAPHS

We can view a geographic map on the plane or the sphere as a plane graph in which the faces are the territories of the map, the vertices are places where several boundaries meet, and the edges are the portions of the boundaries that join two vertices. We allow the full generality of loops and multiple edges. From any plane graph G, we can form another plane graph called its "dual".

Definition: Suppose G is a plane graph. The *dual graph* G* of G is a plane graph having a vertex for each region in G. The edges of G* correspond to the edges of G as follows: if *e* is an edge of G that has region X on one side and region Y on the other side, then the corresponding dual edge

$e^* \in E(G^*)$ is an edge joining the vertices x , y of G* that correspond to the faces X, Y of G.

Example: A simple plane graph and its dual. Below we have drawn a plane graph G with dashed edges and its dual G* with solid edges. Since G has four vertices, four edges, and two faces, G* has four faces, four edges, and two vertices. As in this example, a simple plane graph may have loops and multiple edges arise in its dual. A cut-edge of G becomes a loop in G*, because the faces on both sides of it are the same. Multiple edges arise in the dual when distinct regions of G have more than one common boundary edge. (see the dia)



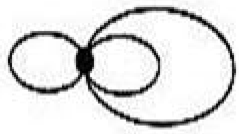
Remark: *Geometry of the dual.* Some arguments require a careful geometric description of the placement of vertices and edges in the dual. For each face F of G, we place the dual vertex for F in the interior of F; hence each face of G contains one vertex of G*. For each edge in the boundary of F, we place a “half–edge” emanating from the dual vertex for F to a point on the edge. These half-edges do not cross each other, and each meets another half-edge at the boundary to form a dual edge. No other edges enter F. Hence G* is a plane graph, and each edge of G* in this layout crosses exactly one edge of G. Such arguments lead to a proof that (G*)* is isomorphic to G if and only if G is connected.

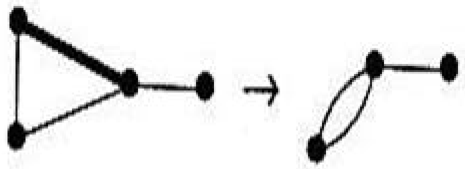
EULER’S FORMULA

Euler’s Formula is the basic computational tool for planar graphs.

Theorem: (Euler [1758]): If a connected plane graph G has n vertices, e edges, and f faces, then $n-e+f=2$.

Proof: Proof by inducation on n. If n=1,then G is a″bouquet″ of loops, Each a closed curve in the embedding. If e=0,then we have one face, And the formula holds. Each added loop passes through a region and partitions it into two regions (by the Jordan Curve Theorem),so the formula holds for n=1 and any $e \geq 0$.





Now suppose n(G)>1. Since G is connected, we can find an edge that is not a loop. When we contract such an edge, we obtain a plane graph G′ With n'vertices, e' edges, and f' faces. The contraction does not change the number of faces (we merely shortened boundaries),but it reduces the number of edges and vertices by one.Applying the induction hypothesis, we find $n-e+f=n'+1-(e'+1)+f=2$.

REMARK

1) Euler’s formula implies that all planar embeddings of a connected graph G have the same number of faces. Thus, although the dual may depend on the embedding chosen for G, the number of vertices in the dual does not.
2) Contracting a non-loop edge of G has the effect of deleting an edge in G*. Similarly, deleting a non-cut edge of G has the effect of contracting an edge in G*, as two faces of G merge into a single face.
3) Euler’s formula as stated fails for disconnected graphs. If a plane graph G has k components, then we can adjust Euler’s formula by observing that adding k–1 edges to G will yield a connected plane graph without changing the components is $n-e+f$=k+1 (for example, consider a graph with n vertices and no edges).

Euler’s formula has many applications, particulary for simple plane graphs, where all faces have length at least 3.

Theorem: If G is a simple planar graph with at least three vertices, then $e(G) \leq 3n(G)-6$. If also G is triangle-free, then $e(G) \leq 2n(G)-4$.

Proof: It suffices to consider connected graphs, since otherwise we could add edges. We can use Euler’s formula to relate n(G) and e(G) if , We can dispose of f. By the known proposition Proposition provides an inequality between e and f . Every face boundary in a simple graph contains at least three edges (if n(G)≥ 3). If $\{ f_i \}$ is the sequence of face-lengths, this yields angle-free, then the faces have length at least four. In this case $2e = \Sigma f_i \geq 4f$, and we obtain $e \leq 2n-4$.

EXAMPLE: K_5 and $K_{3,3.}$ Euler’s formula incorportates the earlier geometric reasoning we used to show that K_5 and $K_{3,3,}$ are nonplanar. The nonplanarity follows immediately from the edge bound. For K_5 , we have e=10> 9 = 3n–6, and for the triangle-free $K_{3,3}$ we have e = 9>8 = 2n –4.

DETECTION OF PLANARITY BY ELEMENTARY REDUCTION

A disconnected graph is planar if each of its components is planar. A seperable graph is planar if each of the blocks is coplanar. The planarity of each is determined by the procedure given below.

Stage 1: Addition or removal of a loop does not affect the planarity.

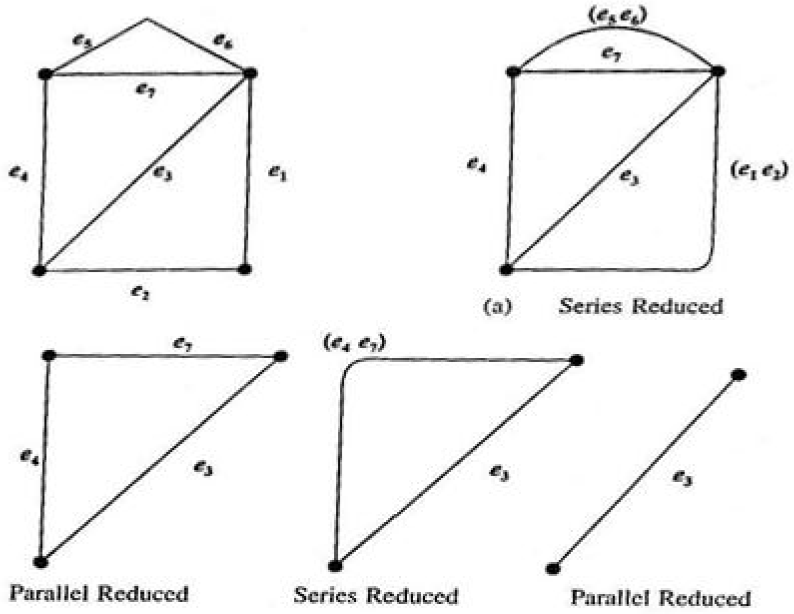
Therefore loops are deleted.

Stage 2: If there are more than one edge joining two points u and v, remove all edges except one.

Stage 3: When there are only two edges meeting at a vertex, remove the vertex and merge the edges and treat it as one edge.

Repeat the stages 2 and 3. The graph will be reduced drastically.

Let the graph G be reduced to graph H.



If G is reduced to a graph K_2 or K_4, that is if H = K_2 or K_4, we conclude that G is planar, since K_2 or K_4 are planar.

If G is reduced to a non separable simple graph with n $\geq$5 and $e \geq 7$, i.,e H has $n \geq 5$ and $e \geq 7$, test whether $e \leq 3n - 6$. If this inequality is not satisfied, the graph H is nonplanar. If this inequality is satisfied, we can't conclude otherwise, because the above inequality is a necessary condition. In such cases the following characterizations may be applied.

1) If a graph G is planar, then G does not contain a subdivision of K_5 or $K_{3,3}$. Assume that G contains L which is a subdivision of K_5, $K_{3,3}$. Since K_5, $K_{3,3}$ are nonplanar, L is also nonplanar. Therefore G is nonplanar. This is a contradiction. Therefore the assumption is wrong. Therefore G does not contain a subdivision of K_5 or $K_{3,3}$.

2) If G doesnot contain the contraction of K_5 or $K_{3,3}$, prove that G is planar. Assume that G is nonplanar. Therefore G contains a subdivision L of K_5 or $K_{3,3}$. By contracting one of the two edges incident at a vertex of degree two in L, we get the contraction of K_5 or $K_{3,3}$. Therefore the assumption is wrong. Therefore we conclude that G is planar.

Note: The converses of the above two, are also true. And the above characterizations are briefly discussed in the next chapter.

Characterization of Planar Graphs

Which graphs embed in the plane? In the previous discussion we have shown that K_5 AND $K_{3,3}$ do not. In a natural sense, these are the critical graphs and yield a characterization of planarity. Before 1930, the most actively sought result in graph theory was the characterization of planar graphs, and the characterization using K_5 and $K_{3,3}$ is known as Kuratowski's theorem.

Kasimir Kuratowski's once asked Harary who originated the notation for K_5 and $K_{3,3}$; harary replied that " K_5 stands for Kasimir and $K_{3,3}$ stands for Kuratowski!" Recall that *subdividing* an edge or performing an *elementary subdivision* means replacing the edge with a path of length 2. A subdivision of G is a graph obtained from G by a sequence of elementary subdivision, turning edges into paths through new vertices of degree 2. If $\delta(G)\geq 3$ and *H* is a subdivision of G, then vertices of *H* having degree at least 3 are the *branch vertices*; these are the images of the original vertices. Subdividing edges does not affect planarity, so we seek a characterization by finding the *topological minimal* non planar graphs-those that are not subdivisions of other nonplanar graphs. We already know that a graph containing any subdivision of K_5

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or $K_{3,3}$ is nonplanar. Kuratowski [1930] proved that G is planar if and only if G contains no subdivision of K_5 or $K_{3,3}$.

Wagner [1937] proved another characterization. Deletion and contraction of edges preserve planarity, so we can seek the minimal nonplanar graphs using this operations. Wegner proved that G is planar if and only if it as no subgraph contractible to K_5 or $K_{3,3}$.

PREFERATION FOR KURATOWSKI'S THEOREM

Thomassen's contribution concern the 3-connected graphs, where the stronger results hold. We first reduce the problem to the 3-connected case. For convenience, we call a subgraph of G that is a subdivision of K_5 or $K_{3,3}$ a *Kuratowski subgraph* of G . a *minimal nonplanar graph* is a nonplanar graph such that every proper subgraph is planar. We reduce problem to the 3-connected case by proving that a minimal nonplanar graph having no Kuratowski subgraph must be 3-connected. Then a proof of planarity 3-connected graphs with no Kuratowski subgraph completes the proof of Kuratowski's Theorem.

Lemma1: If E is the edge set of a face in some planar embedding of G, then G has an embedding in which E is the edge set of the unbounded face.

Proof: Project the embedding onto the sphere, where the edge sets of regions remain the same and all regions are bounded, and then return to the plane by projecting from inside the face bounded by E .

Lemma 2: Every minimal nonplanar graph is 2-connected

Proof: Suppose G is a *minimalnonplanar graph*. If G is disconnected, we can embed one component of G inside one face of an embedding of the rest of G . If G has a cut-vertex v, $G_1,\ldots,G_k$ let be the subgraphs of G induced by v together with a component of G-v. By the minimality of G, these subgraphs are planar. By Lemma (previous), we can embed each with v on the outside face. We can squeeze each embedding to fit in an angle smaller than 360/k degrees at v , after which we can merge the embedding at v to obtain an embedding of G .

Lemma 3: Suppose S={x, y} is a 2-cut of G and G_1, G_2 are subgraphs of G such that $G_1 \cup G_2 = G$ and $V(G_1) \cap V(G_2) = S$. Let $H_I = G_I \cup xy$. If G is nonplanar, then at least one of H_1, is nonplanar.

Proof: Suppose H_1 and H_2 are planar. Then the edge xy occurs on some face in a planar embedding of H_1 . By Lemma 1, H_1 has a planar embedding with xy on the outside face. This allows H_1 to the attached to an embedding of of H_2 , embedded in a face with xy on the boundary. Deleting the edge xy if it does not appear in G, we have constructed a planar embedding of G.

Lemma 4 : Suppose G is a nonplanar graph with no Kurotowski subgraph, and G has the fewest edges among such graphs. Then G is 3-connected.

Proof : If we delete an edge of G, we cannot create a Kuratowski subgraph. Therefore, the hyphotheses on G guarantee that deleting one edge produces a planar subgraph, and hence G is s minimal nonplanar graph. By Lemma 2 G is 2-connected. Suppose G has a 2-cut S={x, y}. Since G is nonplanar, Lemma 3 guarantees that H_1 has fewer edges than G, the choice of G forces H_1 to have a Kuratowski subgraph. All of H_1 appears in G, except possibly the edge xy. The role of xy in the Kuratowski subgraph of H_1 can be played by an x , y-path H_2 to obtain a Kuratowski subgraph of G. This contradicts the hypothesis that G has no Kuratowski subgraph , so G has no 2-cut.

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A SURVEY ON DIFFERENTIAL COUNT OF BLOOD IN HUMAN USING TOPSIS METHOD

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ABSTRACT
This paper is a survey dealing with various differential count of blood in Human beings in day-to-day life situations . The TOPSIS method is used in inferring the abnormality of a patient who truly needs special attention.

Index Terms: Vague matrices, Normalised Euclidean distance, TOPSIS

I. INTRODUCTION
Fuzzy set was proposed by Zadeh [11] in 1965 using only one membership degree for the first time. Later vague set was developed by expanding the fuzzy sets by Gau [4].The common task in human activities is Multi Attribute Decision Making method. This method finds the most preferred alternative from a set of alternatives. Esfandyar[3] proposed two multiple - attribute decision making methods in solving a plant layout problem. Deepa Joshi, Sanjay kumar [1] proposed Intuitionistic fuzzy entropy and distance measure based TOPSIS method for Multi-criteria decision making. Deng- feng Li and Jiang -Xia Nan [2] extended the TOPSIS for multi-attribute group decision making under Atanassov IFS Environments.

TOPSIS is an approach of viability for the case study. A survey is done in this paper considering the differential count of four patients such as Neutrophil, Lymphocytes, Eosinophils and Monocytes with respect to the percentages. Their respective percentages are then converted to the vague matrices and TOPSIS method is used to find the abnormal condition among the four types of differential count of the patients with respect to the percentage values, stating that a particular patient with abnormality needs further improvement.

II. PRELIMINARIES
DEFINITION 2.1[4]
A vague set A in the universe of discourse U is characterized by two membership functions given by:

(i) A true membership function $t_A: U \to [0,1]$ and

(ii) A false membership function $f_A: U \to [0,1]$ where t_A is the lower bound on the grade of membership of x derived from the "evidence for x" , f_A is the lower bound on the negation of x derived from the "evidence against x" , and $t_A + f_A \leq 1$. Thus the grade of membership of μ is the vague set A is bounded by a subinterval $[t_A, 1 - f_A]$ of [0,1]. This indicates that if the grade of membership of x is $\mu(x)$ then, $t_A(x) \leq \mu(x) \leq 1 - f_A(x)$.

DEFINITION 2.2[8]
A vague matrix is a matrix pair A=[< t_{ij},$1 - f_{ij}$ >] of non- negative real numbers satisfying $0 \leq t_{ij} + f_{ij} \leq 1$ for all i , j with vague hesitation degree $\pi_A = 1 - (t_A(x) + f_A(x))$.

DEFINITION 2.3[8]
Let A and B be vague sets defined as A= $\{<x, [t_A(x), 1 - f_A(x)] > / x\epsilon X\}$ and B= $\{<x, [t_B(x), 1 - f_B(x)] > / x\epsilon X\}$ then the vague normalized Euclidean distance between the vague sets A and B is given as follows:

$$d(A,B) = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}\left[\left(t_A(x) - t_B(x)\right)^2 + \left((1 - f_A(x)) - (1 - f_B(x))\right)^2 + \left(\pi_A(x) - \pi_B(x)\right)^2\right]}$$

ALGORITHM OF TOPSIS [8]
This algorithm includes several steps in vague TOPSIS as follows:

Let $\{B = B_1, B_2, \dots \dots B_n\}$ be the alternatives and $\{S = S_1, S_2, \dots \dots S_n\}$ be the set of criteria.

STEP 1: Set the vague preference relation matrix.

Let A = $(a_{ij})_{m \times n}$

(i.e)., $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{32} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

be a vague preference matrix of criteria, where

$$\left(t_{ij}\right)^{*} = \max\left\{t_{ij}, \max_{p}\left\{\frac{t_{ip}t_{pj}}{t_{ip}t_{pj}+(1-t_{ip})(1-t_{pj})}\right\}\right\} \quad \ldots\ldots(1)$$

$$\left(1-f_{ij}\right)^{*} = \max\left\{1-f_{ij}, \max_{p}\left\{\frac{(1-f_{ip})(1-f_{pj})}{(1-f_{ip})(1-f_{pj})+f_{ip}f_{pj}}\right\}\right\} \quad \ldots\ldots(2)$$

where $\left(t_{ij}\right)^{*}$ and $\left(1-f_{ij}\right)^{*}$ are components of (A^{*}) matrix representing the values of vague membership of the alternatives x_i over x_j respectively and $0 \le \left(t_{ij}\right)^{*} \le \left(f_{ij}\right)^{*} \le 1$ for all i,j=1,2,3,….,n then we call A as a multiplicative consistent vague preference relation.

STEP 2: Obtain the priority vector of criteria:

The priority vector of criteria $w = (w_1, w_2, \ldots \ldots w_n)^{T}$ after obtaining aggregated vague preference matrix. The determination of priority vector is given by

$$w_j = \left[w_j^{L}, w_j^{U}\right]$$

$$= \left[\frac{1}{\sum_{j=1}^{n}\frac{1-t_{ij}^{*}}{t_{ij}^{*}}}, \frac{1}{\sum_{j=1}^{n}\frac{\left(1-(1-f_{ij}^{*})\right)}{\left(1-f_{ij}^{*}\right)}}\right] \quad \ldots\ldots(3)$$

STEP 3: Construct a vague decision matrix:

R = $(r_{ij})_{m\times n}$ be a vague decision matrix such that

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ r_{32} & r_{32} & \cdots & r_{3n} \\ \vdots & \vdots & & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{bmatrix}$$

where $r_{ij} = \left[t_{ij}, 1-f_{ij}, \pi_{ij}\right]$ (i=1,2,3,….,m) (j=1,2,3,….,n) which are contained in a vague decision matrix.

STEP 4: Obtain the vague positive and negative ideal solution:

$J_1 \to$ the set of benefit criteria

$J_2 \to$ the set of cost criteria

$B^{*} \to$ the vague positive ideal solution

$B^{-} \to$ the vague negative ideal solution

Then B^{*} and B^{-} are obtained as following

$$B^{*} = (r_1^{*}, r_2^{*}, \ldots, r_n^{*})$$

$$r_j^{*} = \left[t_j^{*}, 1-f_j{}^{*}, \pi_j{}^{*}\right] \quad j=1,2,\ldots..n \quad \ldots\ldots(4)$$

$$B^{-} = (r_1^{-}, r_2^{-}, \ldots, r_n^{-})$$

$$r_j^{-} = \left[t_j^{-}, 1-f_j{}^{-}, \pi_j{}^{-}\right] \quad j=1,2,\ldots..n \quad \ldots\ldots(5)$$

where

$t_j^* = \{(\max_i\{t_{ij} / j \in J_1\}), (\min_i\{t_{ij} / j \in J_2\})\}$ ……………..…………...……..(6)

$1 - f_{ij}^* = \{(\max_i\{1 - f_{ij} / j \in J_1\}), (\min_i\{1 - f_{ij} / j \in J_2\})\}$ …………...……..(7)

$t_j^- = \{(\min_i\{t_{ij} / j \in J_1\}), (\max_i\{t_{ij} / j \in J_2\})\}$ ……………..………...……..(8)

$1 - f_{ij}^- = \{(\min_i\{1 - f_{ij} / j \in J_1\}), (\max_i\{1 - f_{ij} / j \in J_2\})\}$ …………...……..(9)

$\pi_j^* = \left\{\left(1 - (\max_i\{t_{ij}\} + \max_i\{f_{ij}\} / j \in J_1)\right), \left(1 - (\min_i\{t_{ij}\} + \min_i\{f_{ij}\} / j \in J_2)\right)\right\}$ …(10)

$\pi_j^- = \left\{\left(1 - (\min_i\{t_{ij}\} + \min_i\{f_{ij}\} / j \in J_1)\right), \left(1 - (\max_i\{t_{ij}\} + \max_i\{f_{ij}\} / j \in J_2)\right)\right\}$ …(11)

STEP 5: Calculate the weighted separation measures:

The weighted normalized Euclidean distance is used to acquire the separation measures. The weighted lower and upper separation measures $(E_i^*)^L, (E_i^*)^U$ and $(E_i^-)^L, (E_i^-)^U$ for each alternatives from the vague positive ideal solution are calculated respectively.

$(E_i^*)^L = \sqrt{\frac{1}{2n}\sum_{j=1}^n w_j^L \left[(t_{ij} - t_j^*)^2 + ((1 - f_{ij}) - (1 - f_{ij}^*))^2 + (\pi_{ij} - \pi_j^*)^2\right]}$ ……..…(12)

$(E_i^*)^U = \sqrt{\frac{1}{2n}\sum_{j=1}^n w_j^U \left[(t_{ij} - t_j^*)^2 + ((1 - f_{ij}) - (1 - f_{ij}^*))^2 + (\pi_{ij} - \pi_j^*)^2\right]}$ ………(13)

$(E_i^-)^L = \sqrt{\frac{1}{2n}\sum_{j=1}^n w_j^L \left[(t_{ij} - t_j^-)^2 + ((1 - f_{ij}) - (1 - f_{ij}^-))^2 + (\pi_{ij} - \pi_j^-)^2\right]}$ ….....…(14)

$(E_i^-)^U = \sqrt{\frac{1}{2n}\sum_{j=1}^n w_j^U \left[(t_{ij} - t_j^-)^2 + ((1 - f_{ij}) - (1 - f_{ij}^-))^2 + (\pi_{ij} - \pi_j^-)^2\right]}$ …….(15)

STEP 6: Calculate the relative closeness co-efficient:

The relative closeness co-efficient of an alternative B_i with respect to vague positive ideal solution B^* and the vague negative ideal solution B^- is as follows

$((C_i^*)^L, (C_i^*)^U) = \left\{\left(\frac{(E_i^-)^L}{(E_i^*)^U (E_i^-)^U}\right), \left(\frac{(E_i^-)^U}{(E_i^*)^L (E_i^-)^L}\right)\right\}$ ……………………………(16)

STEP 7: Rank the alternatives:

Rank the alternatives of the relative closeness co-efficient according to the descending order.

III. CASE STUDY

There are four different patients namely P_1, P_2, P_3, P_4 and the differential count of the patient such as neutrophil, lymphocytes, eosinophils and monocytes are considered. The differential count of these patients are taken in terms of percentages, which are then converted to vague matrices with respect to the reference range considered.

REAL SURVEY OF DIFFERENTIAL COUNT OF PATIENTS WITH THE REFERENCE RANGE IS GIVEN BELOW

PATIENTS	DIFFERENTIAL COUNT WITH REFERENCE RANGE			
	NEUTROPHIL (40-75)%	**LYMPHOCYTES (20-40)%**	**EOSINOPHILS (1-6)%**	**MONOCYTES (1-10)%**
P_1	43.5	51.8	0.9	3.8
P_2	59.3	30.8	2.4	7.5
P_3	62.2	33.2	1.3	3.3
P_4	57.3	35.9	1.8	4.9

Step 1

Construct a vague preference relation matrix:

A = $(a_{ij})_{4x4}$ be a vague preference matrix of criteria,

$$A = \begin{bmatrix} [0.44,0.75] & [0.20,0.52] & [0.01,0.06] & [0.04,0.10] \\ [0.59,0.75] & [0.31,0.40] & [0.02,0.06] & [0.08,0.10] \\ [0.62,0.75] & [0.33,0.40] & [0.01,0.06] & [0.03,0.10] \\ [0.57,0.75] & [0.36,0.40] & [0.02,0.06] & [0.05,0.10] \end{bmatrix}$$

where A = $(a_{ij})_{4x4}$ has been the consistent vague preference matrix since it is satisfying the condition (1),(2) the following matrices exist as follows:

$$A^* = \begin{bmatrix} [0.44,0.75] & [0.20,0.52] & [0.01,0.06] & [0.04,0.10] \\ [0.59,0.75] & [0.31,0.40] & [0.02,0.06] & [0.08,0.10] \\ [0.62,0.75] & [0.33,0.40] & [0.01,0.06] & [0.03,0.10] \\ [0.57,0.75] & [0.36,0.40] & [0.02,0.06] & [0.05,0.10] \end{bmatrix}$$

Step 2
Obtain the priority vector of criteria by using (3).

w_1= [0.008,0.040], w_2= [0.016,0.038], w_3= [0.009,0.038], w_4= [0.014,0.038].

Step 3
Construction of a vague decision matrix:

Table-1: The vague decision matrix

PATIENTS	DIFFERENTIAL COUNT WITH REFERENCE RANGE			
	NEUTROPHIL	LYMPHOCYTES	EOSINOPHILS	MONOCYTES
P_1	[0.004,0.030,0.026]	[0.003,0.198,0.017]	[0,0.002,0.002]	[0.001,0.004,0.003]
P_2	[0.005,0.030,0.005]	[0.005,0.015,0.010]	[0,0.002,0.002]	[0.001,0.004,0.003]
P_3	[0.005,0.030,0.025]	[0.005,0.015,0.010]	[0,0.002,0.002]	[0.000,0.004,0.004]
P_4	[0.005,0.030,0.025]	[0.006,0.015,0.010]	[0,0.002,0.002]	[0.000,0.004,0.004]

Step 4
Determine the vague positive ideal solution and vague negative ideal solution:

The vague positive and negative ideal solution have been obtained by using equations (4-11),we get,

$$B^* = \begin{cases} [0.005,0.030,0.026], & [0.006,0.198,0.017] \\ [0.000,0.002,0.002], & [0.000,0.004,0.003] \end{cases}$$

$$B^- = \begin{cases} [0.004,0.030,0.025], & [0.003,0.015,0.010] \\ [0.000,0.002,0.002], & [0.001,0.004,0.004] \end{cases}$$

Step 5
Vague positive and negative separation measures based on the weighted lower and upper Normalized Euclidean distance for each type of differential count is calculated by using equations (12-15).

PATIENTS	$(E_i^*)^L$	$(E_i^*)^U$	$(E_i^-)^L$	$(E_i^-)^U$
P_1	0.000041	0.000229	0.008190	0.012622
P_2	0.008190	0.012622	0.000104	0.000170
P_3	0.008190	0.064749	0.000104	0.000170
P_4	0.008190	0.012622	0.000144	0.000229

Step 6
Calculate the relative closeness coefficient of each alternative to the vague negative and positive ideal solution:

The relative closeness coefficient of each patient is calculated by using equation (16) as follows:

$((C_1^*)^L, (C_1^*)^U)$ = {[0.6373,1.5335]}

$((C_2^*)^L, (C_2^*)^U)$ = {[0.0081,0.0205]}

$((C_3^*)^L, (C_3^*)^U)$ = {[0.0016,0.0205]}

$((C_4^*)^L, (C_4^*)^U)$ = {[0.0112,0.0275]}

Step 7
Rank the types of differential count of patients according to the descending order of the relative coefficients.Ranking by the possibility degree formula is constructed below:

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$$P = \begin{bmatrix} 0.5 & 0 & 0 & 0.1 \\ 1.0 & 0.5 & 0.4 & 0.7 \\ 1.0 & 0.6 & 0.5 & 0.7 \\ 1.0 & 0.3 & 0.3 & 0.5 \end{bmatrix}$$

The value 0 confirms the abnormality of patient P_1 in the differential count regarding lymphocytes and eosinophils ,according to the data of real survey taken initially.

IV. CONCLUSION

In this case study, we have confirmed that the patient P_1 requires further improvement in the Lymphocytes and Eosinophils regarding the differential count .Such a case is confirmed by using vague TOPSIS method.

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AN EOQ MODEL OF ITEM HAS FUZZY COSTS UNDER CONDITIONAL DELAY IN PAYMENT

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ABSTRACT

In this universe, probably all the items follow a natural order of deterioration. So, the inventory management of these items needs a solution model to have minimal cost. From the early years of 19th century (See [1]), Economic Ordering Quantity Model serves as the best solution for inventory management of deteriorating items. Further, the costs associated with inventory management are taken as constant. But, there are fluctuations within a limit for various reasons. So, in this article, we consider an EOQ model for deteriorating items having exponential demand with fuzzy costs under conditional delay in payment offered by suppliers. Further, triangular fuzzy numbers and Graded Mean Integration Representation (GMIR) method is used to formulate a mathematical model and its solution for the proposed model. Finally, numerical examples are provided, sensitivity of key parameters carried out and the results are tabulated.

Keywords: EOQ model, Exponential demand, Deterioration, Delay in payment, Triangular fuzzy number, Graded Mean Integration Representation method, Defuzzification.

INTRODUCTION

The main objective of any retailer, wholesaler, distributor, stockist, trader or any business personal is to earn maximum profit by maintaining goodwill, timely service, no shortages and providing quality items in their day to day business. So, they need to maintain an inventory system in such a way that total inventory cost becomes minimum. Further, as deterioration is a natural phenomenon, it affects almost all items. Thus, the researchers working in inventory control and management can't ignore the effect of deterioration in their models. Hence, from last century, many researchers have considered EOQ Model as the best tool to their inventory problems despite various demands and deteriorations. Also, in the last few years, Vandana and Srivastava [14], Sarkar [12], Kundu and Chakrabarti [7], Mishra et al. [9], Jaggi and Mittal [3] and Khanra et al. [6] have developed EOQ models for their inventory problems having different type of demands and deteriorations.

Due to development in technology, communication, transportation and modernization, the living standards of human beings are increasing with their income capabilities, which results in demand of items like mobiles, electronic peripherals, cosmetics, vehicles and fashionable items etc, which opens a competitive environment in business. As a result, to overcome the competition and loss of customers both suppliers and retailers use the strategy of conditional delay in payment. By offering conditional delay in payment, suppliers or retailers may attract more orders or/and customers to enhance their business. Initially, Haley and Higgins [2] used trade credit financing (i.e. delay in payment) in their inventory models. Further, many researchers have developed their inventory models by incorporating a conditional delay in payment. Recently, Singh and Kumar [13], Jaggi et al. [5] and Khanra et al. [6] have developed their inventory models by considering the delay in payment.

The costs associated with inventory are taken as constant by many researchers, but due to some limitations of market strategies these costs are fluctuating within reasonable limits, for example, as it is known that the rate of petroleum products are daily fluctuating, as a result, it affects the transportation cost, then ordering cost of the inventory system. Similarly, the fluctuations of different costs occur due to different reasons. But these reasonable fluctuations of costs significantly affect the total inventory (i.e. some crores of rupees depends on inventory considered). So, it is advisable not to ignore this fluctuation of costs. To deal such costs, fuzzy technique is the best tool. The concept of fuzzy was introduced by Zadeh [16], but Lee and Yao [8] are incorporated fuzziness in their inventory model. Recently, Varadharajan and Sangma [15], Jaggi et al. [5], Misra and Mishra [10] and Jaggi et al. [4] are used fuzziness in their inventory models.

Hence, by taking into account EOQ technique, trade credit strategy and the effect of fuzziness, we have developed an EOQ model for deteriorating items with an exponential demand under conditional delay in payment by incorporating fuzzy holding cost and fuzzy interest rates. Further, to find the optimal results, triangular fuzzy numbers and Graded Mean Integration Representation (GMIR) method are used. Finally, the mathematical model is verified by considering examples. Also, the effect of change in key parameters studied and the results are tabulated.

FUNDAMENTAL ASSUMPTIONS AND NOTATIONS

Assumptions

The following assumptions are taken into account to develop the proposed model.

(i) Demand is an exponential function of time.

(ii) Replenishment rate is instantaneous and infinite in size.

(iii) Shortages are not allowed.

(iv) Deterioration rate is constant per unit time throughout the cycle.

(v) There is no technique to make deteriorated items to serve for their original purpose.

(vi) The time limit is infinite.

Notations

Following notations are used throughout the article:

(i) The Demand rate is $D(t) = \alpha e^{\beta t}$ $(\alpha > 0, \beta \neq 0)$.

(ii) p_c is the purchase cost (per unit) of an item.

(iii) $\widetilde{h_c}$ is the fuzzy inventory holding cost per rupee per unit time.

(iv) γ $(0 < \gamma < 1)$ is the constant rate of deterioration of an item.

(v) A is the replenishment cost.

(vi) $\widetilde{i_c}$ is the fuzzy interest charges per rupee investment per year.

(vii) $\widetilde{i_e}$ is the fuzzy interest earned per rupee in a year.

(viii) M is the trade credit period provided by the supplier.

(ix) $\widetilde{F_{Ci}}(T)$ $(i = 1, 2, 3)$ is the fuzzy total average cost functions of i^{th} case raised due to trade credit offer.

(x) $\widetilde{T_{fc}}(T)$ is total fuzzy average cost functions.

(xi) $\widetilde{T_{fcG}}(T)$ is the defuzzified total average cost.

(xii) T is the complete duration of inventory cycle.

(xiii) $\widetilde{v_0}$ is the initial order quantity in the inventory.

(xiv) T^* is the optimal cycle length.

(xv) Q^* is the optimal Economic Ordering Quantity.

(xvi) $\widetilde{T_{fcG}}(T^*)$ is the optimal defuzzified total average cost.

Mathematical Model Formulation

The inventory level $v(t)$ at any time t during the complete cycle time T is defined by the following differential equation.

$$\frac{dv(t)}{dt} + \gamma v(t) = -D(t) \quad (v(0) = v_0 \text{ and } v(T) = 0), \quad \text{.......................(3.1)}$$

$$D(t) = \alpha e^{\beta t} \quad (\alpha > 0, \beta \neq 0).$$

Solving equation (3.1), we get

$$v(t) = \frac{\alpha}{\beta+\gamma}\left[e^{(\beta+\gamma)T-\gamma t} - e^{\beta t}\right], \; 0 \leq t \leq T. \quad \text{.......................... (3.2)}$$

The inventory level at t=0 be the initial ordering quantity v_0. So using t=0 in the equation (3.2),

We get

$$v_0 = v(0) = \frac{\alpha}{\beta+\gamma}\left[e^{(\beta+\gamma)T} - 1\right]. \quad \text{............................(3.3)}$$

Total demand during the complete cycle is

$$D_T = \int_0^T D(t)\ dt = \frac{\alpha}{\beta}[e^{\beta T} - 1]. \qquad \text{......................(3.4)}$$

Total number of deteriorating unit in the complete cycle is

$$U_D = v_0 - \int_0^T D(t)\ dt = \alpha\left[\frac{1}{\beta+\gamma}\left(e^{(\beta+\gamma)T} - 1\right) - \frac{1}{\beta}\left(e^{\beta T} - 1\right)\right]. \qquad \text{................(3.5)}$$

Total deterioration cost of the inventory is

$$C_D = p_c\alpha\left[\frac{1}{\beta+\gamma}\left(e^{(\beta+\gamma)T} - 1\right) - \frac{1}{\beta}\left(e^{\beta T} - 1\right)\right]. \qquad \text{................(3.6)}$$

Total holding cost of the inventory is

$$C_H = \frac{\alpha\widetilde{h_c}p_c}{\beta+\gamma}\left[\frac{1}{\gamma}\left(e^{(\beta+\gamma)T} - e^{\beta T}\right) - \frac{1}{\beta}\left(e^{\beta T} - 1\right)\right]. \qquad \text{...............(3.7)}$$

Conditions of Delay in Payment

In this model, we have assumed that the supplier offers a delay in payment period M. If $T > M$, within the time interval $[0, M]$ the retailers need not to pay any interest for outstanding due and after time period M, i.e. for the interval $[M, T]$ the retailer is charged with an interest rate $\widetilde{\iota_c}$ per unit amount on outstanding due of stock in the inventory. At the same time during the interval [0, M] retailer earns an interest rate $\widetilde{\iota_e}$ on revenue accumulated by selling the items. If $T < M$, retailer need not to pay any interest and only earns an interest rate $\widetilde{\iota_e}$ on revenue accumulated by selling the items. Now the total interests payable and earned by the retailer for different cases are

Case 1. (T > M)

The total interest payable by the retailer during the interval [T, M] is

$$Int_{p1} = p_c\widetilde{\iota_c}\int_M^T v(t)dt = \frac{\alpha\widetilde{\iota_c}p_c}{\beta+\gamma}\left[\frac{e^{\beta T}}{\gamma}\left(e^{\gamma(T-M)} - 1\right) - \frac{1}{\beta}\left(e^{\beta T} - e^{\beta M}\right)\right]. \qquad \text{.......... (3.8)}$$

The total interest earned in the interval [0, M] is

$$IE_1 = p_c\widetilde{\iota_e}\int_0^M t\,D(t)dt = \frac{\alpha\,\widetilde{\iota_e}\,p_c}{\beta}\left[Me^{\beta M} - \frac{1}{\beta}\left(e^{\beta M} - 1\right)\right]. \qquad \text{..................... (3.9)}$$

Case 2. (T < M)

As $T < M$, retailer need not pay any interest. At the same time, retailer earns the interest. So, the total interest earned in the interval [0, M] is

IE_2 = Interest earned in the interval [0, T] + Interest earned in the interval [T, M]. That is,

$$IE_2 = p_c\,\widetilde{\iota_e}\int_0^T t\,D(t)dt + p_c\widetilde{\iota_e}\,(M-T)\int_0^T D(t)dt$$

$$= \frac{\alpha\,\widetilde{\iota_e}\,p_c}{\beta}\left[T\,e^{\beta T} - \frac{1}{\beta}\left(e^{\beta T} - 1\right)\right] + \frac{\alpha\,p_c\widetilde{\iota_e}(M-T)}{\beta}\left(e^{\beta T} - 1\right)$$

$$= \frac{\alpha\,\widetilde{\iota_e}\,p_c}{\beta^2}\left[(e^{\beta T} - 1)(\beta M - 1) + \beta T\right]. \qquad \text{....................(3.10)}$$

Now the total fuzzy average cost per unit time in the complete cycle is

$\widetilde{T_{fc}}(T) = \frac{1}{T}$ [Replenishment cost + Inventory holding cost + Deterioration cost + Interest payable - Earned interest].

According to the trade credit period, we have the following cases for fuzzy total average cost

$$\widetilde{T_{fc}}(T) = \begin{cases} \widetilde{F_{C1}}(T), & T > M \\ \widetilde{F_{C2}}(T), & T < M \end{cases} \qquad \text{..........................(3.11)}$$

Here,

$$\widetilde{F_{C1}}(T) = \frac{1}{T}[A + C_H + C_D + Int_{p1} - IE_1]$$

$$= \frac{1}{T}\begin{bmatrix} A + \frac{\alpha\,\widetilde{h_c}\,p_c}{\beta+\gamma}\left[\frac{1}{\gamma}\left(e^{(\beta+\gamma)T} - e^{\beta T}\right) - \frac{1}{\beta}\left(e^{\beta T} - 1\right)\right] \\ + p_c\alpha\left[\frac{1}{\beta+\gamma}\left(e^{(\beta+\gamma)T} - 1\right) - \frac{1}{\beta}\left(e^{\beta T} - 1\right)\right] \\ + \frac{\alpha\,\widetilde{\iota_c}\,p_c}{\beta+\gamma}\left[\frac{e^{\beta T}}{\gamma}\left(e^{\gamma(T-M)} - 1\right) - \frac{1}{\beta}\left(e^{\beta T} - e^{\beta M}\right)\right] \\ - \frac{\alpha\,\widetilde{\iota_e}\,p_c}{\beta}\left[Me^{\beta M} - \frac{1}{\beta}\left(e^{\beta M} - 1\right)\right] \end{bmatrix}. \quad ....(3.12)$$

And, $\widetilde{F_{C2}}(T) = \frac{1}{T}[A + C_H + C_D - IE_2]$

$$= \frac{1}{T}[A + \frac{\alpha\,\widetilde{h_c}\,p_c}{\beta+\gamma}\left[\frac{1}{\gamma}\left(e^{(\beta+\gamma)T} - e^{\beta T}\right) - \frac{1}{\beta}\left(e^{\beta T} - 1\right)\right]$$

$$+ p_c\alpha\left[\frac{1}{\beta+\gamma}\left(e^{(\beta+\gamma)T} - 1\right) - \frac{1}{\beta}\left(e^{\beta T} - 1\right)\right]$$

$$-\frac{\alpha\,\widetilde{\iota_e}\,p_c}{\beta^2}[(e^{\beta T} - 1)(\beta M - 1) + \beta T]]. \quad ....(3.13)$$

To defuzzify the total fuzzy average cost $\widetilde{T_{fc}}(T)$, we consider fuzzy parameters $\widetilde{h_c}$, $\widetilde{\iota_c}$ and $\widetilde{\iota_e}$ as triangular fuzzy numbers $(\widetilde{hc}_1, \widetilde{hc}_2, \widetilde{hc}_3)$, $(\widetilde{\iota c}_1, \widetilde{\iota c}_2, \widetilde{\iota c}_3)$ and $(\widetilde{\iota e}_1, \widetilde{\iota e}_2, \widetilde{\iota e}_3)$ respectively and Graded Mean Integration Representation (GMIR) technique as

$$\widetilde{T_{fcG}}(T) = \frac{1}{6}[\widetilde{T_{fc}^1}(T) + 4\,\widetilde{T_{fc}^2}(T) + \widetilde{T_{fc}^3}(T)]. \quad ....(3.14)$$

Here, $\widetilde{T_{fc}^k}(T)$ is the total fuzzy average cost obtained by taking $k^{th}(k = 1, 2, 3)$ fuzzy number for the corresponding fuzzy parameter. Also,

$$\widetilde{T_{fcG}}(T) = \begin{cases} \widetilde{F_{CG1}}(T), & T > M \\ \widetilde{F_{CG2}}(T), & T < M \end{cases} \quad .... (3.15)$$

$\widetilde{F_{CGi}}(T)$ (i = 1, 2) are the defuzzified total average cost of each case can be obtained by using equation (3.14)

Solution Procedure to Obtain EOQ

The total defuzzified average cost $\widetilde{T_{fcG}}(T)$ is minimum, provided that $\frac{d\,\widetilde{T_{fcG}}(T)}{dT} = 0$ and $\frac{d^2\,\widetilde{T_{fcG}}(T)}{dT^2} > 0$. In order to find the optimal solution for the model, we proceed as follows.

Step-1: Find T_1^*, such that $\frac{d\widetilde{F_{CG1}(T_1^*)}}{dT} = 0$ and $\frac{d^2\,\widetilde{F_{CG1}}(T_1^*)}{dT^2} > 0$.

Step-2: Find T_2^*, such that $\frac{d\widetilde{F_{CG1}(T_2^*)}}{dT} = 0$ and $\frac{d^2\,\widetilde{F_{CG1}}(T_2^*)}{dT^2} > 0$.

Step-3: If $T_1^* > M > T_2^*$ then $\widetilde{T_{fcG}}(T^*) = Min\{\widetilde{F_{CG1}}(T_1^*), \widetilde{F_{CG2}}(T_2^*)\}$ and $T^* = argmin\{\widetilde{F_{CG1}}(T_1^*), \widetilde{F_{CG2}}(T_2^*)\}$.

Step-4: If $T_1^* > M$ and $T_2^* > M$ then $\widetilde{T_{fcG}}(T^*) = \widetilde{F_{CG1}}(T_1^*)$ and $T^* = T_1^*$.

Step-5: If $T_1^* < M$ and $T_2^* < M$ then $\widetilde{T_{fcG}}(T^*) = \widetilde{F_{CG2}}(T_2^*)$ and $T^* = T_2^*$.

Thus the optimal solutions are, $\widetilde{T_{fcG}}(T^*)$ is the minimal total average cost, T^* is the optimal cycle length and the optimal economic order quantity (EOQ) Q^* is

$$Q^* = \frac{\alpha}{\beta+\gamma}\left[e^{(\beta+\gamma)T^*} - 1\right]. \quad .... (3.16)$$

(Which is obtained by replacing T *by* T^* in equation (3.3))

Remark. If $T = M$, i.e. total cycle length and conditional delay period are equal, then both the cases discussed above are identical. Thus, to obtain optimal results in this case we may use either of the cases.

NUMERICAL EXAMPLES

Example 1

The values for different parameters involved in the said model are α = 1050, β = 1,

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A = Rs.220, p_c= Rs.15, γ = 0.25, M = 0.27, $(\widetilde{hc_1}, \widetilde{hc_2}, \widetilde{hc_3})$ = (0.008, 0.009, 0.010), $(\widetilde{ic_1}, \widetilde{ic_2}, \widetilde{ic_3})$ = (0.25, 0.35, 0.45), $(\widetilde{ie_1}, \widetilde{ie_2}, \widetilde{ie_3})$ = (0.23, 0.33, 0.43).

By using Graded Mean Integration Representation Method and solution procedure, we get,

M = 0.27 T_1^* = 0.197368 $\widetilde{F_{CG1}}(T)$ = 522.035

M = 0.27 T_2^*= 0.210535 $\widetilde{F_{CG2}}(T)$ = 574.274

And the optimal results are

T^* = 0.197368 $\widetilde{T_{fcG}}(T^*)$ = 522.035 Q^* = 235.039

Example 2

The values for different parameters involved in the said model are α = 1050, $\beta = 1$,

A = Rs.220, p_c= Rs.15, γ = 0.25, M = 0.15, $(\widetilde{hc_1}, \widetilde{hc_2}, \widetilde{hc_3})$ = (0.008, 0.009, 0.010), $(\widetilde{ic_1}, \widetilde{ic_2}, \widetilde{ic_3})$ = (0.25, 0.35, 0.45), $(\widetilde{ie_1}, \widetilde{ie_2}, \widetilde{ie_3})$ = (0.23, 0.33, 0.43).

By using Graded Mean Integration Representation Method and solution procedure, we get,

M = 0.15 T_1^* = 0.194861 $\widetilde{F_{CG1}}(T)$ = 1292.00

M = 0.15 T_2^*= 0.204018 $\widetilde{F_{CG2}}(T)$ = 1267.30

And the optimal results are

T^* = 0.204018 $\widetilde{T_{fcG}}(T^*)$ = 1267.30 Q^* = 244.012

Sensitivity of Key parameters

Table-1: Sensitivity of holding cost $\widetilde{h_c}$

$\widetilde{h_c}$	T^*	$\widetilde{T_{fcG}}(T^*)$
(0.007, 0.008, 0.009)	0.210739	572.328
(0.008, 0.009, 0.010)	0.210535	574.274
(0.009, 0.010, 0.011)	0.210332	576.217

It is observed that the increase in holding cost affects the optimal results as a decrease in cycle length with increase in total average cost.

Table-2: Sensitivity of earned interest $\widetilde{i_e}$

$\widetilde{i_e}$	T^*	$\widetilde{T_{fcG}}(T^*)$
(0.22, 0.32, 0.42)	0.211805	603.758
(0.23, 0.33, 0.43)	0.210535	574.274
(0.24, 0.34, 0.44)	0.209288	544.707

It is observed that the increase in interest earn cost effects the optimal results as a decrease in cycle length and decrease in total average cost.

CONCLUSION

In this article, we have considered an inventory model for items having constant deterioration, exponential demand under the effect of conditional delay in payment and fuzzy costs. With the help of triangular fuzzy numbers and GMIR defuzzification technique a mathematical model is successfully formulated and the optimal EOQ is obtained by finding the total fuzzy average cost. Further, numerical examples with different values of parameters are presented and the effect of change in key parameters discussed. Finally, this article can be extended by considering more inventory aspects, such as inflation, shortages, backlogging, two warehouses and etc.

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ZAGREB INDICES AND THEIR POLYNOMIALS OF WINDMILL GRAPHS

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ABSTRACT
The first three Zagreb indices of a graph G denoted, $M_1(G)$, $M_2(G)$ and $M_3(G)$ are well known. Considering these Zagreb indices, in this paper we computing the first three Zagreb polynomials of certain classes of windmill graphs such as french windmill graphs, dutch windmill graphs, Kulli cycle windmill graphs and Kulli path windmill graphs.

1. INTRODUCTION

In this paper, we consider only finite, connected, undirected without loops and multiple edges. For additional definitions and notations, the reader may refer to [9].

Let G be a connected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The degree $d_G(v)$ of a vertex v is the number of vertices adjacent to v. The edge connecting the vertices u and v will be denoted by uv. Let $d_G(e)$ denote the degree of an edge e in G, which is defined by $d_G(e) = d_G(u) + d_G(v) - 2$ with $e = uv$.

A molecular graph is a graph such that its vertices correspond to the atoms and the edges to the bonds. In Chemical Science, the physico-chemical properties of chemical compounds are often modelled by means of molecular graph based structure descriptors, which are also referred to as topological indices, one can refer [3], [6], [13] and [16].

In [8], the first and second Zagreb indices were introduced to take account of the contributions of pairs of adjacent vertices. The first and Second Zagreb indices of G are defined as

$$M_1(G) = \sum_{v\in V(G)} d_G(v)^2 \text{ or } M_1(G) = \sum_{uv\in E(G)} \left[d_G(v) + d_G(v)\right]$$

and

$M_2(G) = \sum_{uv\in E(G)} \left[d_G(u) d_G(v)\right]$, we refer [2], [10], [14] and [15].

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Key words and phrases: Augmented-Zagreb indices; French windmill graph; Dutch windmill graph; Kulli cycle windmill graph and Kulli path wind mill graph.

In [1], Alberton introduced the irregularity of a graph G as $irr(G) = \sum_{e=uv\in E(G)} imb(e)$, where $imb(e) = \left|d_G(u) - d_G(v)\right|$. This new index was named the third Zagreb index by Fath-Tabar in [5] and is denoted by $M_3(G) = \sum_{e=uv\in E(G)} \left|d_G(u) - d_G(v)\right|$.

Considering the first two Zagreb indices, Fath-Tabar in [4] defined first and the second Zagreb polynomials as $M_1(G,x) = \sum_{uv\in E(G)} x^{\left[d_G(u)+d_G(v)\right]}$ and $M_2(G,x) = \sum_{uv\in E(G)} x^{\left[d_G(u)d_G(v)\right]}$, respectively. Analogously, the third Zagreb polynomial is defined as $M_3(G,x) = \sum_{uv\in E(G)} x^{\left|d_G(u)-d_G(v)\right|}$.

In the following section, we perform some necessary calculations for computing the Zagreb indices and their polynomials of G, we make use of the vertex set partition $V_a = \{v \in V : d_G(v) = a\}$ and edge set partitions $E_b = \{e = uv \in E : d_G(u) + d_G(v) = b\}$ and $E_c^* = \{e = uv \in E : d_G(u) d_G(v) = c\}$.

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2. FRENCH WINDMILL GRAPH

The French windmill graph $F_n^{(m)}$ is the graph by taking $m \geq 2$ copies of the complete graph K_n; $n \geq 2$ with a vertex in common. This graph is shown in Figure-1. The French windmill graph $F_2^{(m)}$ is called a star graph, the French windmill graph $F_3^{(m)}$ is called a friendship graph and the French windmill graph $F_3^{(2)}$ is called a butterfly graph.

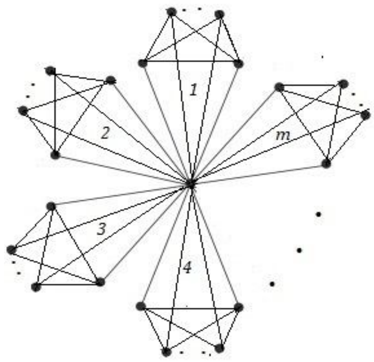

Figure-1: French windmill graph $F_n^{(m)}$.

Let $G = F_n^{(m)}$, where $F_n^{(m)}$ is a French windmill graph. By algebraic method, we get $|V(G)| = m(n-1)+1$ and $|E(G)| = \frac{mn(n-1)}{2}$.

We have two partitions of the vertex set $V(G)$ as follows:

$V_{n-1} = \{v \in V(G) : d_G(v) = n-1\}; |V_{n-1}| = m(n-1)$, and

$V_{(n-1)m} = \{v \in V(G) : d_G(v) = (n-1)m\}; |V_{(n-1)m}| = 1.$

Also we have two partitions of the edge set $E(G)$ as follows:

$$E_{2(n-1)} = E^*_{(n-1)^2} = \{uv \in E(G) : d_G(u) = d_G(v) = n-1\};$$

$$|E_{2(n-1)}| = |E^*_{(n-1)^2}| = m\left[\frac{n(n-1)}{2} - (n-1)\right] = \frac{m(n^2-3n+2)}{2}, \text{ and}$$

$$E_{(n-1)(m+1)} = E^*_{m(n-1)^2} = \{uv \in E(G) : d_G(u) = n-1, d_G(v) = (n-1)m\};$$

$$|E_{(n-1)(m+1)}| = |E^*_{m(n-1)^2}| = (n-1)m.$$

Theorem 2. 1. The first Zagreb index and their polynomial of French windmill graph are

$$M_1\left(F_n^{(m)}\right) = mn^3 + m^2n^2 - 2m^2n - 3mn^2 + 3mn + m^2 - m \text{ and}$$

$$M_1\left(F_n^{(m)}, x\right) = \frac{1}{2}m(n^2-3n+2)x^{2(n-1)} + (n-1)mx^{(n-1)(m+1)}.$$

Proof. Let $G = F_n^{(m)}$, where $F_n^{(m)}$ is a french windmill graph. Consider

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$$
\begin{aligned}
M_1\left(F_n^{(m)}\right) &= \sum_{uv\in E}\left[d_G(u)+d_G(v)\right] \\
&= \sum_{E_{2(n-1)}}\left[d_G(u)+d_G(v)\right]+\sum_{E_{(n-1)(m+1)}}\left[d_G(u)+d_G(v)\right] \\
&= \frac{1}{2}m\left(n^2-3n+2\right)\left[(n-1)+(n-1)\right]+(n-1)m\left[(n-1)+(n-1)m\right] \\
&= m\left(n^2-3n+2\right)(n-1)+(n-1)^2 m(m+1) \\
&= mn^3+m^2n^2-2m^2n-3mn^2+3mn+m^2-m.
\end{aligned}
$$

Now, for first Zagreb polynomial of $F_n^{(m)}$, we have

$$
\begin{aligned}
M_1\left(F_n^{(m)},x\right) &= \sum_{uv\in E} x^{\left[d_G(u)+d_G(v)\right]} \\
&= \sum_{E_{2(n-1)}} x^{\left[d_G(u)+d_G(v)\right]}+\sum_{E_{(n-1)(m+1)}} x^{\left[d_G(u)+d_G(v)\right]} \\
&= \frac{1}{2}m\left(n^2-3n+2\right)x^{\left[(n-1)+(n+1)\right]}+(n-1)mx^{\left[(n-1)+(n-1)m\right]} \\
&= \frac{1}{2}m\left(n^2-3n+2\right)x^{2(n-1)}+(n-1)mx^{(n-1)(m+1)}.
\end{aligned}
$$

Corollary 2.1. The first Zagreb index and their polynomial of a friendship graph $F_3^{(m)}$ are

$$M_1\left(F_3^{(m)}\right)=4m(m+2),$$

$$M_1\left(F_3^{(m)},x\right)=mx^4+2mx^{2(m+1)}.$$

Corollary 2.2. The first Zagreb index and their polynomial of a star graph $F_2^{(m)}$ are

$$M_1\left(F_2^{(m)}\right)=m^2+m,$$

$$M_1\left(F_2^{(m)},x\right)=\frac{m}{2}x^2+mx^{(m+1)}.$$

Corollary 2.3. The first Zagreb index and their polynomial of a butterfly graph $F_2^{(3)}$ are

$$M_1\left(F_2^{(3)}\right)=32,$$

$$M_1\left(F_2^{(3)},x\right)=2x^4+4x^6.$$

Theorem 2. 2. The second Zagreb index and their polynomial of french windmill graph are

$$M_2\left(F_n^{(m)}\right)=\frac{1}{2}(mn^4+2m^2n^3-5mn^3-6m^2n^2+6m^2n+9mn^2+7mn-2m^2+2m),$$ and

$$M_2\left(F_n^{(m)},x\right)=\frac{1}{2}\left(mn^2-3mn+2m\right)x^{(n-1)^2}+2(mn-m)x^{(n-1)^2m}.$$

Proof. Let $G=F_n^{(m)}$, where $F_n^{(m)}$ is a french windmill graph. Consider

$$\begin{aligned}M_2\left(F_n^{(m)}\right) &= \sum_{uv\in E}\left[d_G(u)d_G(v)\right]\\ &= \sum_{E_{2(n-1)}}\left[d_G(u)d_G(v)\right]+\sum_{E_{(n-1)(m+1)}}\left[d_G(u)d_G(v)\right]\\ &= \frac{1}{2}m\left(n^2-3n+2\right)\left[(n-1)(n-1)\right]+(n-1)m\left[(n-1)(n-1)m\right]\\ &= \frac{1}{2}\left(mn^4+2m^2n^3-5mn^3-6m^2n^2+6m^2n+9mn^2-7mn-2m^2+2m\right).\end{aligned}$$

Now, for first Zagreb polynomial of $F_n^{(m)}$, we have

$$\begin{aligned}M_2\left(F_n^{(m)},x\right) &= \sum_{uv\in E}x^{\left[d_G(u)d_G(v)\right]}\\ &= \sum_{E_{2(n-1)}}x^{\left[d_G(u)d_G(v)\right]}+\sum_{E_{(n-1)(m+1)}}x^{\left[d_G(u)d_G(v)\right]}\\ &= \frac{1}{2}m\left(n^2-3n+2\right)x^{\left[(n-1)(n+1)\right]}+(n-1)mx^{\left[(n-1)(n-1)m\right]}\\ &= \frac{1}{2}\left(mn^2-3mn+2m\right)x^{(n-1)^2}+2\left(mn-m\right)x^{(n-1)^2m}.\end{aligned}$$

Corollary 2.4. The second Zagreb index and their polynomial of a friendship graph $F_3^{(m)}$ are

$$M_2\left(F_3^{(m)}\right)=8m^2+4m,$$

$$M_2\left(F_3^{(m)},x\right)=mx^4+4mx^{4m}.$$

Corollary 2.5. The second Zagreb index and their polynomial of a star graph $F_2^{(m)}$ are

$$M_2\left(F_2^{(m)}\right)=m^2,$$

$$M_2\left(F_2^{(m)},x\right)=2mx^4.$$

Corollary 2.6. The second Zagreb index and their polynomial of a butterfly graph $F_2^{(3)}$ are

$$M_2\left(F_2^{(3)}\right)=40,$$

$$M_2\left(F_2^{(3)},x\right)=2x^4+8x^8.$$

Theorem 2. 3. The third Zagreb index and their polynomial of french windmill graph are

$$M_3\left(F_n^{(m)}\right)=(mn)^2-2m^2n-mn^2+2mn+m^2-m, \quad \text{and}$$

$$M_2\left(F_n^{(m)},x\right)=(mn-m)x^{mn-m-n+1}+\frac{1}{2}\left(mn^2-3mn+2m\right).$$

Proof. Let $G=F_n^{(m)}$, where $F_n^{(m)}$ is a french windmill graph. Consider

$$
\begin{aligned}
M_3\left(F_n^{(m)}\right) &= \sum_{uv\in E} \left|d_G(u)-d_G(v)\right| \\
&= \sum_{E_{2(n-1)}} \left|d_G(u)-d_G(v)\right| + \sum_{E_{(n-1)(m+1)}} \left|d_G(u)-d_G(v)\right| \\
&= \frac{1}{2}m\left(n^2-3n+2\right)\left|(n-1)-(n-1)\right| + (n-1)m\left|(n-1)-(n-1)m\right| \\
&= (mn)^2 - 2m^2n - mn^2 + 2mn + m^2 - m.
\end{aligned}
$$

Now, for first Zagreb polynomial of $F_n^{(m)}$, we have

$$
\begin{aligned}
M_3\left(F_n^{(m)}, x\right) &= \sum_{uv\in E} x^{\left|d_G(u)-d_G(v)\right|} \\
&= \sum_{E_{2(n-1)}} x^{\left|d_G(u)-d_G(v)\right|} + \sum_{E_{(n-1)(m+1)}} x^{\left|d_G(u)-d_G(v)\right|} \\
&= \frac{1}{2}m\left(n^2-3n+2\right)x^{\left|(n-1)-(n-1)\right|} + (n-1)mx^{\left|(n-1)-(n-1)m\right|} \\
&= (mn-m)x^{mn-m-n+1} + \frac{1}{2}\left(mn^2-3mn+2m\right).
\end{aligned}
$$

Corollary 2.7. The third Zagreb index and their polynomial of a friendship graph $F_3^{(m)}$ are

$$M_3\left(F_3^{(m)}\right) = 4m(m-1),$$
$$M_3\left(F_3^{(m)}, x\right) = (3m-m)x^{2m-2} + m.$$

Corollary 2.8. The third Zagreb index and their polynomial of a star graph $F_2^{(m)}$ are

$$M_2\left(F_2^{(m)}\right) = m^2 - m,$$
$$M_2\left(F_2^{(m)}, x\right) = mx^{(m-1)}.$$

Corollary 2.9. The third Zagreb index and their polynomial of a butterfly graph $F_2^{(3)}$ are

$$M_3\left(F_2^{(3)}\right) = 8,$$
$$M_3\left(F_2^{(3)}, x\right) = 4x^2 + 2.$$

Corollary 2.10. The Zagreb index and their polynomial of French windmill graph $F_n^{(m)}$ are

$$M_2\left(F_n^{(m)}\right) = \frac{1}{2}(n-1)M_1\left(F_n^{(m)}\right) - \frac{1}{2}(n-1)M_3\left(F_n^{(m)}\right),$$
$$M_1\left(F_n^{(m)}, x\right) = x^{(2n-2)}M_3\left(F_n^{(m)}, x\right).$$

3. DUTCH WINDMILL GRAPH

The Dutch windmill graph $D_n^{(m)}$ is the graph obtained by taking m copies of the cycle C_n with a vertex in common. This graph is shown in Figure-2. The Dutch windmill graph $D_3^{(m)} = F_3^{(m)}$ is called a friendship graph. For more details on windmill graph, see [7].

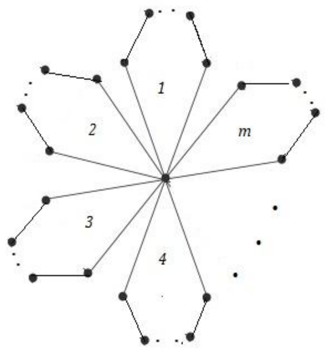

Figure-2: Dutch windmill graph $D_n^{(m)}$.

Let $G = D_n^{(m)}$, where $D_n^{(m)}$ is a Dutch windmill graph. By algebraic method, we get $|V(G)| = m(n-1)+1$ and $|E(G)| = mn.$

We have two partitions of the vertex set $V(G)$ as follows:

$V_2 = \{v \in V(G) : d_G(v) = 2\}; |V_2| = (n-1)m,$ and

$V_{2m} = \{v \in V(G) : d_G(v) = 2m\}; |V_{2m}| = 1.$

Also we have two partitions of the edge set $E(G)$ as follows:

$E_4 = E_4^* = \{uv \in E(G) : d_G(u) = d_G(v) = 2\}; |E_4| = |E_4^*| = (n-2)m,$ and

$E_{2m+2} = E_{2(2m)}^* = \{uv \in E(G) : d_G(u) = 2, d_G(v) = 2m\}; |E_{2m+2}| = |E_{2(2m)}^*| = 2m.$

Theorem 3.1. The first Zagreb index and their polynomial of Dutch windmill graph are

$$M_1\left(D_n^{(m)}\right) = 4mn + 4m^2 - 4m,$$
$$M_1\left(D_n^{(m)}, x\right) = (n-2)mx^4 + 2mx^{2m+2}.$$

Proof. Let $G = D_n^{(m)}$, where $D_n^{(m)}$ be a Dutch windmill graph. Consider

$$\begin{aligned} M_2\left(D_n^{(m)}\right) &= \sum_{uv \in E} \left[d_G(u) + d_G(v)\right] \\ &= \sum_{E_4} \left[d_G(u) + d_G(v)\right] + \sum_{E_{2m+2}} \left[d_G(u) + d_G(v)\right] \\ &= (n-2)m[2+2] + 2m[2+2m] \\ &= 4mn + 4m^2 - 4m. \end{aligned}$$

Now, for first Zagreb polynomial of a $D_n^{(m)}$, we have

$$\begin{aligned} M_1\left(D_n^{(m)}\right) &= \sum_{uv \in E} x^{\left[d_G(u)+d_G(v)\right]} \\ &= \sum_{E_4} x^{\left[d_G(u)+d_G(v)\right]} + \sum_{E_{2m+2}} x^{\left[d_G(u)+d_G(v)\right]} \\ &= (n-2)mx^4 + 2mx^{2m+2}. \end{aligned}$$

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Theorem 3.2. The second Zagreb index and their polynomial of Dutch windmill graph are

$$M_2\left(D_n^{(m)}\right)=4mn+8m^2-8m,$$
$$M_2\left(D_n^{(m)},x\right)=(n-2)mx^4+2mx^{4m}.$$

Proof. Let $G=D_n^{(m)}$, where $D_n^{(m)}$ be a Dutch windmill graph. Consider

$$\begin{aligned}M_2\left(D_n^{(m)}\right)&=\sum_{uv\in E}\left[d_G(u)d_G(v)\right]\\&=\sum_{E_4}\left[d_G(u)d_G(v)\right]+\sum_{E_{2m+2}}\left[d_G(u)d_G(v)\right]\\&=(n-2)m[2\times 2]+2m[2\times 2m]\\&=4mn+8m^2-8m.\end{aligned}$$

Now, for second Zagreb polynomial of a $D_n^{(m)}$, we have

$$\begin{aligned}M_2\left(D_n^{(m)}\right)&=\sum_{uv\in E}x^{\left[d_G(u)d_G(v)\right]}\\&=\sum_{E_4}x^{\left[d_G(u)d_G(v)\right]}+\sum_{E_{2m+2}}x^{\left[d_G(u)d_G(v)\right]}\\&=(n-2)mx^4+2mx^{4m}.\end{aligned}$$

Theorem 3.3. The third Zagreb index and their polynomial of Dutch windmill graph $D_n^{(m)}$ with $m\geq 2$ and $n\geq 2$ are

$$M_3\left(D_n^{(m)}\right)=4m^2-4m,$$
$$M_3\left(D_n^{(m)},x\right)=2mx^{2(m-1)}+(n-2)m.$$

Proof. Let $G=D_n^{(m)}$, where $D_n^{(m)}$ be a Dutch windmill graph. Consider

$$\begin{aligned}M_3\left(D_n^{(m)}\right)&=\sum_{uv\in E}\left|d_G(u)-d_G(v)\right|\\&=\sum_{E_4}\left|d_G(u)-d_G(v)\right|+\sum_{E_{2m+2}}\left|d_G(u)-d_G(v)\right|\\&=(n-2)m\left|2-2\right|+2m\left|2-2m\right|\\&=4m^2-4m.\end{aligned}$$

Now, for third Zagreb polynomial of a $D_n^{(m)}$, we have

$$\begin{aligned}M_3\left(D_n^{(m)}\right)&=\sum_{uv\in E}x^{\left|d_G(u)-d_G(v)\right|}\\&=\sum_{E_4}x^{\left|d_G(u)-d_G(v)\right|}+\sum_{E_{2m+2}}x^{\left|d_G(u)-d_G(v)\right|}\\&=(n-2)mx^{|2-2|}+2mx^{|2-2m|},\sin ce\ m\geq 2\\&=2mx^{2(m-1)}+(n-2)m.\end{aligned}$$

Corollary 3.1. The Zagreb index and their polynomial of dutch windmill graph $D_n^{(m)}$ are

$$M_3\left(D_n^{(m)}\right)=M_2\left(D_n^{(m)}\right)-M_1\left(D_n^{(m)}\right),$$

$$M_1\left(D_n^{(m)},x\right)=x^4M_3\left(F_n^{(m)},x\right).$$

4. KULLI CYCLE WINDMILL GRAPH

The Kulli cycle windmill graph $C_{n+1}^{(m)}$ is the graph obtained by taking m copies of the graph $K_1 + C_n$ for $n \geq 3$ with a vertex K_1 in common. This graph is shown in Figure-3. This Kulli cycle windmill graph $C_4^{(m)}$ is a French windmill graph and it is denoted by $F_3^{(m)}$. This type of windmill graph is initiated by Kulli et al., in [11].

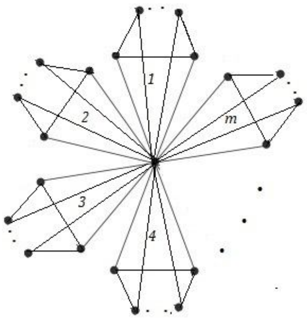

Figure-3: Kulli cycle windmill graph $C_{n+1}^{(m)}$.

Let $G=C_{n+1}^{(m)}$, where $C_{n+1}^{(m)}$ is a Kulli cycle windmill graph. By algebraic method, we get $\left|V\left(G\right)\right|=mn+1$ and $\left|E\left(G\right)\right|=2mn.$ We have two partitions of the vertex set $V(G)$ as follows:

$V_3=\left\{v\in V\left(G\right):d_G\left(v\right)=3\right\};\left|V_3\right|=mn,$ and

$V_{mn}=\left\{v\in V\left(G\right):d_G\left(v\right)=mn\right\};\left|V_{mn}\right|=1.$

Also we have two partitions of the edge set $E(G)$ as follows:

$E_6=\left\{uv\in E\left(G\right):d_G\left(u\right)=d_G\left(v\right)=3\right\};\left|E_6\right|=mn,$ and

$E_{mn+3}=\left\{uv\in E\left(G\right):d_G\left(u\right)=mn,d_G\left(v\right)=3\right\};\left|E_{mn+3}\right|=mn.$

Theorem 4.1. The first Zagreb index and their polynomial of a Kulli cycle windmill graph are

$$M_1\left(C_{n+1}^{(m)}\right)=\left(mn\right)^2+9mn,$$

$$M_1\left(C_{n+1}^{(m)},x\right)=mnx^6+mnx^{mn+3}.$$

Proof. Let $G=C_{n+1}^{(m)}$ where $C_{n+1}^{(m)}$ is a Kulli cycle windmill graph. Consider

$$\begin{aligned}M_1\left(C_{n+1}^{(m)}\right)&=\sum_{uv\in E}\left[d_G\left(u\right)+d_G\left(v\right)\right]\\&=\sum_{E_6}\left[d_G\left(u\right)+d_G\left(v\right)\right]+\sum_{E_{mn+3}}\left[d_G\left(u\right)+d_G\left(v\right)\right]\\&=mn\left[3+3\right]+mn\left[mn+3\right]\\&=\left(mn\right)^2+9mn.\end{aligned}$$

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Now, for first Zagreb polynomial of a $C_{n+1}^{(m)}$, we have

$$\begin{aligned} M_1\left(C_{n+1}^{(m)}, x\right) &= \sum_{uv\in E} x^{\left[d_G(u)+d_G(u)\right]} \\ &= \sum_{E_6} x^{\left[d_G(u)+d_G(u)\right]} + \sum_{E_{mn+3}} x^{\left[d_G(u)+d_G(u)\right]} \\ &= mnx^6 + mnx^{mn+3}. \end{aligned}$$

Theorem 4.2. The second Zagreb index and their polynomial of a Kulli cycle windmill graph are

$$M_2\left(C_{n+1}^{(m)}\right) = 3(mn)^2 + 9mn,$$

$$M_2\left(C_{n+1}^{(m)}, x\right) = mnx^9 + mnx^{3mn}.$$

Proof. Let $G = C_{n+1}^{(m)}$ where $C_{n+1}^{(m)}$ is a Kulli cycle windmill graph. Consider

$$\begin{aligned} M_2\left(C_{n+1}^{(m)}\right) &= \sum_{uv\in E} \left[d_G(u)d_G(v)\right] \\ &= \sum_{E_6} \left[d_G(u)d_G(v)\right] + \sum_{E_{mn+3}} \left[d_G(u)d_G(v)\right] \\ &= mn[3\times 3] + mn[mn\times 3] \\ &= 3(mn)^2 + 9mn. \end{aligned}$$

Now, for second Zagreb polynomial of a $C_{n+1}^{(m)}$, we have

$$\begin{aligned} M_2\left(C_{n+1}^{(m)}, x\right) &= \sum_{uv\in E} x^{\left[d_G(u)d_G(u)\right]} \\ &= \sum_{E_6} x^{\left[d_G(u)d_G(u)\right]} + \sum_{E_{mn+3}} x^{\left[d_G(u)d_G(u)\right]} \\ &= mnx^9 + mnx^{3mn}. \end{aligned}$$

Theorem 4.3. The third Zagreb index and their polynomial of a Kulli cycle windmill graph are

$$M_3\left(C_{n+1}^{(m)}\right) = (mn)^2 - 3mn,$$

$$M_3\left(C_{n+1}^{(m)}, x\right) = mnx^{mn-3} + mn.$$

Proof. Let $G = C_{n+1}^{(m)}$, where $C_{n+1}^{(m)}$ is a Kulli cycle windmill graph. Consider

$$\begin{aligned} M_3\left(C_{n+1}^{(m)}\right) &= \sum_{uv\in E} \left|d_G(u) - d_G(v)\right| \\ &= \sum_{E_6} \left|d_G(u) - d_G(v)\right| + \sum_{E_{mn+3}} \left|d_G(u) - d_G(v)\right| \\ &= mn|3-3| + mn|mn-3|, \text{since } m \geq 2 \text{ and } n \geq 3 \\ &= (mn)^2 - 3mn. \end{aligned}$$

Now, for third Zagreb polynomial of a $C_{n+1}^{(m)}$, we have

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$$\begin{aligned} M_3\left(C_{n+1}^{(m)},x\right) &= \sum_{uv\in E} x^{|d_G(u)-d_G(v)|} \\ &= \sum_{E_6} x^{|d_G(u)-d_G(v)|} + \sum_{E_{mn+3}} x^{|d_G(u)-d_G(v)|} \\ &= mnx^{|3-3|} + mnx^{|mn-3|} \\ &= mnx^{mn-3} + mn. \end{aligned}$$

Corollary 4.1. The Zagreb index and their polynomial of Kulli cycle windmill $C_{n+1}^{(m)}$ are

$i)\, M_2\left(C_{n+1}^{(m)}\right) = M_1\left(C_{n+1}^{(m)}\right) + 2(mn)^2,$

$ii)\, M_3\left(C_{n+1}^{(m)}\right) = M_1\left(C_{n+1}^{(m)}\right) - 12mn,$

$iii)\, M_1\left(C_{n+1}^{(m)},x\right) = x^6\left[M_3\left(C_{n+1}^{(m)},x\right)\right].$

5. KULLI PATH WINDMILL GRAPH

The Kulli path windmill graph $P_{n+1}^{(m)}$ is the graph obtained by taking m copies of the graph $K_1 + P_n$ for $n \geq 2$ with a vertex K_1 in common. This graph is shown in Figure-4. The Kulli path windmill graph $P_3^{(m)}$ is a friendship graph and it is denoted by $F_3^{(m)}$. This type of windmill graph is initiated by Kulli et al., in [12].

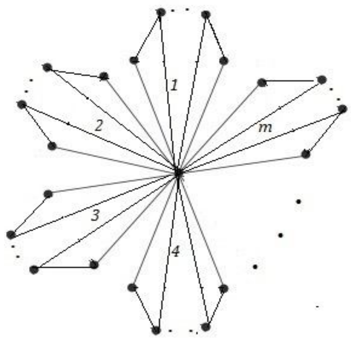

Figure-4: Kulli path windmill graph $P_{n+1}^{(m)}$.

Let $G = P_{n+1}^{(m)}$, where $P_{n+1}^{(m)}$ is a Kulli path windmill graph with $m \geq 2$ and $n \geq 4$. By algebraic method, we get $|V(G)| = mn+1$ and $|E(G)| = 2mn - m.$

We have three partitions of the vertex set $V(G)$ as follows:

$V_2 = \{v \in V(G) : d_G(v) = 2\}; |V_2| = 2m,$

$V_3 = \{v \in V(G) : d_G(v) = 3\}; |V_3| = mn - 2m,$ and

$V_{mn} = \{v \in V(G) : d_G(v) = mn\}; |V_{mn}| = 1.$

Also we have four partitions of the edge set $E(G)$ as follows:

$E_5 = \{uv \in E(G) : d_G(u) = 2, d_G(v) = 3\}; |E_5| = 2m,$

$E_6 = \{uv \in E(G) : d_G(u) = 3, d_G(v) = 3\}; |E_6| = mn - 3m,$

$E_{mn+2} = \{uv \in E(G) : d_G(u) = mn, d_G(v) = 2\}; |E_{mn+2}| = 2m,$ and

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$E_{mn+3}=\left\{uv\in E(G):d_G(u)=mn,d_G(v)=3\right\};\left|E_{mn+3}\right|=mn-2m.$

Theorem 5.1. The first Zagreb index and their polynomial of a Kulli path windmill graph are

$$M_1\left(P_{n+1}^{(m)}\right)=(mn)^2+9mn-10m,$$
$$M_1\left(P_{n+1}^{(m)},x\right)=2mx^5+(mn-3m)x^6+2mx^{mn+2}+(mn-2m)x^{mn+3}.$$

Proof. Let $G=P_{n+1}^{(m)}$ where $P_{n+1}^{(m)}$ is a Kulli path windmill graph with $m,n\geq 2$. Consider

$$\begin{aligned}M_1\left(P_{n+1}^{(m)}\right)&=\sum_{uv\in E}\left[d_G(u)+d_G(v)\right]\\&=\sum_{E_5}\left[d_G(u)+d_G(v)\right]+\sum_{E_6}\left[d_G(u)+d_G(v)\right]\\&+\sum_{E_{mn+2}}\left[d_G(u)+d_G(v)\right]+\sum_{E_{mn+3}}\left[d_G(u)+d_G(v)\right]\\&=2m[2+3]+(mn-3m)[3+3]+2m[mn+2]+(mn-2m)[mn+3]\\&=(mn)^2+9mn-2m.\end{aligned}$$

Now, for first Zagreb polynomial of a $P_{n+1}^{(m)}$, we have

$$\begin{aligned}M_1\left(P_{n+1}^{(m)},x\right)&=\sum_{uv\in E}x^{\left[d_G(u)+d_G(v)\right]}\\&=\sum_{E_5}x^{\left[d_G(u)+d_G(v)\right]}+\sum_{E_6}x^{\left[d_G(u)+d_G(v)\right]}\\&+\sum_{E_{mn+2}}x^{\left[d_G(u)+d_G(v)\right]}+\sum_{E_{mn+3}}x^{\left[d_G(u)+d_G(v)\right]}\\&=2mx^5+(mn-3m)x^6+2mx^{mn+2}+(mn-2m)x^{mn+3}.\end{aligned}$$

Theorem 5.2. The second Zagreb index and their polynomial of a Kulli path windmill graph are

$$M_2\left(P_{n+1}^{(m)}\right)=3(mn)^2-2m^2n+9mn-15m,$$
$$M_2\left(P_{n+1}^{(m)},x\right)=2mx^6+(mn-3m)x^9+2mx^{2mn}+(mn-2m)x^{3mn}.$$

Proof. Let $G=P_{n+1}^{(m)}$ where $P_{n+1}^{(m)}$ is a Kulli path windmill graph with $m,n\geq 2$. Consider

$$\begin{aligned}M_2\left(P_{n+1}^{(m)}\right)&=\sum_{uv\in E}\left[d_G(u)d_G(v)\right]\\&=\sum_{E_5}\left[d_G(u)d_G(v)\right]+\sum_{E_6}\left[d_G(u)d_G(v)\right]\\&+\sum_{E_{mn+2}}\left[d_G(u)d_G(v)\right]+\sum_{E_{mn+3}}\left[d_G(u)d_G(v)\right]\\&=2m[2\times 3]+(mn-3m)[3\times 3]+2m[mn\times 2]+(mn-2m)[mn\times 3]\\&=3(mn)^2-2m^2n+9mn-15m.\end{aligned}$$

Now, for second Zagreb polynomial of a $P_{n+1}^{(m)}$, we have

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$$
\begin{aligned}
M_2\left(P_{n+1}^{(m)},x\right) &= \sum_{uv\in E} x^{\left[d_G(u)d_G(v)\right]} \\
&= \sum_{E_5} x^{\left[d_G(u)d_G(v)\right]} + \sum_{E_6} x^{\left[d_G(u)d_G(v)\right]} \\
&+ \sum_{E_{mn+2}} x^{\left[d_G(u)d_G(v)\right]} + \sum_{E_{mn+3}} x^{\left[d_G(u)d_G(v)\right]} \\
&= 2mx^6 + (mn-3m)x^9 + 2mx^{2mn} + (mn-2m)x^{3mn}.
\end{aligned}
$$

Theorem 5.3. The third Zagreb index and their polynomial of a Kulli path windmill graph with $m,n \geq 2$ are

$$M_3\left(P_{n+1}^{(m)}\right) = (mn)^2 - 3mn + 4m,$$

$$M_3\left(P_{n+1}^{(m)},x\right) = 2m + 2mx^{(mn-2)} + (mn-2m)x^{(mn-3)}.$$

Proof. Let $G = P_{n+1}^{(m)}$ where $P_{n+1}^{(m)}$ is a Kulli path windmill graph with $m,n \geq 2$. Consider

$$
\begin{aligned}
M_3\left(P_{n+1}^{(m)}\right) &= \sum_{uv\in E} \left|d_G(u) - d_G(v)\right| \\
&= \sum_{E_5} \left|d_G(u) - d_G(v)\right| + \sum_{E_6} \left|d_G(u) - d_G(v)\right| \\
&+ \sum_{E_{mn+2}} \left|d_G(u) - d_G(v)\right| + \sum_{E_{mn+3}} \left|d_G(u) - d_G(v)\right| \\
&= 2m|3-2| + (mn-3m)|3-3| + 2m|mn-2| + (mn-2m)|mn-3| \\
&= (mn)^2 - 3mn + 4m.
\end{aligned}
$$

Now, for third Zagreb polynomial of a $P_{n+1}^{(m)}$, we have

$$
\begin{aligned}
M_3\left(P_{n+1}^{(m)},x\right) &= \sum_{uv\in E} x^{\left|d_G(u)-d_G(u)\right|} \\
&= \sum_{E_5} x^{\left|d_G(u)-d_G(u)\right|} + \sum_{E_6} x^{\left|d_G(u)-d_G(u)\right|} \\
&+ \sum_{E_{mn+2}} x^{\left|d_G(u)-d_G(u)\right|} + \sum_{E_{mn+3}} x^{\left|d_G(u)-d_G(u)\right|} \\
&= 2mx + 2mx^{(mn-2)} + (mn-2m)x^{(mn-3)}.
\end{aligned}
$$

Corollary 5.1. The Zagreb index and their polynomial of Kulli path windmill graph $P_{n+1}^{(m)}$ are

$$M_2\left(P_{n+1}^{(m)}\right) = M_1\left(P_{n+1}^{(m)}\right) + M_3\left(P_{n+1}^{(m)}\right) + 3\left(m^2 - 3m\right) + m^2\left(m^2 - 2m\right).$$

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BIPOLAR INTUITIONISTIC FUZZY GENERALISED ALPHA CONTINUOUS AND IRRESOLUTE FUNCTIONS IN BIPOLAR INTUITIONISTIC FUZZY ENVIRONMENT

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ABSTRACT
The idea of this paper is to investigate bipolar intuitionistic fuzzy continuity and irresoluteness of generalised alpha closed sets in bipolar intuitionistic fuzzy environment thereby exploring its related properties.

Keywords: Bipolar intuitionistic fuzzy continuous function, Bipolar intuitionistic fuzzy generalised alpha continuous function, Bipolar intuitionistic fuzzy generalised alpha irresolute function.

1. INTRODUCTION
The concept of fuzzy set was introduced by Zadeh[18] in the year 1965 and later Atanassov[2] generalised this idea to a new class of intuitionistic fuzzy sets using the notions of fuzzy sets. C.L. Chang[3] described the new concept of Fuzzy Topological Spaces and Dogan Coker[6] gave an introduction to intuitionistic fuzzy topological spaces. Bipolar valued fuzzy sets, which was introduced by Lee[8] in 2000,which is an extension of fuzzy sets whose membership degree range is enlarged from the interval [0,1] to [-1,1]. D. Ezhilmaran & K. Sankar[7] in the year 2015,discussed on the morphism of bipolar intuitionistic fuzzy graphs and developed its related properties.Later bipolar intuitionistic Fuzzy sets in a soft environment was put forward by Chiranjibe Jana and Madhumangal Pal[4] in the year 2018. K.Ludi Jancy Jenifer and M.Helen[10] introduced bipolar intuitionistic fuzzy generalised alpha closed sets and bipolar intuitionistic fuzzy generalised alpha open sets via bipolar intuitionistic fuzzy topological spaces and studied some of its properties and relationships.

This paper focuses on bipolar intuitionistic fuzzy continuous and irresolute function through bipolar intuitionistic fuzzy generalised alpha closed set.

2. PRELIMINARIES
Definition 2.1[6]
Let X be a non empty set.A bipolar intuitionistic fuzzy set B={$(x, \mu^P(x), \mu^N(x), \gamma^P(x), \gamma^N(x)) \mid x \in X$} where,

$\mu^P : X \to [0,1]$, $\mu^N : X \to [-1,0]$,$\gamma^P : X \to [0,1]$, $\gamma^N : X \to [-1,0]$ are the mappings such that $0 \le \mu^P(x) + \gamma^P(x) \le 1$ and $-1 \le \mu^N(x) + \gamma^N(x) \le 0$.

Definition 2.2[4]
Let A and B be two bipolar intuitionistic fuzzy sets given by A={<$x, \mu_A^p(x), \mu_A^N(x), \gamma_A^P(x), \gamma_A^N(x)$>;x∈X} and B={< $x, \mu_B^p(x), \mu_B^N(x), \gamma_B^P(x), \gamma_B^N(x)$>;x∈X >} in the universe X .If $\mu_A^p(x) \le \mu_B^p(x), \mu_A^N(x) \ge \mu_B^N(x), \gamma_A^P(x) \ge \gamma_B^P(x), \gamma_A^N(x) \le \gamma_B^N(x)$ then A⊆B.

Definition 2.3[7]
The union of two bipolar intuitionistic fuzzy sets A and B in the universe X written as

C= A∪B which is given by

(A∪B)(x)={< $\mu_A^p(x) \vee \mu_B^p(x), \mu_A^N(x) \wedge \mu_B^N(x), \gamma_A^P(x) \wedge \gamma_B^P(x), \gamma_A^N(x) \vee \gamma_B^N(x)$ >}

Definition 2.4[7]
The intersection of two bipolar intuitionistic fuzzy sets A and B in the universe X written as

C= A∩B which is given by

(A∩B)(x)={<$\mu_A^p(x) \wedge \mu_B^p(x), \mu_A^N(x) \vee \mu_B^N(x), \gamma_A^P(x) \vee \gamma_B^P(x), \gamma_A^N(x) \wedge \gamma_B^N(x)$>}

Definition 2.5[4]
The complement of bipolar intuitionistic fuzzy set A in the universe X written as A^c,which is given by

A^c={1-$\mu_A^p(x)$,-1-$\mu_A^N(x)$,1-$\gamma_A^P(x)$,-1-$\gamma_A^N(x)$}

Definition 2.6
Two bipolar intuitionistic fuzzy sets A and B in the universe X is said to be equal and written as A=B if and only if A⊆B and B⊆A.

Definition 2.7
The bipolar intuitionistic fuzzy empty set may be defined as: $0\sim = \{< x, 0,0,1,-1 >: x \in X\}$

Definition 2.8
The bipolar intuitionistic fuzzy absolute set may be defined as : 1~={<x,1,-1,0,0>:x∈ X}

Definition 2.9: A subset A of a topological space (X,τ) is called a

- preclosed set[13] if cl(int (A))⊆A.
- semi-closed set[9] if int(cl(A))⊆A.
- regular closed set[17] if A=cl(int(A)).
- α closed set if[15] cl(int(cl(A)))⊆A.
- generalized semi-closed set[1] (briefly gs-closed set) scl(A)⊆U whever A⊆U and U is open set in X.
- generalized pre-closed set[12](briefly gp-closed set)pcl(A)⊆U whenever A⊆U and U is open set in X.
- generalised alpha closed set[11] if αcl(A)⊆U whenever A⊆U and U is α open in X.

Definition 2.10: Let (X,τ) and (Y,σ) be any two topological spaces,then a mapping f: (X,τ)→ (Y,σ) is said to be

- pre continuous[13] if $f^{-1}(V)$ is preclosed in (X,τ) for every closed set V of (Y,σ).
- semi continuous[9] if $f^{-1}(V)$ is semi closed in (X,τ) for every closed set V of (Y,σ).
- regular continuous[17] if $f^{-1}(V)$ is regular closed in (X,τ) for every closed set V of (Y,σ).
- α continuous[14] if $f^{-1}(V)$ is alpha closed in (X,τ) for every closed set V of (Y,σ).
- generalised semi continuous[5] if $f^{-1}(V)$ is generalised semi closed set in (X,τ) for every closed set V of (Y,σ).
- generalised pre continuous[16] if $f^{-1}(V)$ is generalised pre closed set in (X,τ) for every closed set V of (Y,σ).
- generalised alpha continuous[11] if $f^{-1}(V)$ is generalised alpha closed set in (X,τ) for every closed set V of (Y,σ).

Definition 2.11[10]
A Bipolar intuitionistic fuzzy topology (BIFTS for short) on a nonempty set X is a family τ of bipolar intuitionistic fuzzy subsets in X satisfying the following axioms

Axiom1:$0\sim, 1\sim \in \tau$
Axiom 2:$\cup G_i \in \tau$ for any $\{G_i / i \in J\} \subseteq \tau$

Axiom 3:A∩B $\in \tau$for any $A, B \in \tau$

In this case the pair (X,τ) is called a bipolar intuitionistic fuzzy topological space(BIFTS for short) and any BIFS in τ is known as a bipolar intuitionistic fuzzy open set (BIFOS for short) in X.The complement A^c of a BIFOS is called a bipolar intuitionistic fuzzy closed set.

3. BIPOLAR INTUITIONISTIC FUZZY CONTINUOUS MAPPINGS
Definition 3.1: Let (X,τ) and (Y,σ) be any two bipolar intuitionistic fuzzy topological spaces, then a mapping f: (X,τ)→ (Y,σ) is said to be

- bipolar intuitionistic fuzzy pre continuous if $f^{-1}(V)$ is bipolar intuitionistic fuzzy preclosed in (X,τ) for every bipolar intuitionistic fuzzy closed set V of (Y,σ).
- bipolar intuitionistic fuzzy semi continuous if $f^{-1}(V)$ is bipolar intuitionistic fuzzy semi closed in (X,τ) for every bipolar intuitionistic fuzzy closed set V of (Y,σ).
- bipolar intuitionistic fuzzy regular continuous if $f^{-1}(V)$ is bipolar intuitionistic fuzzy regular closed in (X,τ) for every bipolar intuitionistic fuzzy closed set V of (Y,σ).
- bipolar intuitionistic fuzzy α continuous if $f^{-1}(V)$ is bipolar intuitionistic fuzzy alpha closed in (X,τ) for every bipolar intuitionistic fuzzy closed set V of (Y,σ).

- bipolar intuitionistic fuzzy generalised semi continuous if $f^{-1}(V)$ bipolar intuitionistic fuzzy is generalised semi closed set in (X,τ) for every bipolar intuitionistic fuzzy closed set V of (Y,σ).
- bipolar intuitionistic fuzzy generalised pre continuous if $f^{-1}(V)$ is bipolar intuitionistic fuzzy generalised pre closed set in (X,τ) for every bipolar intuitionistic fuzzy closed set V of (Y,σ).

4. BIPOLAR INTUITIONISTIC FUZZY GENERALISED ALPHA CONTINUOUS MAPPING

Definition 4.1

Let (X,τ) and (Y,σ) be any two bipolar intuitionistic fuzzy topological spaces, then a mapping f: (X,τ)→ (Y,σ) is said to be bipolar intuitionistic fuzzy generalised alpha continuous if $f^{-1}(V)$ is bipolar intuitionistic fuzzy generalised alpha closed set in (X,τ) for every bipolar intuitionistic fuzzy closed set V of (Y,σ).

Example 4.2

Let X={a,b} and Y={u,v} and

$G_1=\left\{x, \frac{a}{<0.1,-0.2,0.4,-0.4>}, \frac{b}{<0.1,-0.2,0.5,-0.5>}\right\}$,

$G_2=\left\{x, \frac{u}{<0.1,-0.2,0.6,-0.6>}, \frac{v}{<0.1,-0.2,0.7,-0.7>}\right\}$

Let τ={0~,1~,G_1} and σ={0~,1~,G_2} be two BIFTs on X and Y respectively.Define a mapping f: (X,τ)→ (Y,σ) by f(a)=u and f(b)=v.Then f is BIFGα continuous mapping.

Theorem 4.3

Let (X,τ) and (Y,σ) be any two bipolar intuitionistic fuzzy topological spaces.For any bipolar intuitionistic fuzzy continuous function f: (X,τ)→ (Y,σ) we have the following results.

(i) Every BIF continuous mapping is a BIFGα continuous mapping.

(ii) Every BIFα continuous mapping is a BIFGα continuous mapping.

(iii) Every BIFGα continuous mapping is a BIFGP continuous mapping.

(iv) Every BIFGα continuous mapping is a BIFGS continuous mapping.

(v) Every BIFR continuous mapping is a BIFGα continuous mapping.

(vi) Every BIF continuous mapping is a BIFα continuous mapping.

(vii) Every BIF continuous mapping is a BIFP continuous mapping.

(viii) Every BIFα continuous mapping is a BIFP continuous mapping.

(ix) Every BIFR continuous mapping is a BIF continuous mapping.

(x) Every BIFα continuous mapping is a BIFS continuous mapping.

Proof

(i) Let f: (X,τ)→ (Y,σ) be a bipolar intuitionistic fuzzy continuous mapping.Let A be a BIFCS in Y.Then f^{-1}(A) is BIFCS in X.But every BIFCS is BIFGαCS.Therefore f^{-1}(A) is BIFGαCS.Hence f is a BIFGα continuous mapping.

(ii) Let f: (X,τ)→ (Y,σ) be a bipolar intuitionistic fuzzy α continuous mapping.Let A be a BIFCS in Y.Then f^{-1}(A) is BIFαCS in X.But every BIFαCS is BIFGαCS.Therefore f^{-1}(A) is BIFGαCS.Hence f is a BIFGα continuous mapping.

Proof of (iii)-(x) is obvious.

The converse of the above theorem need not be true as shown in the examples below.

Example 4.4

Let X={a,b} and Y={u,v} and

$G_1=\left\{x, \frac{a}{<0.1,-0.2,0.8,-0.2>}, \frac{b}{<0.1,-0.3,0.9,-0.2>}\right\}$,

$G_2=\left\{x, \frac{u}{<0.9,-0.4,0.1,-0.2>}, \frac{v}{<0.8,-0.5,0.1,-0.2>}\right\}$

Let τ={0~,1~,G_1} and σ={0~,1~,G_2}be two BIFTs on X and Y respectively.Define a mapping f: (X,τ)→ (Y,σ) by f(a)=u and f(b)=v.Then f is BIFGα continuous mapping but not BIF continuous mapping,since $f^{-1}(G_2^C)$ is not BIFCS in X.

Example 4.5
Let X={a,b} and Y={u,v} and

$G_1=\left\{x, \frac{a}{<0.5,-0.3,0.1,-0.4>}, \frac{b}{<0.6,-0.3,0.1,-0.4>}\right\}$,

$G_2=\left\{x, \frac{u}{<0.3,-0.3,0.1,-0.2>}, \frac{v}{<0.4,-0.3,0.1,-0.3>}\right\}$

Let τ={0~,1~,G_1} and σ={0~,1~,G_2} be two BIFTs on X and Y respectively.Define a mapping f: (X,τ)→ (Y,σ) by f(a)=u and f(b)=v.Then f is BIFGα continuous mapping but not BIFα continuous mapping,since $BIF\alpha cl\{f^{-1}(G_2^C)\}$=1$\nsubseteq G_2^C$.

Example 4.6
Let X={a,b} and Y={u,v} and

$G_1=\left\{x, \frac{a}{<0.7,-0.6,0.1,-0.2>}, \frac{b}{<0.6,-0.7,0.1,-0.2>}\right\}$,

$G_2=\left\{x, \frac{u}{<0.5,-0.8,0.1,-0.2>}, \frac{v}{<0.4,-0.9,0.1,-0.1>}\right\}$

Let τ={0~,1~,G_1} and σ={0~,1~,G_2} be two BIFTs on X and Y respectively.Define a mapping f: (X,τ)→ (Y,σ) by f(a)=u and f(b)=v.Then f is BIFGP continuous mapping but not BIF$G\alpha$ continuous mapping,since $BIF\alpha cl\{f^{-1}(G_2^C)\}$=1$\nsubseteq G_1$.

Example 4.7
Let X={a,b} and Y={u,v} and

$G_1=\left\{x, \frac{a}{<0.3,-0.4,0.7,-0.6>}, \frac{b}{<0.2,-0.1,0.8,-0.9>}\right\}$,

$G_2=\left\{x, \frac{u}{<0.7,-0.6,0.3,-0.4>}, \frac{v}{<0.8,-0.9,0.2,-0.1>}\right\}$

Let τ={0~,1~,G_1} and σ={0~,1~,G_2} be two BIFTs on X and Y respectively.Define a mapping f: (X,τ)→ (Y,σ) by f(a)=u and f(b)=v.Then f is BIFGS continuous mapping but not BIF$G\alpha$ continuous mapping,since $BIF\alpha cl\{f^{-1}(G_2^C)\}$=$G_1^C \nsubseteq G_1$.

Example 4.8
Let X={a,b} and Y={u,v} and

$G_1=\left\{x, \frac{a}{<0.1,-0.3,0.1,-0.1>}, \frac{b}{<0.1,-0.3,0.1,-0.2>}\right\}$,

$G_2=\left\{x, \frac{u}{<0.1,-0.3,0.1,-0.3>}, \frac{v}{<0.1,-0.3,0.1,-0.4>}\right\}$

Let τ={0~,1~,G_1} and σ={0~,1~,G_2} be two BIFTs on X and Y respectively.Define a mapping f: (X,τ)→ (Y,σ) by f(a)=u and f(b)=v.Then f is BIFR continuous mapping but not BIF$G\alpha$ continuous mapping,since $BIFrcl\{f^{-1}(G_2^C)\}$=0$\neq G_2^C$.

Example 4.9
Let X={a,b} and Y={u,v} and

$G_1=\left\{x, \frac{a}{<0.2,-0.1,0.1,-0.4>}, \frac{b}{<0.2,-0.1,0.1,-0.5>}\right\}$,

$G_2=\left\{x, \frac{u}{<0.2,-0.1,0.1,-0.6>}, \frac{v}{<0.2,-0.1,0.1,-0.7>}\right\}$

Let τ={0~,1~,G_1} and σ={0~,1~,G_2} be two BIFTs on X and Y respectively.Define a mapping f: (X,τ)→ (Y,σ) by f(a)=u and f(b)=v.Then f is BIFα continuous mapping but not BIF continuous mapping,since BIF$cl\{f^{-1}(G_2^C)\}$=$G_1^C \neq G_2^C$.

Example 4.10
Let X={a,b} and Y={u,v} and

$G_1=\left\{x, \frac{a}{<0.9,-0.7,0.1,-0.2>}, \frac{b}{<0.9,-0.7,0.1,-0.3>}\right\}$,

$G_2=\left\{x, \frac{u}{<0.1,-0.3,0.5,-0.4>}, \frac{v}{<0.1,-0.3,0.6,-0.5>}\right\}$

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Let τ={0~,1~,G_1} and σ={0~,1~,G_2} be two BIFTs on X and Y respectively.Define a mapping f: (X,τ)→ (Y,σ) by f(a)=u and f(b)=v.Then f is BIFP continuous mapping but not BIF continuous mapping,since $BIFcl\{f^{-1}(G_2^C)\}=1 \neq G_2^C$.

Example 4.11
Let X={a,b} and Y={u,v} and

$G_1=\left\{x, \frac{a}{<0.1,-0.3,0.7,-0.6>}, \frac{b}{<0.2,-0.1,0.1,-0.2>}\right\}$,

$G_2=\left\{x, \frac{u}{<0.2,-0.1,0.1,-0.3>}, \frac{v}{<0.2,-0.1,0.1,-0.4>}\right\}$

Let τ={0~,1~,G_1} and σ={0~,1~,G_2} be two BIFTs on X and Y respectively.Define a mapping f: (X,τ)→ (Y,σ) by f(a)=u and f(b)=v.Then f is BIFP continuous mapping but not BIFα continuous mapping,since $BIF\alpha cl\{f^{-1}(G_2^C)\}=1 \nsubseteq G_2^C$.

Example 4.12
Let X={a,b} and Y={u,v} and

$G_1=\left\{x, \frac{a}{<0.2,-0.1,0.1,-0.9>}, \frac{b}{<0.2,-0.2,0.1,-0.1>}\right\}$,

$G_2=\left\{x, \frac{u}{<0.2,-0.1,0.1,-0.9>}, \frac{v}{<0.2,-0.2,0.1,-0.4>}\right\}$

Let τ={0~,1~,G_1} and σ={0~,1~,G_2} be two BIFTs on X and Y respectively.Define a mapping f: (X,τ)→ (Y,σ) by f(a)=u and f(b)=v.Then f is BIF continuous mapping but not BIFR continuous mapping,since $BIFrcl\{f^{-1}(G_2^C)\}=0 \nsubseteq G_2^C$.

Example 4.13
Let X={a,b} and Y={u,v} and

$G_1=\left\{x, \frac{a}{<0.2,-0.1,0.1,-0.4>}, \frac{b}{<0.2,-0.1,0.1,-0.5>}\right\}$,

$G_2=\left\{x, \frac{u}{<0.2,-0.1,0.1,-0.6>}, \frac{v}{<0.2,-0.1,0.1,-0.7>}\right\}$

Let τ={0~,1~,G_1} and σ={0~,1~,G_2} be two BIFTs on X and Y respectively.Define a mapping f: (X,τ)→ (Y,σ) by f(a)=u and f(b)=v.Then f is BIFα continuous mapping but not BIF continuous mapping,since $BIFcl\{f^{-1}(G_2^C)\}=G_1^C \nsubseteq G_2^C$.

Remark 4.14
BIFP continuous mapping and BIFGα continuous mapping are independent to each other.

Example 4.15
Let X={a,b} and Y={u,v} and

$G_1=\left\{x, \frac{a}{<0.1,-0.3,0.3,-0.2>}, \frac{b}{<0.1,-0.3,0.4,-0.3>}\right\}$,

$G_2=\left\{x, \frac{u}{<0.9,-0.8,0.1,-0.1>}, \frac{v}{<0.9,-0.7,0.1,-0.1>}\right\}$

Let τ={0~,1~,G_1} and σ={0~,1~,G_2}be two BIFTs on X and Y respectively.Define a mapping f: (X,τ)→ (Y,σ) by f(a)=u and f(b)=v.Then f is BIFP continuous mapping but not BIFGα continuous mapping,since BIF$\alpha cl\{f^{-1}(G_2^C)\}=1 \nsubseteq G_2^C$.

Example 4.16
Let X={a,b} and Y={u,v} and

$G_1=\left\{x, \frac{a}{<0.2,-0.2,0.8,-0.8>}, \frac{b}{<0.4,-0.2,0.6,-0.8>}\right\}$,

$G_2=\left\{x, \frac{u}{<0.7,-0.7,0.3,-0.3>}, \frac{v}{<0.5,-0.7,0.5,-0.3>}\right\}$

Let τ={0~,1~,G_1} and σ={0~,1~,G_2} be two BIFTs on X and Y respectively.Define a mapping f: (X,τ)→ (Y,σ) by f(a)=u and f(b)=v.Then f is BIFGα continuous mapping but not BIFP continuous mapping,since BIF$\alpha cl\{f^{-1}(G_2^C)\}=G_1^C \nsubseteq G_2^C$.

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Remark 4.17
BIFS continuous mapping and BIFGα continuous mapping are independent to each other.

Example 4.18
Let X={a,b} and Y={u,v} and

G_1=$\left\{x, \frac{a}{<0.1,-0.3,0.1,-0.5>}, \frac{b}{<0.1,-0.3,0.1,-0.6>}\right\}$,

G_2=$\left\{x, \frac{u}{<0.1,-0.3,0.1,-0.7>}, \frac{v}{<0.1,-0.3,0.2,-0.1>}\right\}$

Let τ={0~,1~,G_1} and σ={0~,1~,G_2} be two BIFTs on X and Y respectively.Define a mapping f: (X,τ)→ (Y,σ) by f(a)=u and f(b)=v.Then f is BIFGα continuous mapping but not BIFS continuous mapping,since $BIFscl\{f^{-1}(G_2^C)\}$=1 $\nsubseteq G_2^C$.

Example 4.19
Let X={a,b} and Y={u,v} and

G_1=$\left\{x, \frac{a}{<0.2,-0.4,0.8,-0.6>}, \frac{b}{<0.1,-0.4,0.9,-0.6>}\right\}$,

G_2=$\left\{x, \frac{u}{<0.8,-0.6,0.2,-0.4>}, \frac{v}{<0.9,-0.6,0.1,-0.4>}\right\}$

Let τ={0~,1~,G_1} and σ={0~,1~,G_2}be two BIFTs on X and Y respectively.Define a mapping f: (X,τ)→ (Y,σ) by f(a)=u and f(b)=v.Then f is BIFS continuous mapping but not BIFGα continuous mapping,since $BIF\alpha cl\{f^{-1}(G_2^C)\}$=$G_1^C \nsubseteq G_2^C$.

Remark 4.20
The composition of two BIFGα continuous mappings need not be a BIFGα continuous mapping.

Example 4.21
Let X={a,b} and Y={u,v} and Z={w,x} and

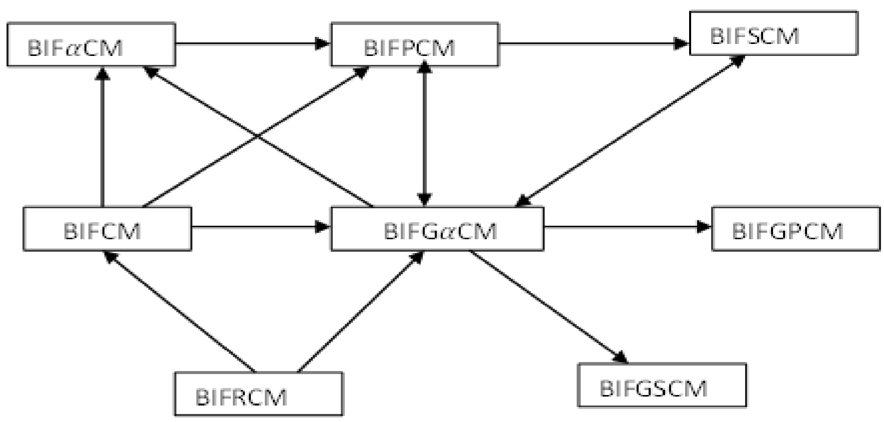
G_1=$\left\{x, \frac{\mathrm{a}}{<0.5,-0.1,0.1,-0.1>}, \frac{\mathrm{b}}{<0.2,-0.1,0.2,-0.2>}\right\}$,

G_2=$\left\{x, \frac{u}{<0.4,-0.1,0.2,-0.4>}, \frac{v}{<0.1,-0.1,0.3,-0.3>}\right\}$,

G_3=$\left\{x, \frac{w}{<0.6,-0.9,0.4,-0.1>}, \frac{x}{<0.8,-0.9,0.2,-0.1>}\right\}$.

Let τ={0~,1~,G_1} and σ={0~,1~,G_2} and μ={0~,1~,G_3}be the BIFTs on X,Y and Z respectively. Define a mapping f: (X,τ)→ (Y,σ) by f(a)=u and f(b)=v and g: (Y,σ)→ (Z,μ) by g(u)=w and g(v)=x. Then the mappings f and g are BIFGα continuous mappings but the mapping, g∘f: (X,τ)→ (Z,μ) is not BIFGα continuous mapping.

From the above theorems we have the following implications as shown in the diagram:



In this diagram A→B means A implies B but not conversely.A↮B means A and B are independent of each other. None of them are reversible.

Definition 4.22
A bipolar intuitionistic fuzzy topological space (X,τ) is said to be $T_{g\alpha}$ space if every BIFGαCS(BIFGαOS) is BIFCS(BIFOS) in X.

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Theorem 4.23
Let f: (X,τ)→ (Y,σ) be a continuous mapping from a BIFTS X into a BIFTS Y.Then the following conditions are equivalent if X is a $T_{g\alpha}$ space.

(i)f is a BIFGα continuous mapping.

(ii)$f^{-1}(B)$is a BIFGαCS in X for every BIFCS B in Y.

(iii)$BIFcl(BIFint(BIFcl(f^{-1}(B)))) \subseteq f^{-1}(BIFcl(B))$ for every BIFCS B in Y.

Proof
(i)⇒(ii):straightforward from the definitions.

(ii)⇒(iii):Let B be a BIFCS in Y.then BIFcl(B)=B.By hypothesis $f^{-1}(BIF\ \mathrm{cl}\ (\mathrm{B}))$ is a BIFGαCS in X.Since X is $T_{g\alpha}$ space,$f^{-1}(\mathrm{BIFcl(B)})$ is a BIFCS in X.Therefore BIFcl(f^{-1}(BIFcl(B)))= $f^{-1}\big(\mathrm{BIFcl(B)}\big)$.Therefore $BIFcl\left(BIFint\left(BIFcl\left(f^{-1}(B)\right)\right)\right) \subseteq$ $BIFcl(BIFint(BIFcl(f^{-1}(BIFcl(B))))) \subseteq f^{-1}(BIFcl(B))$.

(iii)⇒(i)Let B be a BIFCS in Y.By hypothesis $BIFcl(BIFint(BIFcl(f^{-1}(B)))) \subseteq f^{-1}(BIFcl(B))$=$f^{-1}$(B).This implies f^{-1}(B) is a BIFGαCS in X.Therefore f is BIFGα continuous mapping.

Theorem 4.24
A mapping f: (X,τ)→ (Y,σ) is a BIFGα continuous mapping if and only if the inverse image of each BIFOS in Y is BIFGαOS in X.

Proof
Necessity:Let A be a BIFOS in Y.Therefore A^c is BIFCS in Y. Since f is BIFGαCS in X,$f^{-1}(A^c)$=$(f^{-1}(A))^c$,f^{-1}(A) is a BIFGαOS in X.

Sufficient: Straightforward from the definition.

Theorem 4.25
Let f: (X,τ)→ (Y,σ) be a mapping and let f^{-1}(A) is a BIFRCS in X,for every BIFCS A in Y.Then f is BIFGα continuous mapping.

Proof
Let A be a BIFCS in Y,then f^{-1}(A) is a BIFRCS in X.But every BIFRCS is BIFGαCS.Therefore f^{-1}(A) is BIFGαCS in X.Hence f is BIFGα continuous mapping.

Theorem 4.26
Let f: (X,τ)→ (Y,σ) be a continuous mapping from a BIFTS X into a BIFTS Y.Then the following conditions are equivalent if X is a $T_{g\alpha}$ space.

(i)f is a BIFGα continuous mapping.

(ii)$f^{-1}(B)$is a BIFGαOS in X for every BIFOS B in Y.

(iii)$f^{-1}(BIFint(B)) \subseteq BIFint(BIFcl(BIFint(f^{-1}(B))))$for every BIFS B in Y.

Proof
(i)⇒(ii):straightforward from the definitions.

(ii)⇒(iii):Let B be a BIFOS in Y,then BIFint(B)=B.By hypothesis $f^{-1}(BIF\ \mathrm{int(B)})$ is a BIFGαOS in X.Since X is $T_{g\alpha}$ space,$f^{-1}(\mathrm{BIFint(B)})$ is a BIFOS in X.Therefore f^{-1}(BIF int(B))=BIF int($f^{-1}\big(\mathrm{BIF\ int(B)}\big) \subseteq$ $BIF\ int(BIF\ cl(BIF\ int(f^{-1}(B))))$.

(iii)⇒(i)Let B be a BIFCS in Y.Then B^c is BIFOS in Y,then BIF int(B^c)=B^c.By hypothesis

$f^{-1}\left(BIFint(B^c)\right) \subseteq BIFint\left(BIFcl\left(BIFint\left(f^{-1}(B)\right)\right)\right)$ *which implies* $f^{-1}(B^c)\subseteq$ $BIFint(BIFcl(BIFint(f^{-1}(B))))$.This implies $f^{-1}(B^c)$ is a BIFOS in X.But every BIFOS is BIFGαOS.Therefore $f^{-1}(B^c)$ is a BIFGαOS in X.Therefore f is BIFGα continuous mapping.

Theorem 4.27
Let f: (X,τ)→ (Y,σ) be a BIFGα continuous mapping,then f is a BIF continuous mapping if X is a $T_{g\alpha}$ space.

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Proof
Let A be a BIFCS in Y.Then f^{-1}(A) is a BIFGαCS in X.Since X is $T_{g\alpha}$ space,f^{-1}(A) is a BIFCS in X.Hence f is a BIF continuous mapping.

Theorem 4.28
Let f: (X,τ)→ (Y,σ) be a BIFGα continuous mapping and g: (Y,σ)→(Z,μ) is a BIF continuous mapping then g∘f: (X,τ)→ (Z,μ) is a BIFGα continuous mapping.

Proof
Let A be a BIFCS in Z.Then g^{-1}(A) is a BIFCS in Y.By hypothesis,since f is BIFGα continuous mapping,$f^{-1}(g^{-1}(A))$ is BIFGαCS in X.We know that,$(g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A))$.Therefore g∘f: (X,$\tau$)→ (Z,$\mu$) is a BIFG$\alpha$ continuous mapping.

Definition 4.29
Let (X,τ) be a BIFTS.The bipolar intuitionistic fuzzy generalised alpha closure (BIFGαcl(A) in short) for any BIFS A is defined as

BIFGαcl(A)=∩{K/K is an BIFGαCS in X and A⊆K }. If A is a BIFGαCS,then BIFGαcl(A)=A.

Remark 4.30
It is clear that A⊆BIFGαcl(A)⊆BIFcl(A).

Theorem 4.31
Let f: (X,τ)→ (Y,σ) be a BIFGα continuous mapping.Then the following statements hold.

(i) f(BIFGαcl(A))⊆BIFcl(f(A)) for every BIFS, A in X.

(ii)BIFGαcl($f^{-1}(B)$)⊆ $f^{-1}(BIFcl(B))$ for every BIFS, B in X.

Proof
(i)Let A⊆X.Then BIF(cl(A)) is BIFCS in Y.Since f is a BIFGα continuous mapping,f^{-1}(BIF cl(A)) is BIFGα in X.(i.e)BIFGαcl(A)⊆ f^{-1}(BIF cl (f(A))).Since A⊆ $f^{-1}(f(A)) \subseteq f^{-1}BIFcl(f(A))$ and f^{-1}(BIF cl (f(A))) is a BIFGα closed,this implies

BIFGαcl(A)⊆ f^{-1} (BIF cl f(A)).Hence f(BIFGαcl(A))⊆BIFcl(f(A)).

(ii)Replacing A by f^{-1}(B) in (i),we get f(BIFGαcl($f^{-1}(B)$))⊆BIFcl(f($f^{-1}(B)$))⊆BIF cl(B).

Hence BIFGαcl($f^{-1}(B)$)⊆ $f^{-1}(BIFcl(B))$ for every BIFS, B in X.

Definition 4.32
A bipolar intuitionistic fuzzy topological space (X,τ) is said to be T_α space if every BIFGαCS is BIFαCS in X.

Theorem 4.33
Let f: (X,τ)→ (Y,σ) be a mapping from a BIFTS X into a BIFTS Y.If X is a T_α space, then f is BIFGα continuous mapping if and only if f: (X,τ)→ (Y,σ) is a BIFα continuous mapping.

Proof
Let f be a BIFGα continuous mapping and let A be a BIFCS in Y.Then,f^{-1}(A) is a BIFGαCS in X.Since X is a T_α space,we have from the definition $f^{-1}(A)$ is a BIFαCS in X.Hence f is a BIFGα continuous mapping.

Conversely, assume that f is BIFGα continuous mapping by theorem 4.3(ii),f is a BIFGα continuous mapping.

5. BIPOLAR INTUITIONISTIC FUZZY GENERALISED ALPHA IRRESOLUTE MAPPING

Definition 5.1: Let (X,τ) and (Y,σ) be any two bipolar intuitionistic fuzzy topological spaces, then a mapping f: (X,τ)→ (Y,σ) is said to be bipolar intuitionistic fuzzy generalised alpha irresolute if $f^{-1}(V)$ is bipolar intuitionistic fuzzy generalised alpha open(closed) set in (X,τ) for every bipolar intuitionistic fuzzy generalised alpha open(closed) set V of (Y,σ).

Theorem 5.2
Let f: (X,τ)→ (Y,σ) be a mapping from a BIFTS X into a BIFTS Y.Then every BIFGα irresolute mapping is a BIFGα continuous mapping but not conversely.

Proof
Let A be a BIFCS in Y.We know that every BIFCS is a BIFGαCS in Y.Since f is a BIFGαCS in Y.Since f is a BIFGα irresolute mapping, by definition $f^{-1}(A)$ is a BIFGαCS in X.Hence f is a BIFGα continuous mapping.

Example 5.3: Let X={a,b} and Y={u,v}

$G_1=\left\{x,\frac{a}{<0.9,-0.5,0.1,-0.1>},\frac{b}{<0.9,-0.4,0.1,-0.2>}\right\}$,

$G_2=\left\{x,\frac{u}{<0.2,-0.4,0.1,-0.1>},\frac{v}{<0.2,-0.5,0.1,-0.1>}\right\}$

Let $\tau=\{0\sim,1\sim,G_1\}$ and $\sigma=\{0\sim,1\sim,G_2\}$be two BIFTs on X and Y respectively.Define a mapping f: (X,τ)→ (Y,σ) by f(a)=u and f(b)=v.Then f is BIFGα continuous mapping.Let

B=$\left\{x,\frac{u}{<0.2,-0.6,0.1,-0.1>},\frac{v}{<0.2,-0.7,0.1,-0.1>}\right\}$ is a BIFGαCS in Y.But f^{-1}(B) is not a BIFGαCS in X.Therefore f is not a BIFGα irresolute mapping.

Theorem 5.4
Let f: (X,τ)→ (Y,σ) be a mapping from a BIFTS X into a BIFTS Y.Then the following are equivalent.

(i)f is a BIFGα irresolute mapping.

(ii)f^{-1}(B) is a BIFGαOS in X for every BIFGαOS B in Y.

(iii)BIFGαcl($f^{-1}(B)$)⊆ $f^{-1}(BIFG\alpha\ cl(B))$,for every BIFS B in Y.

(iv)f^{-1}(BIFGαint(B))⊆BIFGα int($f^{-1}(B)$),for every BIFS B in Y.

Proof
(i)⟹(ii) straight forward from the definitions.

(ii)⟹(iii) Let B be a BIFS in Y.Then B⊆BIFcl(B) and so f^{-1}(B)⊆ f^{-1}(BIFGα cl (B)).Since BIFGαcl(B) is a BIFGαCS in Y. f^{-1}(BIFGα cl (B)) is a BIFGαCS in X.Therefore BIFGαcl($f^{-1}(B)$)⊆ $f^{-1}(BIFG\alpha\ cl(B))$.

(iii)⇒(iv) Let B be a BIFOS in Y.Then BIF int(B) is a BIFOS in Y.Then f^{-1}(BIF int(B)) is a BIFGαOS in X.Since BIFGα int(B) is a BIFGαOS in Y, f^{-1}(BIFGα int(B)) is a BIFGαOS in X.Therefore BIFGαcl(B) is a BIFGαCS in Y,f^{-1}(BIFGαCS) is a BIFGαCS in X.Therefore f^{-1}(BIFGαint(B))⊆BIFGα int($f^{-1}(B)$).

(iv)⟹(i)Let B be any BIFGαOS in Y,Then we have BIF Gαint (B)=B.By our assumption we have f^{-1}(B)=f^{-1}(BIFGα int (B))⊆ BIFGα int f^{-1}(B),so f^{-1}(B) is a BIFGαOS in X.Hence f is BIFGα irresolute mapping.

Theorem 5.5
If f: (X,τ)→ (Y,σ) be a BIFGα irresolute mapping,then f is a BIF irresolute mapping if X is a $T_{g\alpha}$ space.

Proof
Let A be a BIFCS in Y.Then A is a BIFGαCS in Y.Therefore f^{-1}(A) is a BIFGαCS in X,by assumption.Since X is a $T_{g\alpha}$ space, f^{-1}(A) is a BIFCS in X.Hence f is a BIF irresolute mapping.

Theorem 5.6
Let f: (X,τ)→ (Y,σ) be a BIFGα irresolute mapping,then f is a BIFGα irresolute mapping if X is a T_α space.

Proof
Let B be a BIFαCS in Y.Then B is a BIFGαCS in Y.Since f is a BIFGα irresolute mapping,f^{-1}(B) is a BIFGαCS in X,by assumption.Since X is a T_α space, f^{-1}(B) is a BIFαCSin X.Hence f is a BIFGαirresolute mapping.

Theorem 5.7
Let f: (X,τ)→ (Y,σ) and g: (Y,σ)→(Z,μ) be two BIFGα irresolute mapping,where X,Y and Z are BIFTS then g∘f: (X,τ)→ (Z,μ) is a BIFGα is a irresolute mapping.

Proof
Let A be a BIFGα closed set in Z.Since g is BIFGα irresolute mapping,g^{-1}(A) is a BIFGαCS in Y.Since X is BIFGα irresolute mapping,f^{-1} (g^{-1}(A)) is a BIFGαCS in X.$(g\circ f)^{-1}$(A)= f^{-1} (g^{-1}(A)) for each A in Z. Hence g∘f: (X,τ)→ (Z,μ) is a BIFGα irresolute mapping.

Theorem 5.8
Let f: (X,τ)→ (Y,σ) be a BIFGα irresolute mapping and g: (Y,σ)→(Z,μ) be BIFGα continuous mapping,where X,Y and Z are BIFTS then g∘f: (X,τ)→ (Z,μ) is a BIFGα is a continuous mapping.

Proof

Let A be a BIF closed set in Z.Since g is BIFGα continuous mapping,g^{-1}(A) is a BIFGαCS in Y.Since X is BIFGα irresolute mapping,f^{-1} (g^{-1}(A)) is a BIFGαCS in X.$(g \circ f)^{-1}$(A)= f^{-1} (g^{-1}(A)) for each A in Z. Hence g∘f: (X,τ)→ (Z,μ) is a BIFGα is a irresolute mapping.

Definition 5.9

Bipolar Intuitionistic fuzzy point

Let (α,β)∈ (0,1) and (γ,δ)∈(-1,0) such that α+β ≤1 and γ+δ ≥-1.A bipolar intuitionstic fuzzy point(BIFP in short) p^x of X is a bipolar intutionistic fuzzy set of X defined by p^x=<x,$\mu^p(x), \mu^N(x)$,$\gamma^p(x), \gamma^N$(x)> .For x∈X,we have

$$\mu^p(y)=\begin{cases}\alpha & if\ y = x\\ 0 & if\ y \neq x\end{cases} \qquad \mu^N(y)=\begin{cases}\gamma & if\ y = x\\ 0 & if\ y \neq x\end{cases}$$

$$\gamma^p(y)=\begin{cases}\beta & if\ y = x\\ 0 & if\ y \neq x\end{cases} \qquad \gamma^N(y)=\begin{cases}\delta & if\ y = x\\ 0 & if\ y \neq x\end{cases}$$

In this case,x is called the support of p^x.A bipolar intutionistic fuzzy point p^x is said to belong to a bipolar intuitionistic fuzzy set A=<x,$\mu^p(x), \mu^N(x)$,$\gamma^p(x), \gamma^N$(x)> of X,defined by c($\alpha,\beta,\gamma,\delta$)∈A if $\alpha \leq \mu^p(x)$,$\beta \leq \gamma^p(x), \mu^N(x) \geq \gamma, \gamma^N$(x)≥ δ.

Definition 5.10

Bipolar intuitionistic fuzzy neighbourhood

Let c($\alpha,\beta,\gamma,\delta$) be a bipolar intuitionistic fuzzy point of a BIFTS(X,τ).A BIFS,A of X is called a bipolar intuitionistic fuzzy neighbourhood (BIFN) of c($\alpha,\beta,\gamma,\delta$) if there is a BIFOS,B in X such that c($\alpha,\beta,\gamma,\delta$)∈B⊆A.

Theorem 5.11

A mapping f: (X,τ)→ (Y,σ) from an bipolar intuitionistic fuzzy topological space X into an bipolar intuitionistic fuzzy topological spacey is said to be Bipolar intuitionistic fuzzy irresolute if and only if for each bipolar intuitionistic fuzzy point c($\alpha,\beta,\gamma,\delta$) in X and bipolar intuitionistic fuzzy generalised alpha open set B in Y such that f(c($\alpha,\beta,\gamma,\delta$))∈B,there exists an bipolar intuitionistic fuzzy generalised alpha open set A in X such that c($\alpha,\beta,\gamma,\delta$)∈A and f(A)⊆B.

Proof

Let f be any bipolar intuitionistic fuzzy generalised alpha irresolute mapping, c($\alpha,\beta,\gamma,\delta$) be an bipolar intuitionistic fuzzy point in X and B be any bipolar intuitionistic fuzzy generalised alpha set in Y such that f((c($\alpha,\beta,\gamma,\delta$))∈B.Then (c($\alpha,\beta,\gamma,\delta$))∈ $f^{-1}(B)$ =BIF Gαint($f^{-1}(B)$) .Let A=BIF int ($f^{-1}(B)$).Then A is an bipolar intuitionistic fuzzy generalised alpha open set in X containing bipolar intuitionistic fuzzy point c($\alpha,\beta,\gamma,\delta$) and f(A)=f(BIF int($f^{-1}(B)$))⊆f($f^{-1}$(B))=B.

Conversely,let B be an bipolar intuitionistic fuzzy generalised alpha open set in Y and c($\alpha,\beta,\gamma,\delta$) be a bipolar intuitionistic fuzzy point in X such that c($\alpha,\beta,\gamma,\delta$)∈A and f(A)⊆B.Hence c($\alpha,\beta,\gamma,\delta$)∈A⊆ $f^{-1}(B)$ and c($\alpha,\beta,\gamma,\delta$)ϵA=BIFG$\alpha$ int(A)⊆BIFGαint ($f^{-1}(B)$).Since c($\alpha,\beta,\gamma,\delta$) be any arbitrary bipolar intutuinistic fuzzy point and f^{-1}(B) is the union of all bipolar intuitionistic fuzzy point contained in f^{-1}(B),we obtain that f^{-1}(B) =BIFGαint(f^{-1}(B)).So f is a bipolar intuitionistic fuzzy generalised alpha irresolute mapping.

Definition 5.12

A mapping f: (X,τ)→ (Y,σ) is said to an bipolar intuitionistic fuzzy contra Gα irresolute mapping if $f^{-1}(A)$ is an BIFGαCS in X for every BIFGαOS in Y.

Defintion 5.13

A mapping f: (X,τ)→ (Y,σ) is said to an bipolar intuitionistic fuzzy perfectly contra BIFGα-irresolute mapping if $f^{-1}(A)$ is both BIFGαCS,BIFGαOS in X for every BIFGαOS in Y.

Definition 5.14

A mapping f: (X,τ)→ (Y,σ) is said to an bipolar intuitionistic fuzzy contra Gα-closed if $f(A)$ is an BIFGαOS in Y for every BIFGαCS A in Y.

Remark 5.15

Bipolar intuitionistic fuzzy contra Gα irresoluteness and bipolar intuitonistic fuzzy Gα irresoluteness are independent of each other.

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Example 5.16
Let X={a,b} and Y={u,v}

G_1=$\left\{x, \frac{a}{<0.2,-0.2,0.1,-0.1>}, \frac{b}{<0.2,-0.3,0.1,-0.1>}\right\}$,

G_2=$\left\{x, \frac{u}{<0.2,-0.3,0.2,-0.2>}, \frac{v}{<0.2,-0.4,0.2,-0.2>}\right\}$

Let τ={0~,1~,G_1} and σ={0~,1~,G_2}be two BIFTs on X and Y respectively.Define a mapping f: (X,τ)→ (Y,σ) by f(a)=u and f(b)=v.Then f is BIFGα irresolute mapping.Let

B=$\left\{x, \frac{u}{<0.1,-0.1,0.2,-0.2>}, \frac{v}{<0.1,-0.2,0.3,-0.4>}\right\}$ is a BIFGαCS in Y.But $BIF\alpha cl\{f^{-1}$(B)} is not a BIFGαCS in X.Therefore f is not contra BIFGα irresolute mapping.

Example 5.17
Let X={a,b} and Y={u,v}

G_1=$\left\{x, \frac{a}{<0.2,-0.4,0.8,-0.6>}, \frac{b}{<0.3,-0.4,0.7,-0.6>}\right\}$,

G_2=$\left\{x, \frac{u}{<0.3,-0.1,0.1,-0.1>}, \frac{v}{<0.4,-0.1,0.1,-0.1>}\right\}$

Let τ={0~,1~,G_1} and σ={0~,1~,G_2} be two BIFTs on X and Y respectively.Define a mapping f: (X,τ)→ (Y,σ) by f(a)=u and f(b)=v.Then f is contra BIFGα irresolute mapping.Let

B=$\left\{x, \frac{u}{<0.9,-0.7,0.1,-0.3>}, \frac{v}{<0.7,-0.7,0.3,-0.3>}\right\}$ is a BIFGαOS in Y.But $BIF\alpha cl\{f^{-1}$(B)} is not a BIFGαOS in X.Therefore f is not BIFGα irresolute mapping.

Theorem 5.18
Let f: (X,τ)→ (Y,σ) be a mapping from a BIFTS X into a BIFTS Y.Then the following conditions are equivalent:

(i)f is a bipolar intuitonistic fuzzy contra Gα irresolute mapping.

(ii)The inverse image of each bipolar intuitionistic fuzzy Gα closed set in Y is an bipolar intuitionistic fuzzy Gα open set in X.

Proof: Straight forward from the definition.

Theorem 5.19
Every bipolar intuitionistic fuzzy perfectly contra Gα irresolute mapping is an bipolar intuitionistic fuzzy contra Gα irresolute mapping.

Proof
Let f: (X,τ)→ (Y,σ) be a bipolar intuitionistic fuzzy perfectly contra Gα irresolute mapping and let A be a bipolar intuitionistic fuzzy Gα open set in Y.Then by our assumption,f^{-1}(B) is an bipolar intuitionistic fuzzy Gα clopen set in X.Thus f^{-1}(B) is an bipolar intuitionistic fuzzy Gα closed set in X.Hence f is bipolar intuitionistic fuzzy contra Gα irresolute mapping.

The converse of the above theorem need not be true as shown in the example below.

Example 5.20
Let X={a,b} and Y={u,v}

G_1=$\left\{x, \frac{a}{<0.6,-0.1,0.3,-0.7>}, \frac{b}{<0.5,-0.1,0.3,-0.6>}\right\}$,

G_2=$\left\{x, \frac{u}{<0.2,-0.7,0.2,-0.2>}, \frac{v}{<0.2,-0.8,0.2,-0.2>}\right\}$

Let τ={0~,1~,G_1} and σ={0~,1~,G_2} be two BIFTs on X and Y respectively.Define a mapping f:(X,τ)→(Y,σ) by f(a)=u and f(b)=v.Then f is contra BIFGα irresolute mapping.Let B=$\left\{x, \frac{u}{<0.9,-0.9,0.1,-0.1>}, \frac{v}{<0.6,-0.9,0.4,-0.1>}\right\}$ be a BIFGαOS in Y.But $BIF\alpha cl\{f^{-1}$(B)} is not a BIFGαOS in X.Therefore f is not perfectly contra BIFGα irresolute mapping.

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Theorem 5.21
Every bipolar intuitionistic fuzzy perfectly contra Gα irresolute mapping is an bipolar intuitionistic fuzzy Gα irresolute mapping.

Proof: Straight from the definition.

The converse of the above theorem need not be true as shown in the example below.

Example 5.22
Let X={a,b} and Y={u,v}

$G_1=\left\{x,\frac{a}{<0.6,-0.2,0.2,-0.1>},\frac{b}{<0.5,-0.3,0.3,-0.2>}\right\}$,

$G_2=\left\{x,\frac{u}{<0.2,-0.5,0.2,-0.2>},\frac{v}{<0.2,-0.6,0.2,-0.2>}\right\}$

Let τ={0~,1~,G_1} and σ={0~,1~,G_2}be two BIFTs on X and Y respectively.Define a mapping f: (X,τ)→ (Y,σ) by f(a)=u and f(b)=v.Then f is BIFGα irresolute mapping.Let B=$\left\{x,\frac{u}{<0.4,-0.1,0.3,-0.2>},\frac{v}{<0.5,-0.2,0.4,-0.3>}\right\}$ be a BIFGαOS in Y.But $BIF\alpha cl\{f^{-1}$(B)} is not a BIFGαCS in X.Therefore f is not perfectly contra BIFGα irresolute mapping.

Theorem 5.23
Let f: (X,τ)→ (Y,σ) be a mapping from a BIFTS(X,τ) into an BIFTS(Y,σ).Then the following conditions are equivalent.

(i)f is an bipolar intuitionistic fuzzy perfectly contra Gα irresolute mapping.

(ii)f is an bipolar intuitionistic fuzzy Gα contra –irresolute mapping and bipolar intuitionistic fuzzy Gα-irresolute mapping.

Proof
(i)⇒(ii) Let f be a bipolar intuitionistic fuzzy perfectly Gα irresolute mapping.Let B be a BIFGαOS in Y.By our assumption,f^{-1}(B) is a bipolar intuitionistic fuzzy Gα clopen set in X.Thus f^{-1}(B) is a bipolar intuitionistic fuzzy Gα closed and bipolar intuitionistic fuzzy Gα open set in X.Hence f is a bipolar intuitionistic fuzzy Gα contra –irresolute mapping and bipolar intuitionistic fuzzy Gα-irresolute mapping.

(ii)⇒(i)Let f be both bipolar intuitionistic fuzzy Gα contra –irresolute mapping and bipolar intuitionistic fuzzy Gα-irresolute mapping and let B be a bipolar intuitionistic fuzzy Gα-open set in Y.By our assumption,f^{-1}(B) is both BIFGαCS and BIFGαOS in X.(i.e) f^{-1}(B) is bipolar intuitionistic fuzzy perfectly contra Gα irresolute mapping.

Theorem 5.24: Let f: (X,τ)→ (Y,σ) and g: (Y,σ)→(Z,μ) be two mappings.Then

(i) g∘f is a bipolar intuitionistic fuzzy perfectly contra Gα-irresolute if f and g are bipolar intuitionistic fuzzy perfectly contra Gα-irresolute mapping.

(ii) g∘f is a bipolar intuitionistic fuzzy contra Gα-irresolute if f is bipolar intuitionistic fuzzy perfectly contra Gα-irresolute mapping and g is bipolar intuitionistic fuzzy contra Gα-irresolute mapping.

(iii) g∘f is a bipolar intuitionistic fuzzy Gα-irresolute if f is bipolar intuitionistic fuzzy perfectly contra Gα-irresolute mapping and g is bipolar intuitionistic fuzzy contra Gα-irresolute mapping.

(iv) g∘f is a bipolar intuitionistic fuzzy contra Gα-irresolute if f is bipolar intuitionistic fuzzy perfectly contra Gα-irresolute mapping and g is bipolar intuitionistic fuzzy Gα-irresolute mapping.

(v) g∘f is a bipolar intuitionistic fuzzy Gα-irresolute if f is bipolar intuitionistic fuzzy contra Gα-irresolute mapping and g is bipolar intuitionistic fuzzy contra Gα-irresolute mapping.

(vi) g∘f is a bipolar intuitionistic fuzzy contra Gα-irresolute if f is bipolar intuitionistic fuzzy contra Gα-irresolute mapping and g is bipolar intuitionistic fuzzy Gα-irresolute mapping.

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MULTIPLICATIVE CONNECTIVITY INDICES OF SILICATE AND HEXAGONAL NETWORKS

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ABSTRACT
Topological indices a numeric quantity from the structural graph of a molecule. Topological index can be regarded as a score function which maps each molecular structure to a real number. In this paper, we compute the multiplicative product connectivity, multiplicative sum connectivity, First multiplicative atom bond connectivity and multiplicative geometric- Arithmetic Revan indices of silicate and hexagonal networks.

Keywords: Revan vertex degree, Multiplicative product connectivity Revan index, Multiplicative sum connectivity Revan index, First multiplicative atom bond connectivity Revan index, Multiplicative geometric-arithmetic Revan index silicate network, hexagonal network.

1. INTRODUCTION
Chemical graph theory is a branch of Mathematical chemistry which has an important effect on the development of the chemical sciences. Numerous topological indices have been considered in theoretical chemistry, and have found some applications in QSPR/QSAR study [2]. Recently many multiplicative topological indices were studied, for example, in [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 14, 15, 16]. Also some connectivity indices were studied, for example, in [17, 18, 19, 20, 21, 22]. In this paper we are finding the Multiplicative product connectivity Revan index, Multiplicative sum connectivity Revan index, First multiplicative atom bond connectivity Revan index and Multiplicative geometric-arithmetic Revan index of silicate network and hexagonal network.

Let G be a graph which is finite only, simple and connected with vertex set V(G) and edge set E(G). The degree deg(v) of a vertex v is the number of vertices adjacent to v. Let Δ(G) and δ(G) denote the maximum and minimum degree among the vertices of G. The Revan vertex degree of a vertex v in G is defined as $r_G(v) = \Delta(G) + \delta(G) - \deg(v)$. The edge connecting the Revan vertices u and v will be denoted by uv. For additional definitions and notations, the reader may refer to [13].

Best known and used topological indices are the multiplicative connectivity indices, introduced by Kulli in [2]. Motivated by the definitions of the multiplicative connectivity indices and their wide applications, we introduce the multiplicative product connectivity Revan index, multiplicative sum connectivity Revan index, multiplicative atom bond connectivity Revan index and multiplicative geometric-arithmetic Revan index of a molecular graph as follows:

The multiplicative product connectivity Revan index of a graph *G* is defined as

$$PRII(G) = \prod_{uv\in E(G)} \frac{1}{\sqrt{r_G(U)\, r_G(V)}}$$

The multiplicative sum connectivity Revan index of a graph *G* is defined as

$$SRII(G) = \prod_{uv\in E(G)} \frac{1}{\sqrt{r_G(U) + r_G(V)}}$$

The first multiplicative atom bond connectivity Revan index of a graph *G* is defined as

$$ABCRII(G) = \prod_{uv\in E(G)} \sqrt{\frac{r_G(U) + r_G(V) - 2}{r_G(U)\, r_G(V)}}$$

The multiplicative geometric-arithmetic Revan index of a graph *G* is defined as

$$GARII(G) = \prod_{uv\in E(G)} \frac{2\sqrt{r_G(U)\, r_G(V)}}{r_G(U) + r_G(V)}$$

Recently many multiplicative topological indices were studied. we compute multiplicative connectivity Revan indices of Silicate and Hexagonal networks.

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2. RESULTS FOR SILICATE NETWORKS

In this section we consider silicate network SL_n. From Figure 1, the vertices of SL_n are either of degree 3 or 6. By algebraic method, we obtain that V (SL_n)j = $15n^2 + 3n$ and E(SL_n)j = $36n^2$.

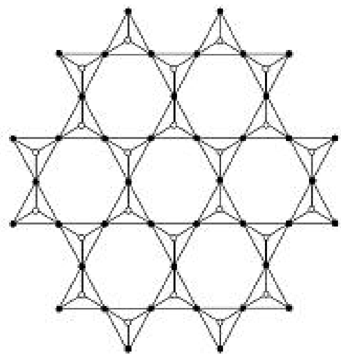

Figure-1: A 2-dimensional Silicate network

In SL_n, by algebraic method, there are three types of edges based on the degree of the end vertices of each edge as follows:

Edge Partition	Number of Edges
$E_{33} = \{u, v \in E(SL_n) / d_{SL_n}(u) = d_{SL_n}(v) = 3\}$	$\|E_{33}\| = 6n$
$E_{36} = \{u, v \in E(SL_n) / d_{SL_n}(u) = 3, d_{SL_n}(v) = 6\}$	$\|E_{36}\| = 18n^2 + 6n$
$E_{66} = \{u, v \in E(SL_n) / d_{SL_n}(u) = d_{SL_n}(v) = 6\}$	$\|E_{66}\| = 18n^2 - 12n$

Clearly $\Delta(G) = 6$ and $\delta(G) = 3$. Therefore $r_{SL_n}(u) = 9 - d_{SL_n}(u)$. Thus we ensure that there are three types of Revan edges based on the degree of end Revan vertices of each Revan edge as follows:

Revan Edge Partition	Number of Edges
$RE_{66} = \{u, v \in E(SL_n) / r_{SL_n}(u) = r_{SL_n}(v) = 6\}$	$\|RE_{66}\| = 6n$
$RE_{36} = \{u, v \in E(SL_n) / r_{SL_n}(u) = 6, r_{SL_n}(v) = 3\}$	$\|RE_{63}\| = 18n^2 + 6n$
$RE_{66} = \{u, v \in E(SL_n) / r_{SL_n}(u) = r_{SL_n}(v) = 6\}$	$\|RE_{33}\| = 18n^2 - 12n$

In the following theorem, we compute the multiplicative product connectivity Revan index of SL_n.

Theorem 1: The multiplicative product connectivity Revan index of a Silicate network SL_n is given by

$$PRII(SL_n) = \left(\frac{1}{6}\right)^{6n} \left(\frac{1}{3\sqrt{2}}\right)^{18n^2+6n} \left(\frac{1}{3}\right)^{18n^2-12n}$$

Proof: By definition, we have $PRII(G) = \prod_{uv\in E(G)} \frac{1}{\sqrt{r_G(U)\, r_G(V)}}$

Thus

$$PRII(SL_n) = \left(\frac{1}{\sqrt{6\times 6}}\right)^{6n} \left(\frac{1}{\sqrt{6\times 3}}\right)^{18n^2+6n} \left(\frac{1}{\sqrt{3\times 3}}\right)^{18n^2-12n}$$

$$= \left(\frac{1}{6}\right)^{6n} \left(\frac{1}{3\sqrt{2}}\right)^{18n^2+6n} \left(\frac{1}{3}\right)^{18n^2-12n}$$

In the following theorem, we compute the multiplicative sum connectivity Revan index of SL_n.

Theorem 2: The multiplicative sum connectivity Revan index of a Silicate network SL_n is given by

$$SRII(SL_n) = \left(\frac{1}{\sqrt{12}}\right)^{6n} \left(\frac{1}{\sqrt{9}}\right)^{18n^2+6n} \left(\frac{1}{\sqrt{6}}\right)^{18n^2-12n}$$

Proof: By definition, we have $SRII(G) = \prod_{uv\in E(G)} \frac{1}{\sqrt{r_G(U) + r_G(V)}}$

Thus

$$SRII(SL_n) = \left(\frac{1}{\sqrt{6+6}}\right)^{6n} \left(\frac{1}{\sqrt{6+3}}\right)^{18n^2+6n} \left(\frac{1}{\sqrt{3+3}}\right)^{18n^2-12n}$$

$$= \left(\frac{1}{\sqrt{12}}\right)^{6n} \left(\frac{1}{\sqrt{9}}\right)^{18n^2+6n} \left(\frac{1}{\sqrt{6}}\right)^{18n^2-12n}$$

In the following theorem, we compute the multiplicative atom bond connectivity Revan index of SL_n.

Theorem 3: The multiplicative atom bond connectivity Revan index of a Silicate network SL_n is given by

$$ABC_1RII(SL_n) = \left(\sqrt{\frac{5}{6}}\right)^{6n} \left(\sqrt{\frac{7}{18}}\right)^{18n^2+6n} \left(\sqrt{\frac{4}{9}}\right)^{18n^2-12n}$$

Proof: By definition, we have $ABC_1RII(G) = \prod_{uv\in E(G)} \sqrt{\frac{r_G(U)+r_G(V)-2}{r_G(U)\, r_G(V)}}$

Thus

$$ABC_1RII(SL_n) = \left(\sqrt{\frac{6+6-2}{6\times 6}}\right)^{6n} \left(\sqrt{\frac{6+3-2}{6\times 3}}\right)^{18n^2+6n} \left(\sqrt{\frac{3+3-2}{3\times 3}}\right)^{18n^2-12n}$$

$$= \left(\sqrt{\frac{5}{6}}\right)^{6n} \left(\sqrt{\frac{7}{18}}\right)^{18n^2+6n} \left(\sqrt{\frac{4}{9}}\right)^{18n^2-12n}$$

In the following theorem, we compute the multiplicative geometric-arithmetic Revan index of SL_n.

Theorem 4: The multiplicative geometric-arithmetic Revan index of a Silicate network SL_n is given by

$$GARII(SL_n) = \left(\frac{2\sqrt{18}}{9}\right)^{18n^2+6n}$$

Proof: By definition, we have $GARII(G) = \prod_{uv\in E(G)} \frac{2\sqrt{r_G(U)\, r_G(V)}}{r_G(U)+r_G(V)}$

Thus

$$GARII(SL_n) = \left(\frac{2\sqrt{6\times 6}}{6+6}\right)^{6n} \left(\frac{2\sqrt{6\times 3}}{6+3}\right)^{18n^2+6n} \left(\frac{2\sqrt{3\times 3}}{3+3}\right)^{18n^2-12n}$$

$$= \left(\frac{2\sqrt{18}}{9}\right)^{18n^2+6n}$$

3. RESULTS FOR HEXAGONAL NETWORKS

In this section we consider hexagonal network HX_n. From Figure 1, the vertices of HX_n are either of degree 3 or 6. By algebraic method, we obtain that $V(HX_n) = 3n^2 - 3n+1$ and $E(HX_n) = 9n^2-15n+6$.

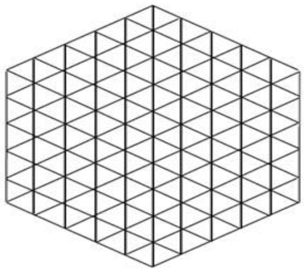

Figure-2: A 6-dimensional hexagonal network

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In HX_n, by algebraic method, there are five types of edges based on the degree of the end vertices of each edge as follows:

Edge Partition	**Number of Edges**
$E_{34} = \{u, v \in E(HX_n) / d_{HX_n}(u) = 3, d_{HX_n}(v) = 4\}$	$\lvert E_{34}\rvert = 12$
$E_{36} = \{u, v \in E(HX_n) / d_{HX_n}(u) = 3, d_{HX_n}(v) = 6\}$	$\lvert E_{36}\rvert = 6$
$E_{44} = \{u, v \in E(HX_n) / d_{HX_n}(u) = d_{HX_n}(v) = 4\}$	$\lvert E_{44}\rvert = 6n - 18$
$E_{46} = \{u, v \in E(HX_n) / d_{HX_n}(u) = 4, d_{HX_n}(v) = 6\}$	$\lvert E_{46}\rvert = 12n - 24$
$E_{66} = \{u, v \in E(HX_n) / d_{HX_n}(u) = d_{HX_n}(v) = 6\}$	$\lvert E_{66}\rvert = 9n^2 - 33n + 30$

Clearly $\Delta(G) = 6$ and $\delta(G) = 3$. Therefore $r_{HX_n}(u) = 9 - d_{HX_n}(u)$. Thus we ensure that there are three types of Revan edges based on the degree of end Revan vertices of each Revan edge as follows:

Revan Edge Partition	**Number of Edges**
$RE_{65} = \{u, v \in E(HX_n) / r_{HX_n}(u) = 6, r_{HX_n}(v) = 5\}$	$\lvert RE_{65}\rvert = 12$
$RE_{63} = \{u, v \in E(HX_n) / r_{HX_n}(u) = 6, r_{HX_n}(v) = 3\}$	$\lvert RE_{63}\rvert = 6$
$RE_{55} = \{u, v \in E(HX_n) / r_{HX_n}(u) = r_{HX_n}(v) = 5\}$	$\lvert RE_{55}\rvert = 6n - 18$
$RE_{53} = \{u, v \in E(HX_n) / r_{HX_n}(u) = 5, r_{HX_n}(v) = 3\}$	$\lvert RE_{53}\rvert = 12n - 24$
$RE_{33} = \{u, v \in E(HX_n) / r_{HX_n}(u) = r_{HX_n}(v) = 3\}$	$\lvert RE_{33}\rvert = 9n^2 - 33n + 30$

In the following theorem, we compute the multiplicative product connectivity Revan index of HX_n.

Theorem 5: The multiplicative product connectivity Revan index of a hexagonal network HX_n is given by

$$PRII(HX_n) = \left(\frac{1}{\sqrt{30}}\right)^{12} \left(\frac{1}{\sqrt{18}}\right)^{6} \left(\frac{1}{\sqrt{25}}\right)^{6n-18} \left(\frac{1}{\sqrt{15}}\right)^{12n-24} \left(\frac{1}{3}\right)^{9n^2-33n+30}$$

Proof: By definition, we have $PRII(G) = \prod_{uv \in E(G)} \frac{1}{\sqrt{r_G(U)\, r_G(V)}}$

Thus

$$PRII(HX_n) = \left(\frac{1}{\sqrt{6 \times 5}}\right)^{12} \left(\frac{1}{\sqrt{6 \times 3}}\right)^{6} \left(\frac{1}{\sqrt{5 \times 5}}\right)^{6n-18} \left(\frac{1}{\sqrt{5 \times 3}}\right)^{12n-24} \left(\frac{1}{\sqrt{3 \times 3}}\right)^{9n^2-33n+30}$$

$$= \left(\frac{1}{\sqrt{30}}\right)^{12} \left(\frac{1}{\sqrt{18}}\right)^{6} \left(\frac{1}{\sqrt{25}}\right)^{6n-18} \left(\frac{1}{\sqrt{15}}\right)^{12n-24} \left(\frac{1}{3}\right)^{9n^2-33n+30}$$

In the following theorem, we compute the multiplicative sum connectivity Revan index of HX_n.

Theorem 6: The multiplicative sum connectivity Revan index of a hexagonal network HX_n is given by

$$SRII(HX_n) = \left(\frac{1}{\sqrt{11}}\right)^{12} \left(\frac{1}{3}\right)^{6} \left(\frac{1}{\sqrt{10}}\right)^{6n-18} \left(\frac{1}{\sqrt{8}}\right)^{12n-24} \left(\frac{1}{\sqrt{6}}\right)^{9n^2-33n+30}$$

Proof: By definition, we have $SRII(G) = \prod_{uv \in E(G)} \frac{1}{\sqrt{r_G(U) + r_G(V)}}$

Thus

$$SRII(HX_n) = \left(\frac{1}{\sqrt{6+5}}\right)^{12} \left(\frac{1}{\sqrt{6+3}}\right)^{6} \left(\frac{1}{\sqrt{5+5}}\right)^{6n-18} \left(\frac{1}{\sqrt{5+3}}\right)^{12n-24} \left(\frac{1}{\sqrt{3+3}}\right)^{9n^2-33n+30}$$

$$= \left(\frac{1}{\sqrt{11}}\right)^{12} \left(\frac{1}{3}\right)^{6} \left(\frac{1}{\sqrt{10}}\right)^{6n-18} \left(\frac{1}{\sqrt{8}}\right)^{12n-24} \left(\frac{1}{\sqrt{6}}\right)^{9n^2-33n+30}$$

In the following theorem, we compute the multiplicative atom bond connectivity Revan index of HX_n.

Theorem 7: The multiplicative atom bond connectivity Revan index of a hexagonal network HX_n is given by

$$ABC_1RII(HX_n) = \left(\sqrt{\frac{9}{30}}\right)^{12} \left(\sqrt{\frac{7}{18}}\right)^{6} \left(\sqrt{\frac{8}{25}}\right)^{6n-18} \left(\sqrt{\frac{6}{8}}\right)^{12n-24} \left(\sqrt{\frac{4}{9}}\right)^{9n^2-33n+30}$$

Proof: By definition, we have $ABC_1RII(G) = \prod_{uv\in E(G)} \sqrt{\frac{r_G(U)+r_G(V)-2}{r_G(U)\,r_G(V)}}$

Thus $ABC_1RII(HX_n)$

$$= \left(\sqrt{\frac{6+5-2}{6\times 5}}\right)^{12} \left(\sqrt{\frac{6+3-2}{6\times 3}}\right)^{6} \left(\sqrt{\frac{5+5-2}{5\times 5}}\right)^{6n-18} \left(\sqrt{\frac{5+3-2}{5\times 3}}\right)^{12n-24} \left(\sqrt{\frac{3+3-2}{3\times 3}}\right)^{9n^2-33n+30}$$

$$= \left(\sqrt{\frac{9}{30}}\right)^{12} \left(\sqrt{\frac{7}{18}}\right)^{6} \left(\sqrt{\frac{8}{25}}\right)^{6n-18} \left(\sqrt{\frac{6}{8}}\right)^{12n-24} \left(\sqrt{\frac{4}{9}}\right)^{9n^2-33n+30}$$

In the following theorem, we compute the multiplicative geometric-arithmetic Revan index of HX_n.

Theorem 8: The multiplicative geometric-arithmetic Revan index of a hexagonal network HX_n is given by

$$GARII(HX_n) = \left(\frac{2\sqrt{30}}{11}\right)^{12} \left(\frac{2\sqrt{18}}{9}\right)^{6} \left(\frac{2\sqrt{15}}{8}\right)^{12n-24}$$

Proof: By definition, we have $GARII(G) = \prod_{uv\in E(G)} \frac{2\sqrt{r_G(U)\,r_G(V)}}{r_G(U)+r_G(V)}$

Thus

$$GARII(HX_n) = \left(\frac{2\sqrt{6\times 5}}{6+5}\right)^{12} \left(\frac{2\sqrt{6\times 3}}{6+3}\right)^{6} \left(\frac{2\sqrt{5\times 5}}{5+5}\right)^{6n-18} \left(\frac{2\sqrt{5\times 3}}{5+3}\right)^{12n-24} \left(\frac{2\sqrt{3\times 3}}{3+3}\right)^{9n^2-33n+30}$$

$$\left(\frac{2\sqrt{30}}{11}\right)^{12} \left(\frac{2\sqrt{18}}{9}\right)^{6} \left(\frac{2\sqrt{15}}{8}\right)^{12n-24}$$

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ON FUZZY STRONGLY GENERALISED b-CLOSED SETS IN FUZZY TOPOLOGICAL SPACES

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ABSTRACT
Present article introduces a new form of fuzzy generalized b-closed sets namely fuzzy strongly generalized b-closed sets in fuzzy topological spaces and investigate their characterizations. As an application to fuzzy strongly generalized b-closed sets, we introduce fuzzy spaces namely fuzzy sgb-$T_{1/2}$ and fuzzy sbg-$T_{1/2}$ spaces and their properties.*

Keywords and Phrases: Fuzzy generalized closed sets, fsgb-closed sets, fsgb-$\mathrm{T}_{1/2}$,fsbg-$\mathrm{T}_{1/2}$ spaces and fts.*

1. INTRODUCTION

After Zadeh [10] and Chang [5] introduced the concept of a fuzzy subset and fuzzy topological space(in short fts). In this article we define a new class of fuzzy generalized b-closed(in short fgb) sets namely, fuzzy strongly generalized b-closed(in short fsgb) sets and investigate their properties. fuzzy b-open(in short fb) sets, fuzzy b-generalized(in short fbg) closed sets , fgb sets, fuzzy b-$\mathrm{T}_{1/2}$ and fuzzy bg-$\mathrm{T}_{1/2}$ spaces have been introduced and investigated its properties by S.S.Benchalli and Jenifer J.K.

2. PRELIMINARIES

Throughout this paper X denotes the fuzzy topological spaces (fts. for short) (X,τ). For a fuzzy set A, the operators fuzzy closure and fuzzy interiors are denoted and defined by $\mathrm{Cl}(\alpha)=\wedge\{\beta:\beta\geq\alpha,1-\beta\in\tau\}$ and $\mathrm{Int}(\alpha)=\vee\{\beta:\beta\leq\alpha,\beta\in\tau\}$.

Definition 2.1 A fuzzy set *D* in a fts *X* is called

(i) Fuzzy semi-open[1] (in short fs-open)set if $D\leq\mathrm{Cl}(\mathrm{Int}D)$.

(ii) Fuzzy α-open[6] (in short fα-open)set if $D\leq\mathrm{Int}\,\mathrm{Cl}(\mathrm{Int}D)$.

(iii) Fuzzy pre-open[9] (in short fp-open)set if $D\leq\mathrm{Int}(\mathrm{Cl}D)$.

(iv) Fuzzy generalized open[2] (in short fg-open)set if $\mathrm{Int}D\leq E$, where *E* is fuzzy open set in *X*.

The complements of the above mentioned sets are their respective closed sets.

Theorem2.2 *In a fuzzy topological space X ,we have the following*

(i) every fuzzy open set is fuzzy α-open set[6].

(ii) every fuzzy α-open set is both fuzzy pre–open and fuzzy semi-open set[6].

Definition 2.3 A fuzzy set *D* in a fts *X* is called

(i) fb open set[3] iff $D\leq(\mathrm{Int}\ \mathrm{Cl}D)\vee(\mathrm{Cl}\ \mathrm{Int}D)$.

(ii) fb closed(in short set[3] iff $D\geq(\mathrm{Int}\ \mathrm{Cl}D)\wedge(\mathrm{Cl}\ \mathrm{Int}D)$.

Theorem 2.4 *For a fuzzy set D in a fuzzy topological space X*

(i) D is a fb open set iff $1-D$ is a fb closed set[3].

(ii) D is a fb closed set iff $1-D$ is a fb open set[3].

Definition2.5 Let D be a fuzzy set in a fts X . Then

(i) $\mathrm{bCl}D=\wedge\{E:E$ is a fb closed set of X and $\mathrm{E}\geq D\}$ [3].

(ii) $\mathrm{bInt}D=\vee\{E:E$ is a fb open set of X and $D\geq E\}$ [3].

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Lemma 2.6 *In a fts X ,we have the following*

(i) every fuzzy open set is fb open[3].

(ii) every fuzzy semi-open set is fb open set[3].

(ii) every fuzzy pre-open set is fb open set[3].

Definition 2.7 A fuzzy set *D* in a fts *X* is called fuzzy generalized b-closed[3] (fgb-closed) if $\mathrm{bCl}(D) \le E$, whenever $D \le E$ and *E* is fuzzy open.

Definition 2.8 A fuzzy set *D* in a fts *X* is called fuzzy b-generalized closed[4] (briefly fbg-closed) if $\mathrm{bCl}(D) \le E$, whenever $D \le E$ and *E* is fb open.

Lemma 2.9 *In a fuzzy topological space X ,we have the following*

(i) every fuzzy open set is fb open [3].

(ii) every fuzzy pre-open set is fb open[3].

(iii) every fuzzy semi-open set is fb open[3] .

(iv) every fb-closed set is fgb-closed [3].

(v) every fbg-closed set is fgb-closed [4].

3. FUZZY STRONGLY GENERALIZED B-CLOSED SETS AND FUZZY STRONGLY GENERALIZED B-OPEN SETS

In this section we define a new class of fuzzy generalized closed sets called a fsgb-closed sets and study its properties.

Definition 3.1 A fuzzy set *D* in a fts X is called a fsgb-closed set if $\mathrm{bCl}D \le E$, whenever $D \le E$ and E is fg-open set in X.

Definition 3.2 A fuzzy set D in a fts X is called a fsgb-open set if $\mathrm{bInt}D \ge E$, whenever $D \ge E$ and E is fg-open set in X.

The following theorem shows that the class of fsgb-closed sets contains the class of fb-closed sets.

Theorem 3.3 *Every fuzzy closed set in a fts X is fsgb-closed set.*

Proof : Let *D* be fuzzy closed set and *E* be any fg-open set such that $D \le E$. Since *D* is fuzzy closed, $Cl(D) = D$,also every fuzzy closed set is fb-closed set which implies that $fbCl(D) = D$ then $Cl(D) = fbCl(D) = D \le E$. It follows that $fbCl(D) \le E$ .Hence *E* is fsgb-closed set.

Theorem 3.4 *Every fb-closed set in a fts X is fsgb-closed.*

Proof: Let *D* be a fb-closed set in a fts *X*. Suppose that $D \le E$ and *E* is a fg-open set in *X*. Since $\mathrm{bCl}D = D$, it follows that $\mathrm{bCl}D = D \le E$. Hence D is fsgb-closed.

The converse of the above theorem is not true which is shown in the following example.

Example 3.5 For a fts *X*, let $X = \{a,b,c\}$ and $\tau = \{0,1,A,B\}$, where $A = \{(a,.6),(b,.4),(c,1)\}$, $B = \{(a,.6),(b,0),(c,0)\}$.

Fuzzy closed sets of X are 0, 1, $\overline{A} = \{(a,.4),(b,.6),(c,0)\}$, $\overline{B} = \{(a,.4),(b,1),(c,1)\}$.

Family of fg-closed sets in X are $\{0,1,\overline{A},\overline{B},(a,\alpha),(b,\beta),(c,\gamma)\}$, for $\alpha > 0.6$ or $\beta > 0.4$.

Family of fg-open sets in X are $\{0,1,A,B,(a,\alpha),(b,\beta),(c,\gamma)\}$, for $\alpha < 0.4$ or $\beta < 0.6$.

$\mathrm{bO}(X) = C = \{(a,1),(b,0.4),(c,0)\}$ is not a fb-closed set in X, but C is fsgb-closed in X.

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Theorem 3.6 *Every fsgb-closed set in a fts X is fgb-closed.*

Proof Let *D* be a fsgb-closed set in a fts (X, τ). Suppose that $D \le E$ and *E* is a fuzzy -open set in X .Then $\mathrm{bCl}D \le E$ and E is fg-open in X . Hence *D* is fgb-closed.

Remark 3.7 Union of two fsgb-closed sets in a fts need not be fsgb-closed set.

Example 3.8 For a fts *X*, let $X = \{a,b,c\}$ and $\tau = \{0,1,A\}$, Fuzzy closed sets of X are 0,1, $A = \{(a,1),(b,.5),(c,0)\}$,.

Family of fg-closed sets in *X* are $\{0,1,\overline{A},(a,\alpha),(b,\beta),(c,\gamma)\}$, for $\alpha > 0.6$ or $\beta > 0.4$.

Family of fg-open sets in *X* are $\{0,1,A,(a,\alpha),(b,\beta),(c,\gamma)\}$, for $\alpha < 0.4$ or $\beta < 0.6$.

Let $C = \{(a,1),(b,0),(c,0)\}$ and $B = \{(a,0),(b,0.5),(c,0)\}$be fuzzy sets in X . Now C and B are fuzzy b-closed set in X and hence by theorem C and B are fsgb-closed set in X .But $C \vee B = \{(a,1),(b,0.5),(c,0)\}$ is not fsgb-closed set in X .Since $bCl(C \vee B) = 1 > C \vee B$ and $C \vee B$ is fg-open .

Theorem 3.9 *In a fts X for a fuzzy D, if D is fg-open and fsgb-closed, then D is fb-closed.*

Proof Let *D* be fg-open and $D \le D$. Then, it follows that $D \vee ((\mathrm{IntCl}(D)) \wedge (\mathrm{ClInt}(D))) \le \mathrm{bCl}(D) \le D$ .Since *D* is fsgb-closed. This implies $(\mathrm{IntCl}(D)) \wedge (\mathrm{ClInt}(D)) \le D$. Hence *D* is fb-closed.

Theorem 3.10 *Let D be a fsgb-closed set in fts X. If E is a fuzzy set in X such that $D \le E \le \mathrm{bCl}(D)$, then E is fsgb-closed .*

Proof : Let $E \le F$ and *F* be fg-open in *X*.Then $D \le F$. Since D is a fsgb-closed set in *X*, it follows from Lemma $\mathrm{bCl}E \le \mathrm{bCl}(\mathrm{bCl}(D)) = \mathrm{bCl}(D) \le F$. Hence E is fsgb-closed .

Definition 3.11 A fuzzy set D in a fts (X,τ) is called a fsgb-open set in X if $1-D$ is fsgb-closed.

Theorem 3.12 *Let D be a fsgb-closed set in fts (X,τ).If E is a fuzzy set in X such that $\mathrm{bCl}(D) \ge E \ge D$, then E is fsgb-closed set.*

Proof : $1-D$ is fsgb-open set in fts X and E is a fuzzy set in X such that $1-D \ge 1-E \ge 1-\mathrm{bCl}(D) = \mathrm{bInt}(1-D)$.That implies $1-E$ is fsgb-open and so *E* is fsgb-closed set.

Theorem 3.13 *Every fp-closed set is fsgb-closed set.*

Proof : Let *D* be fp-closed set and *E* be any fg-open set , such that $D \le E$. Since every fp-closed set is fb-closed, *D* is fb-closed set , then it follows that $\mathrm{bCl}(D) = D \le E$ .Therefore *D* is fsgb-closed set.

The converse of the above theorem need not be true , as it can be seen from the following example.

Example 3.14 Let $X = \{a,b\}$ and the functions $A, B, C : X \to [0,1]$ be defined as

$A(a) = 0.2$, $A(b) = 0.4$, $B(a) = 0.5$, $B(b) = 0.6$, $C(a) = 0.3$, $C(b) = 0.5$ .Let $\tau = \{0, A, 1\}$

Then C is fsgb-closed set but C is not fp-closed set.

Theorem 3.15 *Every fα-closed set is fsgb-closed set.*

Proof : Let *D* be fα-closed set and *E* be any fg-closed set , such that $D \le E$. Since every fα-closed set is fp-closed set, then by Theorem 3.13 , *D* is fsgb-closed set. Therefore , *D* is fsgb-closed set.

Theorem 3.16 *Every fs-closed set is fsgb-closed set.*

Proof : Let *D* be fs-closed set and *E* be any fg-closed set , such that $D \le E$ .Since every fs-closed set is fb-closed set, *D* is fb-closed set , then it follows that $\mathrm{bCl}(D) = D \le E$ .Therefore *D* is fsgb-closed set.

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Let $X=\{\text{a,b,c}\}$ and the functions $A,B,C:X\to[0,1]$ be defined as

$A(a)=0.6$, $A(b)=0.7$, $A(c)=0.9$, $B(a)=0.4$, $B(b)=0.5$, $B(c)=0.8$, $C(a)=0.0$, $C(b)=0.1$, $C(c)=0.2$ .Let $\tau=\{0,A,1\}$

Then B is fsgb-closed set but B is not fp-closed set.

Theorem 3.17 A fuzzy set *C* of (X,τ) is fsgb-closed iff $C\bar{q}D$ which imples $\text{bCl}(C)\le D$,for every fuzzy generalized closed set *D* of (X,τ).

Proof : Let *C* be fuzzy sgb-closed set of (X,τ) such that $C\bar{q}D$.Then from the definition , it implies $C\le 1-D$ and $1-D$ is fg-open set of (X,τ).Hence $\text{bCl}(C)\le \text{bCl}(1-D)\le 1-D$ as *C* is fsgb-closed set. Therefore $\text{bCl}(C)\bar{q}D$.

Conversely , let *E* be fg-open set of (X,τ) such that $C\le E$. Then by definition , $C\bar{q}(1-E)$ and $(1-E)$ is fg-closed set of (X,τ). By hypothesis , $\text{bCl}(C)\bar{q}1-E$, which implies $\text{bCl}(C)\le E$. Hence *C* is fsgb-closed set .

Theorem 3.18 *Let C be fsgb-closed set in fts X and* x_p *be a fuzzy point in fts X such that fuzzy point* $x_p q\text{fbCl}(C)$ *then* $\text{fbCl}(x_p)qC$.

Proof : Let *C* be fuzzy sgb-closed set in fts *X* and x_p be a fuzzy point in fts *X* .Suppose that $\text{bCl}(x_p)\bar{q}C$, then by definition $\text{bCl}(x_p)\le 1-A$ which implies $\text{A}\le 1-\text{bCl}(x_p)$.So that $\text{bCl}(A)\le 1-\text{bCl}(x_p)\le 1-x_p$. Since $\text{bCl}(x_p)$is fg-open in *X* and *C* is fsgb-closed in *X*. Hence $x_p\,\bar{q}\,\text{bCl}(C)$ which is a contradiction .Therefore $\text{bCl}(x_p)qC$.

From the above results and discussions we have the following implications.

fuzzy pre-open(closed) ↑ ↘
f-open(closed) → fuzzy α-open(closed) fb-open(closed) → fsgb-open(closed)
fuzzy semi-open(closed) ↓ ↗

4. FUZZY $sgbT_{1/2}$-SPACES AND FUZZY $sbg*-T_{1/2}$ SPACES

Definition 4.1 A fuzzy topological space (X,τ) is called a fuzzy $sgbT_{1/2}$-spaces if every fsgb-closed set in it is fuzzy b-closed.

Theorem 4.2 *A fts X is* $fsgbT_{1/2}$ space if and only if every fuzzy set in *X* is fsgb-open and fb-open *set.*

Proof Let (X,τ)be a $fsgbT_{1/2}$ space and let *D* be a fsgb-open set in X . Then $1-D$ is fsgb-closed set.By hypothesis, every fsgb-closed set is fb-closed set.Therefore $1-D$is fb-closed set .Hence D is fb-open set in X .

Conversely, let Dbe fsgb-closed set. Then $1-D$ fsgb-open set which implies $1-D$is fb-open set. Therefore D isfb-closed set .Every fsgb-closed set in X is fb-closed set .Hence X is $fsgbT_{1/2}$ space.

Definition 4.4 A fuzzy topological space (X,τ) is called a fuzzy $sbg*-T_{1/2}$ spaces if every fuzzy bg-closed set in it is fsbg-closed.

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Theorem 4.5 *Every fuzzy* $bT_{1/2}$ *space is fuzzy* $sbg*-T_{1/2}$ *space.*

Proof Let (X,τ) be a fuzzy $bT_{1/2}$ space and let D be a fbg-closed set in X .Then D is a fb-closed and hence by theorem 3.4 , D is fsgb-closed set in X and hence it is fuzzy sgb-closed in (fgb-closed set in a fts (X,τ) is fsgb-closed) X . Thus X is fuzzy $sbg*-T_{1/2}$ spaces.

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SIMPLE ITERATIVE TECHNIQUE FOR SOLVING THE MODIFIED BURGERS' EQUATION USING HAAR WAVELETS

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ABSTRACT
In this paper, we present a simple iterative technique for solving the nonlinear modified Burgers' equation (MBE) with the help of Haar wavelets method. Haar wavelets with the aid of collocation method have become very useful in providing highly accurate solution to the MBE. Numerical example is included to demonstrate the validity and applicability of the present technique. The accuracy and efficiency of the proposed method is discussed by computing L_2 *and* L_∞ *error norms.*

Keywords: Collocation method, Haar wavelets, Modified Burgers' equation.

1. INTRODUCTION

The modified Burgers' equation (MBE) finds its applications in many fields such as gas dynamics, heat conduction, fluid dynamics, etc. The MBE is of the following form

$$\frac{\partial y}{\partial t}+y^g\frac{\partial y}{\partial x}=v\frac{\partial^2 y}{\partial x^2},\ a\le x<d, t\ge t_0, \qquad (1.1)$$

subject to the initial condition

$$y(x,t_0)=f(x),\ a\le x<d,$$

and the boundary conditions

$$y(a,t)=q_1,\ y(b,t)=q_2,\ t\ge t_0.$$

Where $y(x,t)$ represents the velocity for spatial dimension $'x'$ and time $'t'$, $'v'$ is a positive parameter which may be interpreted as the kinematic viscosity of the fluid, $'g'$ is a positive integer, q_1 and q_2 are constants. For $g=1$ the equation (1.1) corresponds to the Burgers' equation (BE) which was initially introduced by Bateman, H. [2] and later look upon by Burgers, J. M. [4]. For $g\ge 2$ the equation (1.1) is called the MBE. The MBE has the strong nonlinear form and it is used in many practical transport problems such as those of nonlinear waves in a medium with low-frequency pumping or absorption, transport and dispersion of pollutants in rivers and sediment transport, etc. Finding a solution for the MBE plays an essential aspect in different physical phenomena. In few cases we do not always find the exact solution for the MBE by analytical method. For example analytical solutions for the BE were found for restricted values of $'\nu'$ [6,9]. To overcome from these drawbacks various numerical methods are developed to find the numerical solution for the MBE like Least –square B-spline finite element method[10], Quintic B-spline collocation method [17], Septic B-spline collocation method [15], Petrov-Galerkin method [16], El-Gendi method [19], B-spline differential quadrature method [3], etc.

Wavelet analysis is a very fruitful and an effective technique to obtain the numerical solution of various types of partial differential equations. Many authors like Lepik, U. [11], Hariharan, G. et al. [7], Abd-Elhameed, W.M.et al. [1], Bujurke, N. M.et al. [5] are worked on wavelets to find the numerical solution of partial differential equations using different types of wavelets. Among the wavelet family special attention desired by Haar wavelet because of its mathematical simplicity. The main aim of this paper is to applying the simple iterative technique for finding the numerical solution of the MBE using Haar wavelet collocation method (HWCM). Haar wavelets based collocation method will give high precision numerical solution to the MBE.

The layout of this paper is organized as follows. In section 2, Haar wavelets and their integrals are introduced. In section 3, we applied Haar wavelet method to the MBE. In section 4, the introduced method is applied to numerical example. Finally conclusion has been written in section 5.

2. HAAR WAVELETS AND THEIR INTEGRALS

Alfred Haar initially introduced the Haar wavelet in 1910.The Haar wavelet method has been propitiously used in many applications such as time-frequency analysis, nonlinear approximation and solving different types of

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ordinary differential equations and partial differential equations by various authors [7, 13, 18]. The Haar wavelet family $h_j(x)$ is defined as follows.

$$h_j(x)=\begin{cases} 1, & \text{for } x\in[\alpha(j),\beta(j)), \\ -1, & \text{for } x\in[\beta(j),\gamma(j)), \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

Where $\alpha(j)=\frac{r}{m}$, $\beta(j)=\frac{r+0.5}{m}$ and $\gamma(j)=\frac{r+1}{m}$, $m=2^i$ and $i=0,1,2,...,J$. $'J'$ indicates the maximal level of resolution. The integer $r=0,1,2,...,m-1$ is the translation parameter. The index $'j'$ is evaluated as: $j=m+r+1$. The minimal value of $'j'$ is 2. The maximal value of $'j'$ is 2^{i+1}. $h_j(x)$ is true for $j>2$. When $j=1,2$. $h_1(x)$ and $h_2(x)$ are called father and mother wavelet respectively [11]. It is assumed that the interval $[a,d]$ will be divided into $2L$ subintervals, hence $dx=\frac{d-a}{2L}$. Where $L=2^J$. Let us define the collocation points $x_l=\frac{l-0.5}{2L}$, where $l=1,2,....,2L$.

We introduce the following notations are as follows

$$P_{1,j}(x)=\int_a^x h_j(x)\,dx, \tag{2.2}$$

$$P_{2,j}(x)=\int_a^x P_{1,j}(x)\,dx,$$

....

Integrating the Haar functions for $'\lambda'$ times. We get

$$P_{\lambda,j}(x)=\int_a^x\int_a^x....\int_a^x h_j(s)\,ds^{\lambda},$$

$$P_{\lambda,j}(x)=\frac{1}{(\lambda-1)!}\int_a^x (x-s)^{\lambda-1}\,h_j(s)\,ds. \tag{2.3}$$

All these integrals are evaluated directly as discussed by Lepik, U.[12]. For $j=1$ the equation (2.3) becomes

$$P_{\lambda,1}(x)=\frac{1}{(\lambda)!}(x-a)^{\lambda}. \tag{2.4}$$

For $j\geq 2$ from the equation (2.3), we get

$$P_{\lambda,j}(x)=\frac{1}{\lambda!}\begin{cases} 0, & \text{if } x\in[a,\ \alpha(j)), \\ [x-\alpha(j)]^{\lambda}, & \text{if } x\in[\alpha(j),\beta(j)), \\ \left[[x-\alpha(j)]^{\lambda}-2[x-\beta(j)]^{\lambda}\right], & \text{if } x\in[\beta(j),\gamma(j)), \\ \left[[x-\alpha(j)]^{\lambda}-2[x-\beta(j)]^{\lambda}+[x-\gamma(j)]^{\lambda}\right], & \text{if } x\in[\gamma(j),\ d). \end{cases} \tag{2.5}$$

3. METHOD OF SOLUTION

Consider the MBE (1.1) for $g = 2$ with the following conditions as follows

$$\frac{\partial y}{\partial t}+y^2\frac{\partial y}{\partial x}=v\frac{\partial^2 y}{\partial x^2},\ 0\le x<1, t\ge t_0, \tag{3.1}$$

subject to the initial condition

$$y(x,t_0)=f(x),\ 0\le x<1, \tag{3.2}$$

and the boundary conditions

$$y(0,t)=0,\ y(1,t)=0,\ t\ge t_0. \tag{3.3}$$

To find the solution of the MBE (3.1). It is assumed that $\dot{y}''(x,t)$ can be expanded in terms of Haar wavelets as follows

$$\dot{y}''(x,t)=\sum_{j=1}^{2L}a_j h_j(x). \tag{3.4}$$

Where $'\cdot'$ and $'\,''\,'$ means differentiation with respect to $'t'$ and $'x'$ respectively. Where a_j's are Haar wavelet coefficients in the interval $t\in[t_n,t_{n+1}]$ and $h_j(x)$ is Haar wavelet family.

Integrating equation (3.4) w.r.t. to t from t_n to t and twice w.r.t. x from 0 to x, following equations are obtained

$$y''(x,t)=y''(x,t_n)+(t-t_n)\sum_{j=1}^{2L}a_j(t_n)h_j(x), \tag{3.5}$$

$$y'(x,t)=y'(0,t)+y'(x,t_n)-y'(0,t_n)+(t-t_n)\sum_{j=1}^{2L}a_j(t_n)P_{1,j}(x), \tag{3.6}$$

$$y(x,t)=(t-t_n)\sum_{j=1}^{2L}a_j(t_n)P_{2,j}(x)+y(x,t_n)-y(0,t_n)+x\left[y'(0,t)-y'(0,t_n)\right]+y(0,t), \tag{3.7}$$

$$\dot{y}(x,t)=\sum_{j=1}^{2L}a_j(t_n)P_{2,j}(x)+x\,\dot{y}'(0,t)+\dot{y}(0,t). \tag{3.8}$$

By using boundary conditions from the equation (3.3). We obtain the following equations as follows

$$\dot{y}'(0,t)=-\sum_{j=1}^{2L}a_j(t_n)P_{2,j}(1), \tag{3.9}$$

$$\left[y'(0,t)-y'(0,t_n)\right]=-(t-t_n)\sum_{j=1}^{2L}a_j(t_n)P_{2,j}(1). \tag{3.10}$$

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Substituting the equations (3.9) and (3.10) in (3.5) to (3.8) and discritizing these results $x \to x_l$ and $t \to t_{n+1}$, we obtain the following equations

$$y''(x_l, t_{n+1}) = y''(x_l, t_n) + (t_{n+1} - t_n) \sum_{j=1}^{2L} a_j(t_n) h_j(x_l), \tag{3.11}$$

$$y'(x_l, t_{n+1}) = y'(x_l, t_n) + (t_{n+1} - t_n) \sum_{j=1}^{2L} a_j(t_n) \left[P_{1,j}(x_l) - P_{2,j}(1) \right], \tag{3.12}$$

$$y(x_l, t_{n+1}) = (t_{n+1} - t_n) \sum_{j=1}^{2L} a_j(t_n) \left[P_{2,j}(x_l) - x_l P_{2,j}(1) \right] + y(x_l, t_n), \tag{3.13}$$

$$\dot{y}(x_l, t_{n+1}) = \sum_{j=1}^{2L} a_j(t_n) \left[P_{2,j}(x_l) - x_l P_{2,j}(1) \right]. \tag{3.14}$$

There are many possibilities for solving the nonlinearity in equation (3.1). Here we followed the simple iterative technique as follows.

$$\frac{\partial}{\partial t} y(x_l, t_{n+1}) = -\left[y(x_l, t_n) \right]^2 \frac{\partial}{\partial x} y(x_l, t_n) + v \frac{\partial^2}{\partial x^2} y(x_l, t_n). \tag{3.15}$$

After substituting the equations (3.11) to (3.14) in (3.15). Let us divide the time interval $[t_0, T]$ into 'E' subinterval of equal length $dt = \frac{[T - t_0]}{E}$. Initially we start the iteration by assuming $t_n = t_0$ and $t_{n+1} = t_n + dt$ for $n = 1, 2, ..., E$. After successful iteration in the time interval $t \in [t_0, T]$, we will get the Haar wavelet coefficients a_j's. The Haar wavelet based numerical solution to the MBE (3.1) is obtained by substituting the obtained coefficients a_j's in the equation (3.13).

4. NUMERICAL STUDIES AND DISCUSSION

In this section, we apply the Haar wavelet collocation method (HWCM) to the MBE (3.1) whose analytical solution is known. All numerical computations are carried out by using Matlab software. The errors for the MBE are studied for different values of (kinematic viscosity parameter) and time using two different norms, namely L_2 and L_∞ defined by

$$L_\infty = \max_{1 \le p \le N} \left| y(p) - y_{ex}(p) \right|, \quad L_2 = \sqrt{h \sum_{p=1}^{N} \left| y(p) - y_{ex}(p) \right|^2}.$$

Where $N =$ number of nodes, $h = \frac{(d-a)}{N}$, $y(p) =$ approximate solution, $y_{ex}(p) =$ exact solution at $x = x_p$ for $p = 1, 2, .., N$ respectively.

Example 1. Consider the MBE (3.1) with the following conditions [14],

$$\frac{\partial y}{\partial t} + y^2 \frac{\partial y}{\partial x} = v \frac{\partial^2 y}{\partial x^2}, \; 0 \le x < 1, t \ge 1, \tag{3.1}$$

subject to the initial condition

$$y(x,1)=\frac{x}{1+\left(\frac{1}{c_0}\right)e^{\left(\frac{x^2}{4v}\right)}},0\le x<1,$$

and the boundary conditions

$$y(0,t)=0,\ y(1,t)=0,\ t\ge 1.$$

Where $c_0=0.5$ is a constant.

The analytical solution of the MBE [8] is given by

$$y(x,t)=\frac{\left(\frac{x}{t}\right)}{1+\left(\frac{\sqrt{t}}{c_0}\right)e^{\left(\frac{x^2}{4tv}\right)}},0\le x<1,t\ge 1.$$

The acquired results are outlined in the following Tables and Figures.

In **Table1** and **Table2** the analogy of analytical solution and approximate solution for dt = 0.01, v = 0.01, J = 4 at t = 6 and dt = 0.01, v = 0.005, J = 5 at t = 2 are inserted along with their absolute errors respectively. In **Figure1** and **Figure2** the behavior of the numerical solution of the MBE with analytical solution of the MBE for dt = 0.01, v = 0.01, J = 4 and dt = 0.01, v = 0.005, J = 5 at t = 2 are plotted respectively. It can be recognized from the plots that the analytical solution and approximate solution are in close vicinity.

Table-1: Comparison of the values of analytical, Haar solutions with their absolute errors of example 1 for dt = 0.01, v = 0.01 and J = 4 at t = 6.

X_l / 32	Haar solution	Analytical solution	Absolute error $\times 10^{-3}$
1	0.000465709367680	0.000441087715958	0.024621651721561
3	0.002266405563763	0.002161075086492	0.105330477271350
5	0.003821472071579	0.003708617927754	0.112854143825123
7	0.004974107410660	0.004965469317296	0.008638093363539
9	0.005679947952778	0.005845661912896	0.165713960118093
21	0.003393595218808	0.003801525446118	0.407930227310694
23	0.002618070001499	0.002971676896464	0.353606894965184
25	0.001915989984233	0.002225295475822	0.309305491588336
27	0.001305865677411	0.001598816496520	0.292950819108731
29	0.000782590186048	0.001103647376934	0.321057190885899
31	0.000323156821560	0.000732815676823	0.409658855263038

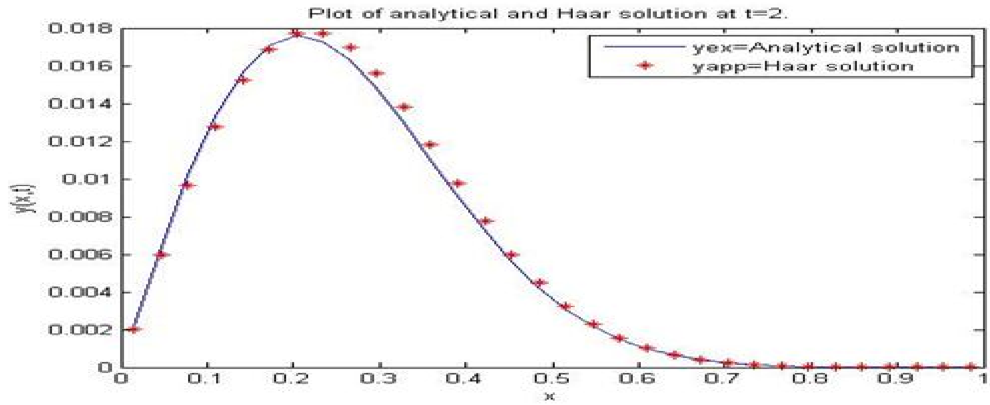

Figure-1: Comparison of analytical and Haar solution of example 1 for v = 0.01, dt = 0.01 and J = 4 at t = 2.

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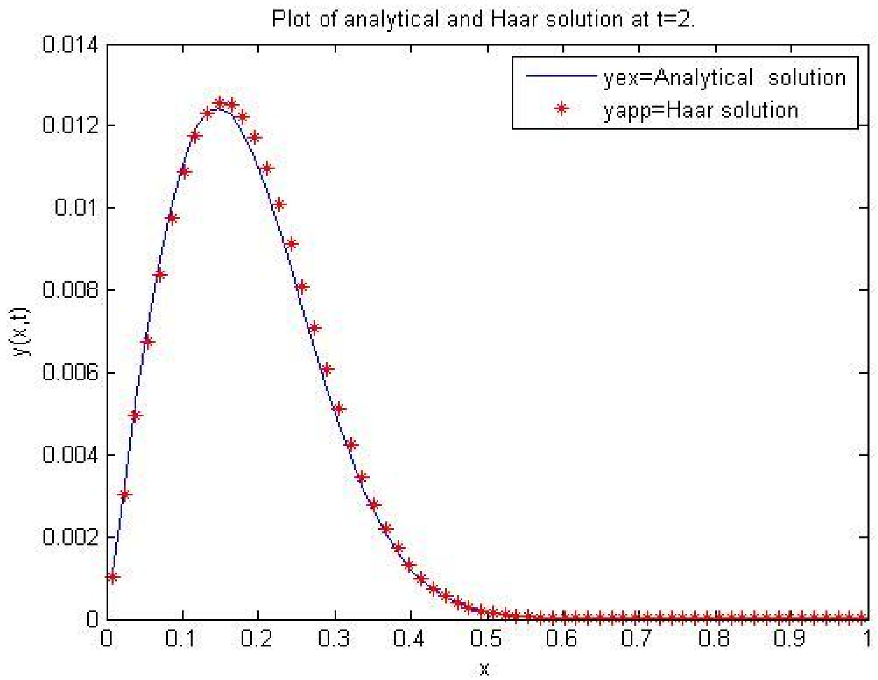

Figure-2: Comparison of analytical and Haar solution of example 1 for v = 0.005, dt = 0.01 and J = 5 at t = 2.

Table-2: Comparison of the values of analytical, Haar solutions with their absolute errors of example 1 for dt = 0.01, v = 0.005 and J = 5 at t = 2.

X_l/64	Haar solution	Analytical solution	Absolute error $\times 10^{-3}$
1	0.001091043246909	0.001019177826720	0.071865420188374
3	0.005274721548915	0.004959174579077	0.315546969838208
5	0.008775876581454	0.008369476114731	0.406400466722547
7	0.011195003611767	0.010896141431058	0.298862180709224
9	0.012345090011275	0.012306522176616	0.038567834659149
25	0.001622609789967	0.001719473505142	0.096863715174887
27	0.000952128014561	0.001002105232636	0.049977218075157
29	0.000527221782917	0.000552029053017	0.024807270099816
31	0.000275592267681	0.000287862227460	0.012269959779524
49	0.000000061155244	0.000000077951260	0.000016796016132
51	0.000000017993304	0.000000024239835	0.000006246531115
53	0.000000004989792	0.000000007167185	0.000002177393770
55	0.000000001303331	0.000000002015260	0.000000711928519
57	0.000000000320402	0.000000000538918	0.000000218515660
59	0.000000000074062	0.000000000137077	0.000000063014688
61	0.000000000016041	0.000000000033166	0.000000017125043
63	0.000000000003012	0.000000000007634	0.000000004621849

Table-3: The L_2 and L_∞ error norms for different values of v with J = 4 and dt = 0.01.

v	$L_\infty \times 10^{-4}$ t = 2	$L_2 \times 10^{-4}$ t = 2	$L_\infty \times 10^{-4}$ t = 4	$L_2 \times 10^{-4}$ t = 4	$L_\infty \times 10^{-4}$ t = 6	$L_2 \times 10^{-4}$ t = 6	$L_\infty \times 10^{-4}$ t = 10	$L_2 \times 10^{-4}$ t = 10
0.015	10.07758	5.11593	8.81855	5.34678	19.778696	8.58100	28.15810	13.21760
0.01	8.403971	3.83983	6.25117	3.31656	4.81811	3.33852	12.21814	5.46523
0.002	4.35176	1.34249	3.49141	1.26385	2.63102	1.13111	2.25273	0.97903

In **Table3**, we tested the suggested technique for different values of v = 0.015, v = 0.01, v = 0.002 at t = 2, 4,6,10 using time step dt =0.01 for J = 4. The L_2 and L_∞ error norms are found to be decreases as 'v' decreases. The errors are satisfactorily tolerable for all values of 'v'.

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5. CONCLUSION

In this paper a simple iterative technique for solving the nonlinear MBE is proposed using HWCM. From the example we proclaim that proposed method is more accurate, simple, fast, computationally efficient and flexible. The proposed method can securely and briskly be used for the solution of a broad number of alike problems. Finally, we culminate that HWCM is competent of solving the nonlinear MBE for small values of viscosity parameters.

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ESTIMATION OF NOISE LEVEL IN NOISY SIGNAL BASED ON STATISTICS AND DISCRETE WAVELET TRANSFORM

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ABSTRACT

Noise level estimation is an important process in signal processing. For example, the performance of signal denoising algorithm can be significantly degraded because of poor noise level estimation. Most of the literature on the subject tends to use the true noise level of a noisy signal when suppressing noise artifacts. In this paper, a new, effective noise level estimation method is proposed based on the study of statistics and discrete wavelet transform. The performance of the proposed method is compared with different type of discrete wavelet transform.

Keywords: Noise level ectimation, Discrete wavelet transform, MAD, Absolute error approximation.

1 INTRODUCTION

Digital signal processing is extensively used in many areas of electronics, communication and information techniques [1, 2]. Noise is inevitable during the process of transmission of signal, storage, processing. It is degrade the information of signal. Therefore, estimation of noise level is an important mission of signal and image processing [3], since the noise level is not always known beforehand, but many signal denoising algorithms [4, 5] take it as an input parameter, and their performance depends heavily on the accuracy of the noise level.

Wavelets have essentially been imposed as a noise level estimation in noisy signal. Discrete wavelet transforms enable us to represent signals with a high degree of sparsely and exciting new tool for statistical signal processing. The remarkable properties of the wavelet transform have led to powerful signal processing methods based on simple scalar transformations of individual wavelet coefficients. These methods implicitly treat each wavelet coefficient as though it were independent of all others. Methods that exploit dependencies between wavelet coefficients should perform even better. Hample [6] shows that basic study of statistics to calculate approximately noise deviation of noisy signal data.

In filters- based method [7], a noisy signal is first filtered by a low-pass filter to suppress the noise. Then the noisy variance is computed from the difference between the noisy image and filtered image. This method is to estimate noise level with minimum influence of a image signal and it require high computational load [8, 9]. This paper help to analysis of noisy signal i.e., we do not know $v(t)$ or $n(t)$, but we are given $y(t)$ and our task is to estimate $v(t)$. One of the steps in this process is to estimate the noise level σ that occurs in $n(t)$. These processes help to reach the main task.

Paper is systematized as follows: we will first study noise model with respect to statistics (as Sections 2 and 3) and signal with noise model in section 4. In the sections 5 and 6 is related to wavelet domain and expand noise estimation algorithm. Finally, sections 7 and 8 provides simulation results and conclusion respectively.

2 DESCRIPTIVE STATISTICS [10, 11]

2.1 Median

Let $x=(x_1,x_2,...,x_n)$ be a list numbers. The median x_{med} is constructed by first sorting the values $x_1,x_2,...,x_n$ from smallest to largest. Then to get $u=(u_1,u_2,...,u_n)$,

$$x_{med}=\begin{cases} u_{\frac{n+1}{2}} & \text{if } n \text{ is odd,} \\ u_{n/2}+u_{n/2+1} & \text{if } n \text{ is even.} \end{cases}$$

2.2 Variance and Standard Deviation

Suppose that the list of numbers $x=(x_1,x_2,...,x_n)$ has mean $\bar{x}$ or μ. Variance of this data set is defined by

$$\sigma^2=\frac{1}{n-1}\sum_{k=1}^{n}\left(x_k-\bar{x}\right)^2.$$

The standard deviation (σ) is defined to be the nonnegative square root of the variance.

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2.3 Mean Absolute Deviation (MAD)

Let $x = (x_1, x_2, ..., x_N)$ and x_{med} denote the median of x. Form the vector $z = (|x_1 - x_{med}|, ..., |x_N - x_{med}|)$. We define the median absolute deviation of x as: $MAD(x) = x_{med}$.

3 ADDITIVE WHITE GAUSSIAN NOISE (AWGN) [7]

It is a basic noise model used in Information theory to mimic the effect of many random processes that occur in nature. The amplitude of noise is of Gaussian distribution:

$$n(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

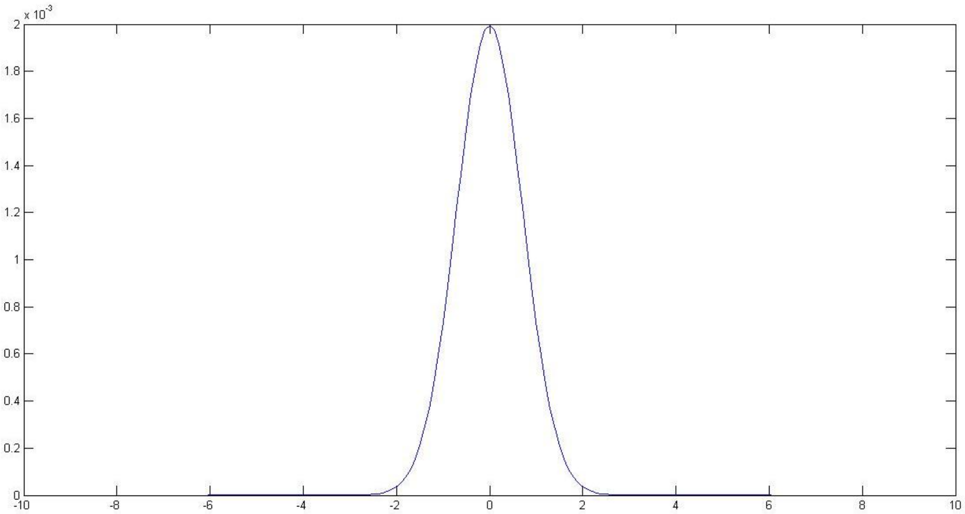

Fig-1: Normal distrubtion curve of Guassian pulse.

Fig. 1 shows normal distribution has a bell-shaped density curve desired by its $\mu = 0$ and σ. Then density curve is symmetrical, centered about its mean, with its widen determined by its standard deviation and most cases, noise can be modeled as Gaussian distribution.

4 SIGNAL WITH NOISE MODEL [2,12]

The function (signal) is defined as

$g(t) = 4\sin(4\pi t) - \text{sgn}(t - 0.3) - \text{sgn}(0.72 - t)$ for $t \in [0,1]$.Here the sign function defined by

$$\text{sgn} = \begin{Bmatrix} 1, & t > 0, \\ 0, & t = 0, \\ -1, & t < 0 \end{Bmatrix}.$$

We form $v \in \mathbb{R}^N$ using the $v_k = g(k/N), k = 1, 2, ..., N-1$. Where N is the signal length $(N = 2048)$. An observed noisy signal is expressed noise model as:

$$y(t) = v(t) + n(t) \qquad (1)$$

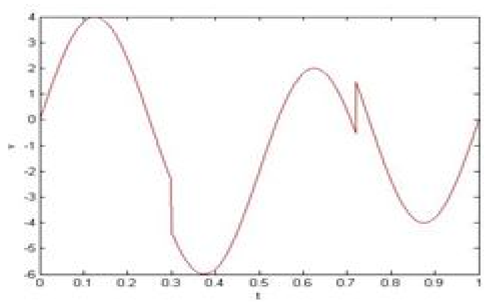

Fig-2: True signal $v(t)$

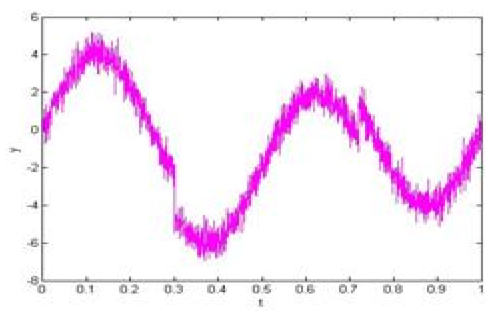

Fig-3: Noisy signal $y(t)$

Where $y(t)$ is received signal, $v(t)$ is represents the transmitted signal, and $n(t)$ is composed of independent samples from a normal distribution with mean zero (i.e., $\mu = 0$) and standard deviation (σ). Such noise is called Gaussian white noise.

5 WAVELET AND MULTIRESOLUTION ANALYSIS

Defintion1: (The space $L^2(\mathbb{R})$) We define the space $L^2(\mathbb{R})$ to be the set

$$L^2(\mathbb{R}) = \left\{ f:\mathbb{R} \to \mathbb{C} \mid \int_{\mathbb{R}} |f(t)|^2 \right\}.$$

$\square$

Defintion 2:(The $L^2(\mathbb{R})$ Norm) Let $f(t) \in L^2(\mathbb{R})$. Then the norm of $f(t)$ is

$$\|f(t)\| = \left(\int_{\mathbb{R}} |f(t)|^2 dt \right)^{\frac{1}{2}}.$$

$\square$

The norm of the function is also referred to as the energy of the function or signal.

5.1 The Haar Space V_j

Let $\varphi(t)$ be the Haar function given

$$\varphi(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & otherwise. \end{cases}$$

We define the vector space

$$V_j = span\left\{ \varphi(2^j t - k) \right\}_{k \in \mathbb{Z}} \cap L^2(\mathbb{R}) \qquad (2)$$

5.1.1 Orthonormal Basis for V_j

Let V_j be given by (2) for $j \in \mathbb{Z}$. For each $k \in \mathbb{Z}$, define the function

$$\varphi_{j,k}(t) = 2^{j/2} \varphi(2^j t - k).$$

Then the set $\left\{ \varphi_{j,k}(t) \right\}_{k \in \mathbb{Z}}$ is an orthonormal basis for V_j.

5.1.2 Dilation Equation for $\varphi_{j,k}(t)$:

For $i, j \in \mathbb{Z}$, we have

$$\varphi_{j,k}(t) = \frac{\sqrt{2}}{2} \varphi_{j+1,2k}(t) + \frac{\sqrt{2}}{2} \varphi_{j+1,2k+1}(t).$$

The function $\varphi(t) = \frac{\sqrt{2}}{2} \varphi_{1,0}(t) + \frac{\sqrt{2}}{2} \varphi_{1,1}(t)$. For this reason, we call $\varphi(t)$ is a scaling function. The coefficients $h_0 = h_1 = \frac{\sqrt{2}}{2}$ are exactly the same numbers are those used in the average portion of the discrete wavelet transform.

5.2 The Haar Wavelet Space W_j

Let $\psi(t)$ be the Haar function given

$$\psi(t) = \varphi(2t) - \varphi(2t+1).$$

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$$\psi(t)=\begin{cases} 1, & 0\le t<\frac{1}{2} \\ -1, & \frac{1}{2}\le t<1 \\ 0, & otherwiese \end{cases}.$$

We define the vector space

$$W_j = span\left\{\psi(2^j t-k)\right\}_{k\in ¢} \cap L^2(¡). \qquad (3)$$

5.2.1 Orhtonormal Basis for W_j

Let W_j be given by (3) for $j \in ¢$. Then the set $\left\{\psi_{j,k}(t)\right\}_{k\in ¢}$ where $\psi_{j,k}(t)=2^{j/2}\psi(2^j t-k),\ i,k\in ¢$ is an orthonormal basis for W_j.

5.2.2 Dilation Equations for $\psi_{j,k}(t)$

For $i,\ k\in ¢$, we have

$$\psi_{j,k}(t)=\frac{\sqrt{2}}{2}\psi_{j+1,2k}(t)+\frac{\sqrt{2}}{2}\psi_{j+1,2k+1}(t).$$

The function $\psi(t)$ also satisfies a dilation equation:

$$\psi(t)=\frac{\sqrt{2}}{2}\varphi_{j,0}(t)-\frac{\sqrt{2}}{2}\varphi_{j,1}(t).$$

As was the case with the scaling function for $\varphi(t)$, the coefficients $g_0=\frac{\sqrt{2}}{2}$ and $g_1=-\frac{\sqrt{2}}{2}$ are exactly the same numbers as those used in the differences portion of discrete wavelet transform [10,12].

5.3 Discrete Haar Wavelet Transform to One Dimensional Signal

Suppose N is an even positive integer. We define Haar wavelet transformation as

$$W_N=\left[\frac{H_{N/2}}{G_{N/2}}\right]=\left[\begin{array}{cccccc} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ M & & & O & & M \\ 0 & 0 & 0 & 0 \ L & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \hline \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 \\ M & & & O & & M \\ 0 & 0 & 0 & 0 \ L & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{array}\right]$$

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The $\frac{N}{2}\times N$ block $H_{N/2}$ is called the averages block and the $\frac{N}{2}\times N$ block $G_{\frac{N}{2}}$ is called the details block.

If we apply $H_{N/2}$ to vector $a=\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}$; where $a_1, a_2, \ldots, a_N \in \mathbb{R}^N$

$$Ha=\sqrt{2}\begin{bmatrix} \frac{a_0+a_1}{2} \\ \frac{a_2+a_3}{2} \\ \vdots \\ \frac{a_{N-2}+a_{N-1}}{2} \end{bmatrix}$$

$H_{N/2}\,a$ computes pair wise averages of consecutive values of a and weight the result by $\sqrt{2}$.

5.4 Multiresolution Analysis

Let V_j, $j\in\mathbb{Z}$, be a sequence of subspace of $L^2(R)$. We say that $\{V_j\}_{j\in\mathbb{Z}}$ is a multiresolution analysis(MRA) of $L^2(\mathbb{R})$ if

$V_j \subset V_{j+1}$ (nested)

$\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$ (density)

$\cap_{j\in\mathbb{Z}} V_j = \{0\}$ (separation)

$f(t)\in V_0 \Leftrightarrow f(2^j t)\in V_j$ (scaling)

and there exists a function $\varphi(t)\in V_0$, with $\int_{\mathbb{R}} \varphi(t)dt \neq 0$, called a scaling function, such that the set $\{\varphi(t-k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis for V_0 [12-16].

Table-1: Key elements of MRA analysis

	Functions	Spaces	Bases	j Z	j]
Approximations	Scaling function φ	V_j	$\{\varphi_{j,k}\}_{k\in\mathbb{Z}}$	Coarser	Finer
Details	Wavelet ψ	W_j	$\{\psi_{j,k}\}_{k\in\mathbb{Z}}$		

5. DAUBECHIES WAVELETS

Even though the Haar wavelets provide local analysis of a signal in frequency as well as in spatial domain, they are not smooth in nature, due to their step function behavior. Daubechies, discovered new wavelet bases which overcome the limitation of the Haar wavelets [17].

5.1 Daubechies Wavelet Transformation Matrix

Suppose that h is the length ($L+1$) Daubechies scaling filter and assume that N is even positive integer with $N>L$. We define the discrete Daubechies wavelet transformation matrix W_N as

$$D_N = \begin{bmatrix} E_{N/2} \\ \hline F_{N/2} \end{bmatrix} = \left[\begin{array}{cccccccccc}
h_0 & h_1 & h_2 & h_3 & \cdots & h_{L-1} & h_L & \cdots & 0 & 0 \\
0 & 0 & h_0 & h_1 & \cdots & h_{L-3} & h_{L-2} & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & h_{L-5} & h_{L-4} & \cdots & 0 & 0 \\
 & & & & \ddots & & & \ddots & & \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & h_{L-1} & h_L \\
h_{L-1} & h_L & 0 & 0 & \cdots & 0 & 0 & \cdots & h_{L-3} & h_{L-3} \\
 & & & & \ddots & & & \ddots & & \\
h_2 & h_3 & h_4 & h_5 & \cdots & 0 & 0 & & h_0 & h_1 \\
\hline
g_0 & g_1 & g_2 & g_3 & \cdots & g_{L-1} & g_L & \cdots & 0 & 0 \\
0 & 0 & g_0 & g_1 & \cdots & g_{L-3} & g_{L-2} & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & g_{L-5} & g_{L-4} & \cdots & 0 & 0 \\
 & & & & \ddots & & & \ddots & & \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & g_{L-1} & g_L \\
g_{L-1} & g_L & 0 & 0 & \cdots & 0 & 0 & \cdots & g_{L-3} & g_{L-3} \\
 & & & & \ddots & & & \ddots & & \\
g_2 & g_3 & g_4 & g_5 & \cdots & 0 & 0 & & g_0 & g_1
\end{array}\right]$$

The $\frac{N}{2}\times N$ block $E_{N/2}$ is called the averages block (Daubechies scaling filters) and the $\frac{N}{2}\times N$ block $F_{\frac{N}{2}}$ is called the details block [12].

6 METHOD OF NOISE ESTIMATION ALGORITHM IN WAVELET DOMAIN

The method is based on wavelet transform and standard deviation, we obtain standard deviation of AWGN using coefficients of wavelet transform.

Input: True signal $v(t)$ and σ with AWGN $n(t)$

Output: Approximate noise level (δ) and Absolute error approximation (e).

Step 1: Collect signal $y(t)$ with noise σ .

Step 2: To perform discrete wavelet transform (i.e., db1, db2) and assemble detail coefficients.

Step 3: Hample show that: $\text{MAD}(x) \to 0.6745\sigma$ [6]

$\delta = \text{MAD}(x)/0.6745$, where x is detail coefficients.

Step 4: Find $e = |\sigma - \delta|$ with different level of noise (σ).

7 SIMULATION RESULTS

The noise estimation error is criterion for objective measurement of the error as compared different wavelet transform. Then results have shown in Table 2.

Table-2: Different noise level results of wavelet transform

Noise level σ	Haar wavelet (db1)		Daubechies wavelet (db2)	
	δ	e	δ	e
0.1	0.1197	0.0197	0.1211	0.0211
0.2	0.2345	0.0345	0.2420	0.0420
0.3	0.3599	0.0599	0.3619	0.0619
0.4	0.4678	0.0678	0.4833	0.0833
0.5	0.5839	0.0839	0.6047	0.1047

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0.6	0.6739	0.0739	0.6929	0.0929
0.7	0.7990	0.0990	0.8295	0.1295
0.8	0.9262	0.1262	0.9629	0.1629
0.9	1.0386	0.1386	1.0820	0.1820

7.1 Performance Comparison

We have compared with 2 methods. Fig.4 gives the simulation results. The x- axis denotes the known noise σ, and the y-axis denote absolute error approximation (e). The comparison results with two different wavelets (Haar and Daubechies), while the error in estimating the noise level is low, we can improve approximation error by using an improved discrete wavelet transformation.

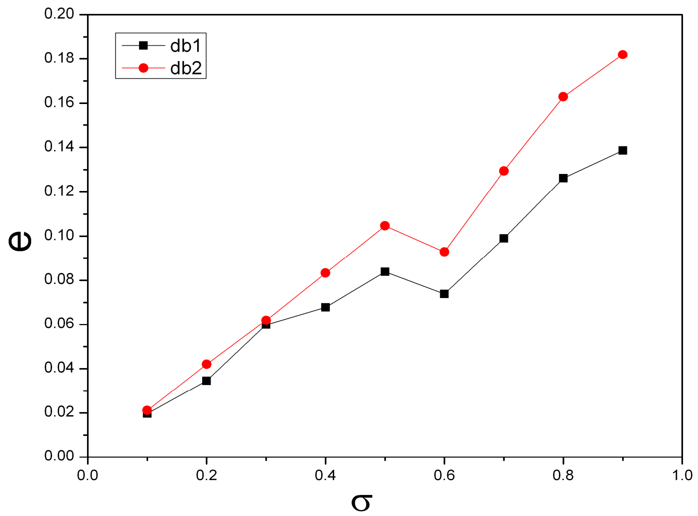

Fig-4: Influence of known σ vs absolute error approximation

8 CONCLUSION

In this work, we have presented a noise level estimation algorithm using wavelet transform. To collect the detail coefficient of noisy signal then apply Hample result is to get approximation level. If the noise level is low, absolute error is also low. It means that signal degrade low noise. The comparison with the several best state of the art methods shows that the accuracy of the proposed approach. Finally, noise estimation error is more, then level of noise is lofty.

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GOAL PROGRAMMING MODELS FOR OPTIMUM ALLOCATION OF RESOURCES TO RURAL SCHOOLS

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ABSTRACT
This paper presents three Goal Programming modals by considering six priorities, which help to avail quality education by travelling minimum distance. The problem raised in one of the rural area of Telangana state.

Keywords: Goal Programming, Multi-Objective models.

INTRODUCTION
Majority of India still lives in villages and so the topic of rural education in India is of utmost importance. Even as the Centre pushes for universal education through various schemes, around 13,000 villages are yet to get a school. A report by the Ministry of Rural Development revealed around 13,511 villages across all states do not have a school. “There could be various reasons for this. Firstly, the lackadaisical attitude of the state governments is resulting in the villages not having schools. Also, there are some villages which do not have the desired population for setting up schools,” a ministry official said. In terms of numbers, Mizoram is the only state where every village has a school, the report states. The highest number of villages without schools is in Uttar Pradesh.

From the survey Jangedu is a Village in Bhupalpalle Mandal in Jaya Shankar District of Telangana State, India. It is located 68 Km to-wards North from District Headquarters Warangal, 1 Km from Bhupalpalle. Jawaharnagar (4 Km) , Kompalle (6 Km) , Gorlaveedu (9 Km) , Kamalapur (10 Km) , Chelpur (10 Km) , are the nearby Villages to Jangedu. And Sub Villages are: Gaddiganipally, Kadwada, Peddakuntapally, Shyamgadda, Seggampally, Pakkiragadda, Gurijalawada, Beddalonipally. For all these Villages near by College is Pavithra Junior College and Schools are three which are listed below.

NEAR COLLEGES/SCHOOLS FOR THE ABOVE MENTIONED VILLAGES:

Pavithra Junior College	jangedu
Singareni Collories Hs	jangedu
Zphs	jangedu
Ups Goodshipyard Fakeer	jangedu
Mpps Kasimpally	jangedu

By using the above survey we developed three models to address their difficulties to educate their children. [1] “Proposed a trial and error modal “Feeder-Bus Network Design (FBNDP)” which gives a solution even for a critical routing issues. The edge of this will rise under variable demand and provided ideal balance between a shrewd and user costs. [3] Presents an overview of several important areas of operations research applications in the air transport industry. Specific areas covered are: the various stages of aircraft and crew schedule planning; revenue management, including overbooking and leg-based and network-based seat inventory management; and the planning and operations of aviation infrastructure (airports and air traffic management). For each of these areas, the paper provides a historical perspective on OR contributions, as well as a brief summary of the state of the art. [4]The optimal design of less-than-truckload (LTL) motor carrier networks is cast as a fixed-charge network design problem. A recently developed lower bound is applied iteratively with a link inclusion heuristic in an implicit enumeration framework. Initial computational results for solving this class of problems are reported. [5]A method for optimizing tractor-trailer and twin-trailer movements in a group line-haul operation has been described. The method is based on a B&B procedure in which the lower bounds are calculated by a Lagrangian relaxation. The first incumbent is generated by a cycle-splicing heuristic and the incumbents at every sub problem are generated by a simple local improvement heuristic. [6] Provides an analysis framework in which future transit systems can be designed. Questions of control, safety and dependa- bility are not addressed.

DATA OF THE PROBLEM
The data for busing costs and mileage are presented in **Table 1**. This table also presents the school capacity to provide “quality” education, number of pupils in each area, and the decision variables, which represent the number of pupils to be transported from each area to each school.

Figure-1

Area 1 A	Area 2 C	Area 3
B Area 4	Area 5	D Area 6

Figure 1, County Areas and School Locations, A, B, C and D represent locations of schools in county. Scale: Each area section = a 3-mile square. The quoted mileage figures represent the average miles traveled by a student within the country on the shortest route basis from strategic community bus stops to schools. These figures only measure in a crude way the inconvenience attached to each school assignment per student, since the estimates will vary with the bus routes and economies of scale encountered. It is apparent from **Table 1** that the demand for classroom space, shown by the total number of pupil residents (2,700), is 200 seats greater than the capacity of the available facilities (2,500). This problem is often encountered by many local school districts. In such cases, it is customary to accept all students in order to provide each child an educational opportunity, while sacrificing some degree of "quality" education.

Table-1: Student Transportation Model in Tabular Format

Average Miles Traveled / Annual Transportation Cost/Student						
Area	Schools / Race	A	B	C	D	Total Resident Pupils
1	RC	20 3 $X\alpha_{11}$	45 8 $X\alpha_{12}$	35 6 $X\alpha_{13}$	55 10 $X\alpha_{14}$ 0	0
	URC	20 3 $X_{\beta 11}$	45 8 $X_{\beta 12}$	35 6 $X_{\beta 13}$	55 10 $X_{\beta 14}$	390
2	RC	25 4 $X\alpha_{21}$	50 9 $X\alpha_{22}$	10 0.5 $X\alpha_{23}$	35 6 $X\alpha_{24}$	3
	URC	25 4 $X_{\beta 21}$	50 9 $X_{\beta 22}$	10 0.5 $X_{\beta 23}$	35 6 $X_{\beta 24}$	437
3	RC	55 10 $X\alpha_{31}$	65 12 $X\alpha_{32}$	10 1 $X\alpha_{33}$	35 6 $X\alpha_{34}$	0
	URC	55 10 $X_{\beta 31}$	65 12 $X_{\beta 32}$	10 1 $X_{\beta 33}$	35 6 $X_{\beta 34}$	546
4	RC	25 4 $X\alpha_{41}$	15 2 $X\alpha_{42}$	45 8 $X\alpha_{43}$	60 11 $X\alpha_{44}$	9
	URC	25 4 $X_{\beta 41}$	15 2 $X_{\beta 42}$	45 8 $X_{\beta 43}$	60 11 $X_{\beta 44}$	425
5	RC	15 2 $X\alpha_{51}$	15 2 $X\alpha_{52}$	30 5 $X\alpha_{53}$	45 8 $X\alpha_{54}$	351
	URC	15 2 $X_{\beta 51}$	15 2 $X_{\beta 52}$	30 5 $X_{\beta 53}$	45 8 $X_{\beta 54}$	37
6	RC	45 8 $X\alpha_{61}$	40 7 $X\alpha_{62}$	35 6 $X\alpha_{63}$	10 1 $X\alpha_{64}$	447
	URC	45 8 $X_{\beta 61}$	40 7 $X_{\beta 62}$	35 6 $X_{\beta 63}$	10 1 $X_{\beta 64}$	55
	School Capacity	500	700	800	500	2500 2700

Gp Model Formulation

The development of a goal-programming model requires a sequence of several steps:

(i) Determination of model objectives and their priorities

(ii) Identification of the decision variables

(iii) Formulation of model constraints

(iv) Analysis of the model solution and its implications. In this section the first three items are discussed in greater detail.

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MODEL OBJECTIVES AND THEIR PRIORITIES

The school-busing problem involves social, political, and economic objectives and implications. Although their importance and priority may vary according to the unique environmental conditions of the locality under consideration, the objectives can be viewed as universal. These are:

1 Provide equal educational opportunity to each child in the community district.

2. Achieve racial balance in each school according to the race composition of children in the community district

3. Minimize the total transportation cost of the district school system.

4. Avoid extremely long transportation times of children on a bus.

5. Balance overcrowding among schools if the number of children in the district exceeds the total capacity of school.

6. Avoid underutilization of school capacities if total number of pupils in the district is more than the total school capacity.

These six goals represent multiple objectives in different dimensions. Furthermore, these objectives are often in conflict. For example, the racial balance objective and transportation cost minimization may be in direct conflict. The goal-programming model of school busing can be viewed, therefore, as a modified transportation problem with multiple objectives. In this study, three different goal-programming models will be presented in order to demonstrate its flexibility in dealing with unique local situations in school busing. **Table 2** presents the descriptive summary of each model's objectives and their preemptive priorities.

Table-2: Priority Structures of Three Alternative Models

Goals	Model I	Model II	Model III
Provide each child an education opportunity by accepting everyone.	P_1	P_1	P_1
Avoid underutilization of school capacities.	P_2	P_2	P_2
Achieve 70:30 racial balances.	P_4	P_3	P_3
Balance overcrowding among schools.	P_5	P_4	P_5
Minimize transportation cost of the system.	P_3 (0)	P_5 (35,010)	P_4 (50,000)
Limit transportation times to 20 minutes.	P_6	P_6	P_6
P_j = preemptive priority weights ($P_j >>> P_{j+1}$)			

DECISION VARIABLES

The primary objective of the school-busing problem is to determine how many children of each race in a given area should be assigned to each school. Therefore, the decision variables can be expressed by X_{rij}, where r = race (either RC or URC), i = area in the district (county), and j = school. The decision variables are shown in **Table 1**.

MODEL CONSTRAINTS:

The goal-programming model usually has two types of constraints, system and goal constraints. The former represents a set of fact-of-life type constraints, which must be adhered to before an optimal solution could be considered. The latter represents a set of constraints, which include the objectives of the problem.

In the goal-programming model, the objective is to achieve all the goals as closely as possible within the convex set defined by the system constraints. This can be achieved by minimizing either the negative (d_1^-) or positive (d_1^+) deviations from the goals with certain preemptive priority weights (P_j). In the school-busing model, the following goal constraints are to be considered.

A. Educational Opportunity: The first objective of any public school system is to provide an educational opportunity for every child within its domain. Therefore, it is necessary to accommodate everyone in the four schools of the district. The goal constraints are:

$$\sum_{j=1}^{4} X_{rij} + d^- = b_{ri} \qquad (i = 1, 2, \ldots\ldots 6), \text{ for each } r, \qquad \text{--------} \qquad 2.1$$

where d^- = number of unaccepted children of race r in area i by schools, and b_{ri} = number of total children of race r in area i. The educational opportunity can be achieved by minimizing d^-. Since there are two races and six areas in the district (county) there will be twelve goal constraints for educational opportunity.

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B.) Racial Balance: The primary objective of the busing program is to achieve a racial balance in each school, corresponding to the racial proportions in the entire district. In the problem presented, the racial mix for the county is 30 percent R.C and 70 percent U.R.C as shown below.

	Pupils	Proportion
Total RC	810	30%
Total URC	1890	70%
Total Population	2700	100%

Consequently, the racial mix in each school should also be 30 percent R.C and 70 percent U.R.C according to the higher courts. The racial balance goal constraint for each school can be expressed by:

$0.3\sum_{i=1}^{6} X_{\alpha ij} - 0.7\sum_{i=1}^{6} X_{\beta ij} + d^{-} - d^{+} = 0$,for each j,-------- 2.2

where d^{-} = number of excess RC children beyond 70:30 racial balance, and, d^{+} = number of excess URC children beyond 70:30 racial balance. The racial balance goal for each school is attempted by minimizing both negative (d^{-}) and positive (d^{+}) deviations. Since there are four schools, there are four racial balance goal constraints.

C.)Transportation Costs: A primary consideration in any public decision making area is the cost involved in implementation of a proposal. The cost function, as a matter of constituency acceptance, must be kept within reasonable bounds. Minimization of total transportation cost can be achieved by minimizing the positive deviation from the following constraint:

$\sum_{i=1}^{6}\sum_{j=1}^{4} C_{ij}\left(X_{\alpha ij} + X_{\beta ij}\right) - d^{+} = 0$ ---------- 2.3

Where C_{ij} = annual per-student transportation cost from area I to school *j*.

D.)Transportation Time: Parents of children who are affected by the busing policy are naturally concerned with the transportation time (or distance). Conversely, school administrators wish to distribute equitably the transportation time borne by each child. Furthermore, there usually exists a strong sentiment for the community school concept whereby children attend schools in their immediate community whenever possible. Thus, school district officials might declare that they would like to avoid having to transport any individual student more than twenty minutes. If we assume an average bus speed of twenty-five kilometers per hour, the maximum distance desired is eight kilometers. Therefore, all transportation blocks showing a mileage factor greater than eight miles should be used as little as possible in the final solution, as shown by the following constraint.

$X_{rij} - d^{+} = 0$, ---------- 2.4

Where X_{rij} = number of children for given race *r* to be transported from area *i* to school *j* where transportation distance exceeds eight miles. In order to achieve the above goal, d^{+} must be minimized. Since there are ten blocks which indicate distance exceeding eight miles (Table 1), there are ten goal constraints. The minimization of d^{+} for each constraint is weighted by the mileage of the associated individual blocks in excess of eight miles, so that transportation for a longer distance is minimized with a greater weight.

E.)Balance Overcrowding: Another important consideration is the allocation of the students to schools in proportion to their capacities. It is assumed that the most equitable way to distribute overcrowding among schools is by allocating an equal proportion of overcrowding to the "normal" load capacities. Thus, overcrowding would be shared among schools in such a manner that each school would have the same percentage of overcrowding (i.e. 8 percent overcrowding for all schools).

The proportions for each school, in terms of its capacity in the school system, are shown below:

****	*****	*****
A	500	0.20
B	700	0.28
C	800	0.32
D	500	0.20
Total	2500	1.00

Applying the load proportions to the total school population, the student load constraint for each school can be determined. For example, the student load for school A will be

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$\sum_{i=1}^{6}\left(X_{\alpha i1} + X_{\beta i1}\right) = 0.2\sum_{i=1}^{6}(X_{\alpha ij} + X_{\beta ij})$ Transforming the above constraint, it becomes:

$0.8\sum_{i=1}^{6}\left(X_{\alpha i1} + X_{\beta i1}\right) - 0.2\sum_{i=1}^{6}\sum_{j=1}^{4}(X_{\alpha ij} + X_{\beta ij})$=0

The general goal programming constraint is now formulated for each school with the objective of minimizing both the negative and positive deviational variables.

$(1 - l_i)\sum_{i=1}^{6}\left(X_{\alpha ij} + X_{\beta ij}\right) - l_j\sum_{i=1}^{6}\sum_{j=2}^{4}\left(X_{\alpha ij} + X_{\beta ij}\right) + d^- - d^+ = 0$, j=1…..4. ……2.5

Where l_j = load proportion for school, j, d^+ = assignment in excess of load proportion, and d^- = assignment below load proportion. Since there are four schools in the district, there will be four constraints for the balancing of overcrowding goal

F.)Underutilization of School Capacities: If the total school capacity in the district exceeds the total number of pupils, there exists the balancing problem of underutilization of schools. This can be treated by a goal constraint similar to the balance of overcrowding discussed above in **E**. However, there may exist more complicated considerations, such as special facilities at various schools (laboratories, physical education facilities, amount of space for extracurricular activities, etc.), the age of the school buildings, and the possibility of closing down a school.

In the problem under consideration, however, the total number of pupils exceeds the school capacities. Consequently, overcrowding will occur for some schools. In order to avoid the problem of underutilization for any given school while others are being overcrowded, the following constraints are formulated for each school (where d^- is minimized).

$\sum_{i=1}^{6}\left(X_{\alpha ij} + X_{\beta ij}\right) + d^- - d^+ = CAP_J$ j=1……..4 …………..2.6

Where CAP$_j$ = normal capacity for school j. The goal-programming model involves a total of thirty-five goal constraints, forty-eight decision variables, and six preemptive priority factors.

Mathematically The Model Constraints are developed as Follows

A. Educational Opportunity

$\boldsymbol{\sum_{j=1}^{4} X_{\alpha 1J} + d_1^-}$=0 2.7

$\boldsymbol{\sum_{j=1}^{4}} X_{\beta 1j} + \boldsymbol{d_2^-}$=390 2.8

$\sum_{j=1}^{4} X_{\beta 2j} + d_3^- = 3$ 2.9

$\sum_{j=1}^{4} X_{\beta 2j} + d_4^- = 437$ 2.10

$\sum_{j=1}^{4} X_{\alpha 3j} + d_5^- = 0$ 2.11

$\sum_{j=1}^{4} X_{\beta 3j} + d_6^- = 546$ 2.12

$\sum_{j=1}^{4} X_{\alpha 4j} + d_7^- = 9$ 2.13

$$\sum_{j=1}^{4} X_{\beta 4j} + d_8^- = 425 \qquad 2.14$$

$\sum_{j=1}^{4} X_{\alpha 5j} + d_9^- = 351$ 2.15

$\sum_{j=1}^{4} X_{\beta 5j} + d_{10}^- = 37$ 2.16

$\sum_{j=1}^{4} X_{\alpha 6j} + d_{11}^- = 447$ 2.17

$\sum_{j=1}^{4} X_{\beta 6j} + d_{12}^- = 55$ 2.18

B. **Racial Balance Of 30: 70** (RC and URC)

$0.7\sum_{l=1}^{4} X_{\beta i1} - 0.3\sum_{i=1}^{6} X_{\alpha i1} + d_{13}^- - d_{13}^+ = 0$ 2.19

$0.7\sum_{l=1}^{6} X_{\beta i2} - 0.3\sum_{i=1}^{6} X_{\alpha i2} + d_{14}^- - d_{14}^+ = 0$ 2.20

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$0.7\sum_{I=1}^{6} X_{\beta i3} - 0.3\sum_{i=1}^{6} X_{\alpha i3} + d_{15}^{-} - d_{15}^{+} = 0$ 2.21

$0.7\sum_{I=1}^{6} X_{\beta i4} - 0.3\sum_{i=1}^{6} X_{\alpha i4} + d_{16}^{-} - d_{16}^{+} = 0$ 2.22

C. TOTAL TRANSPORTATION COST

$\sum_{i=1}^{6}\sum_{j=1}^{4} c_{ij}\left(X_{\alpha ij} + X_{\beta ij}\right) + d_{17}^{-} - d_{17}^{+} = rhs$ 2.23

C_{ij} = annual per student transportation cost (Table 1), and rhs = 0 for Model I; 35,010 for Model II; 50,000 for Model III.

D. BALANCING OF OVERCROWDING

$$0.8\sum_{I=1}^{6}\left(X_{\alpha i1} + X_{\beta i1}\right) - 0.2\sum_{i=1}^{6}\sum_{j=1}^{4}\left(X_{\alpha ij} + X_{\beta ij}\right) + d_{18}^{-} - d_{18}^{+} = 0 \qquad 2.24$$

$$0.72\sum_{I=1}^{6}\left(X_{\alpha i2} + X_{\beta i2}\right) - 0.28\sum_{i=1}^{6}\sum_{j=1}^{4}\left(X_{\alpha ij} + X_{\beta ij}\right) + d_{19}^{-} - d_{19}^{+} = 0 \qquad 2.25$$

$$0.68\sum_{I=1}^{6}\left(X_{\alpha i3} + X_{\beta i3}\right) - 0.32\sum_{i=1}^{6}\left[\sum_{j=1}^{2}\left(X_{\alpha ij} + X_{\beta ij}\right) + \left(X_{\alpha i4} + X_{\beta i4}\right)\right] d_{20}^{-} - d_{20}^{+} = 0 \qquad 2.26$$

$$0.8\sum_{I=1}^{6}\left(X_{\alpha i4} + X_{\beta i4}\right) - 0.2\sum_{i=1}^{6}\sum_{j=1}^{3}\left(X_{\alpha ij} + X_{\beta ij}\right) + d_{21}^{-} - d_{21}^{+} = 0 \qquad 2.27$$

E. LIMIT TRANSPORTATION TIMES TO 20 MINUTES

$X_{\alpha 31} - d_{22}^{+} = 0$ 2.28

$X_{\beta 31} - d_{23}^{+} = 0$ 2.29

$X_{\alpha 22} - d_{24}^{+} = 0$ 2.30

$X_{\beta 22} - d_{25}^{+} = 0$ 2.31

$X_{\alpha 32} - d_{26}^{+} = 0$ 2.32

$X_{\beta 32} - d_{27}^{+} = 0$ 2.33

$X_{\alpha 14} - d_{28}^{+} = 0$ 2.34

$X_{\beta 14} - d_{29}^{+} = 0$ 2.35

$X_{\alpha 44} - d_{\mathbf{30}}^{+} = 0$ 2.36

$X_{\beta 44} - d_{\mathbf{31}}^{+} = 0$ 2.37

F. AVOID UNDERUTILIZATION OF SCHOOL CAPACITIES:

$$\sum_{I=1}^{6}\left(X_{\alpha i1} + X_{\beta i1}\right) + d_{32}^{-} - d_{32}^{+} = 500 \qquad 2.38$$

$$\sum_{I=1}^{6}\left(X_{\alpha i2} + X_{\beta i2}\right) + d_{33}^{-} - d_{33}^{+} = 700 \qquad 2.39$$

$$\sum_{I=1}^{6}\left(X_{\alpha i3} + X_{\beta i3}\right) + d_{34}^{-} - d_{34}^{+} = 800 \qquad 2.40$$

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$$\sum_{I=1}^{6}\left(X_{\alpha i4} + X_{\beta i4}\right) + d_{35}^{-} - d_{35}^{+} = 500 \qquad 2.41$$

THE OBJECTIVE FUNCTION

The objective function of the goal programming model includes minimizing deviations, either negative or positive, from set goals with certain preemptive priority factors Pj ($P_j >>> P_{j+1}$), assigned by the decision maker. Since there are three models presented in this chapter, there will be three objective functions.

Model I

Minimize $Z= P_1\sum_{1}^{12}=1d_1^- + P_2\sum_{1}^{35}=32\,d_1^- + P_3\,d_{17}^+ + P_4\sum_{1}^{16}=13(d_1^- + d_1^+) + P_5\sum_{1}^{21}=18(d_1^- + d_1^+) +$
$P_6\,(2d_{22}^+ + 2d_{23}^+ + d_{24}^+ + d_{25}^+ + 4d_{26}^+ + 4d_{27}^+ + 2d_{28}^+ + 2d_{29}^+ + 3d_{30}^+ + 3d_{31}^+).$

Model II

Minimize $Z = P_1\sum_{1}^{12}=1d_1^- + P_2\sum_{1}^{35}=32\,d_1^- + P_3\sum_{1}^{16}=13(d_1^- + d_1^+) + P_4\sum_{1}^{21}=18(d_1^- + d_1^+) +$
$P_5\,d_{17}^+ + P_6\,(2d_{22}^+ + 2d_{23}^+ + d_{24}^+ + d_{25}^+ + 4d_{26}^+ + 4d_{27}^+ + 2d_{28}^+ + 2d_{29}^+ + 3d_{30}^+ + 3d_{31}^+).$

Model III

Minimize $Z= P_1\sum_{1}^{12}=1d_1^- + P_2\sum_{1}^{35}=32\,d_1^- + P_3\sum_{1}^{16}=13(d_1^- + d_1^+) + P_4\,d_{17}^+ + P_5\sum_{1}^{21}=18(d_1^- + d_1^+) +$
$P_6\,(2d_{22}^+ + 2d_{23}^+ + d_{24}^+ + d_{25}^+ + 4d_{26}^+ + 4d_{27}^+ + 2d_{28}^+ + 2d_{29}^+ + 3d_{30}^+ + 3d_{31}^+).$

RESULT AND ANALYSIS

Model I

The solution will be obtained by using **QM for WINDOWS** package as follows.

The first model analyzed included minimization of total transportation cost to zero as the third goal after the educational opportunity and full utilization of school capacity goals, followed by the remaining three goals [see **Table 2.2**]. The reasoning behind the use of this model initially was to determine the minimum cost possible under a busing program for all students. The secondary purpose was to determine the degree of goal attainments for racial balance, for balancing of overcrowding, and minimization of busing over twenty minutes.

The results of the model were shown in **Table 2.4**. As would be expected, the maximum numbers of children were allocated to the least cost cells with the total transportation cost equal to the minimum of Rs.35, 010. This amount, then, is the right-hand-side value for the transportation cost constraint in Model II. The results of Model I indicate the following goal attainments:

Table-2.3

	Goal Attainment	Achieved/Not Achieved
P_1:	Accept all children	Achieved
P_2:	Avoid underutilization of schools	Achieved
P_3:	Minimize transportation costs	Rs35,010
P_4:	Achieve racial balance	Not achieved
P_5:	Balance overcrowding	Not achieved
P_6:	Limit transportation time to 20 minutes	Not achieved

Since this model attempted to minimize total transportation costs, racial balance and balancing of overcrowding were sacrificed. As can be seen in Table 3, the racial balance and balancing of overcrowding were attempted while transporting children to schools closest to their communities because of the higher order goal of cost minimization. Thus, there were 649 students assigned in violation of racial balance and 244 students were allocated to schools without regard to the balancing of overcrowding. It is interesting to note that the goal concerning the minimization of transportation to 20 minutes was completely attained in this model. Clearly, this result is primarily due to the fact that transportation time is directly related to the minimization of transportation cost. This model was designed primarily to determine the impact of cost minimization on the goals of racial balance and balancing of overcrowding. The results of the model indicate that this model solution is indeed impractical and unacceptable.

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Table-4: Results of Model I

Area	School Race	A	B	C	D	Accepted / Population
1	RC					0 / 0
	URC	390				390 / 390
2	RC			3		3 / 3
	URC			437		437 / 437
3	RC					0 / 0
	URC			546		546 / 546
4	RC		9			9 / 9
	URC		425			425 / 425
5	RC	122	229			351 / 351
	URC		37			37 / 37
6	RC				447	447 / 447
	URC				55	55 / 55
Accepted / Capacity		512 / 500	700 / 700	986 / 800	502 / 500	2700 / 2500

Model II

Model II was developed to ascertain what level of resources would be required in order to achieve most of the social goals involved in the busing program. Therefore, minimization of total transportation costs to Rs.35, 010 derived in Model I was treated as the second lowest priority goal.

The results of the model, presented in **Table 6**, indicate the following goal attainments:

Table-5

	Goal Attainment	Achieved/Not Achieved
P_1:	Accept all children	Achieved
P_2:	Avoid underutilization of schools	Achieved
P_3:	Achieve racial balance	Achieved
P_4:	Balance overcrowding	Achieved
P_5:	Minimize transportation cost to Rs35,010	Not achieved
P_6:	Limit transportation time to 20 minutes	Achieved

All the desired social goals were completely attained: however, the total transportation cost of the model result was Rs.52,862. This amount exceeds the minimum cost derived in Model I by Rs.17,862. The major implication is that in order to achieve two important social goals, racial balance and balancing of overcrowding, the school system is required to absorb an additional Rs.17,852 for busing.

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Table-6: Results of Model II

Area	School / Race	A	B	C	D	Accepted / Population
1	RC					0 / 0
	URC	323	67			390 / 390
2	RC			3		3 / 3
	URC	55		59	323	437 / 437
3	RC					0 / 9
	URC			546		546 / 546
4	RC		9			9 / 9
	URC		425			425 / 425
5	RC	162	189			351 / 351
	URC		37			37 / 37
6	RC		29	256	162	447 / 447
	URC				55	55 / 55
Accepted / Capacity		540 / 500	756 / 700	864 / 800	540 / 500	2700 / 2500

Model III

Model II approximated the real-world priority rankings for the goals. Therefore, the estimated total cost of Rs.50,000 was used as the target cost figure in Model III. Any school system required to implement a busing program must concern itself with two predominant goals: providing educational opportunity to all children in the system and achieving racial balance. At the same time underutilization of any individual school's capacity would be politically unacceptable in a school system with in-adequate overall capacity. Model III attempted to achieve these two primary social goals while insuring no underutilization of individual schools. The budget goal of Rs.50,000 for the busing program was treated as the fourth goal, followed by the balancing of overcrowding and minimization of busing time in excess of twenty minutes. Following is the summary of the goal attainments for Model III:

Table-7

	Goal Attainment	Achieved/Not Achieved
P_1:	Accept all children	Achieved
P_2:	Avoid underutilization of schools	Achieved
P_3:	Achieve racial balance	Achieved
P_4:	Limit the busing budget to Rs50,000	Not achieved
P_5:	Balance overcrowding	Not achieved
P_6:	Limit transportation time to 20 minutes	Achieved

The results of the model are presented in **Table 8**. It is interesting to note the redistribution of students from that proposed by the Model I solution.

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Table-8: Results of Model III

Area	School Race	A	B	C	D	Accepted / Population
1	RC					0 / 0
	URC	362	28			390 / 390
2	RC			3		3 / 3
	URC			142	295	437 / 437
3	RC					0 / 0
	URC			546		546 / 546
4	RC		9			9 / 9
	URC		425			425 / 425
5	RC	155	196			351 / 351
	URC		37			37 / 37
6	RC		5	292	150	447 / 447
	URC				55	55 / 55
Accepted / Capacity		517 / 500	700 / 700	983 / 800	500 / 500	2700 / 2500

The budget goal was not completely attained, as the total transportation cost obtained was Rs.50,536. Complete attainment of the Rs.50,000 target would have conflicted with the goal to avoid underutilization of schools. Also, the balancing of overcrowding of schools was not completely attained, where the solution resulted in 238 students assigned to schools nearest in the communities. Failure to achieve balancing appears to be insignificant in view of the fact that the complete balance of overcrowding would cost an additional Rs.2,326.

Evaluation of the three models leads to the inevitable conclusion that a Rs.50,536 budget for the busing program appears to be the most realistic target for the public administrators to adopt. Acceptance of this Rs.50,536 budget enables the policy-makers to achieve most of the important goals involved in the problem. Model I is an unrealistic model in that it attempts to minimize total transportation cost before satisfying the legal restrictions regarding racial mix in the schools. Model II provides results that are most desirable. However, it requires the total cost of Rs.52,862. Model III accepts Rs.50,536 as the acceptable busing budget. This model offers the most promising results. All the important goals are completely or very closely attained. Since the unattained portions of the budget of Rs.50,000 and balancing of overcrowding are relatively insignificant, the results appear to be easily implement able in the system.

CONCLUSIONS

Virtually all models developed for the school-busing problem have focused upon the analysis of input (resource) requirements, busing schedules, and redistricting. They have generally neglected or often ignored the system outputs (budgetary considerations), unique community situations, and multiple decision criteria. However, these are important environmental factors, which greatly influence the decision process. In this study the goal programming approach is utilized because it allows the optimization of goal attainments while permitting an explicit consideration of the multiple conflicting objectives.

Developing and solving the goal programming model points out where some goals cannot be achieved under the desired structure of goals and hence where tradeoff must occur due to conflicts among the goals. Furthermore, the model allows the administrator to review critically the priority structure in view of the solution derived by the model. Indeed, the most important property of the goal-programming model is its great flexibility, which allows model manipulation with numerous variations of constraints and goal priority structures.

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The goal programming formulation used in this study is simplistic in its structure but could be expanded to include any number of possible variations that would make the model more realistic and adoptable. The model could be extended, of course, to any size district covering the prospective county or area under consideration. The portion of the district could be narrowed down to the scope of neighborhood blocks. Consideration of the possibilities of reallocation procedures for various grades is another method of manipulating the constraints to provide optimal solutions. Such a technique would separate elementary schools into kindergarten-3rd grade schools and 4th-6th grade schools, or a kindergarten school with all other schools being elementary schools. The model is appropriate for use at any educational level through high school. The impact of various new school locations on the existing school system, and its goal structure may be other factors that could be added to the model. Minimization of class size and balancing of these class sizes would require more detailed student census information concerning student grade levels, as would equitable racial balancing within classes in each school. Retention of family groupings is another possibility for extension of the model's constraints. Rezoning within the goal of contiguous school districts could also be incorporated into the model by use of minimization of the sum of the squared distance of each child from each school. One last potential expansion would include the additional variables relating to bus capacities and bus routes. The model formulation would have to consider the number of buses available, the bus capacities, the location of student residents, and the tightness of the route loops.

Many of these elaborations may not be necessary or desirable. They are offered as possibilities for generalization of the model to any one of many specific situations demanding the use of a multiple criteria model to determine the most satisfactory solution, given the constraints and objectives of the system. School busing is indeed a controversial social issue with conflicting goals. However, the problem is susceptible to analysis by the goal programming technique to provide policy-makers with more information upon which to base their public decisions in a more logical and defensible manner.

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A RESOURCE ALLOCATION MODEL FOR INSURANCE MANAGEMENT USING GOAL PROGRAMMING

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ABSTRACT

In a model of a property-liability insurer, Hofflander and Drandell (H-D) constructed a linear programming model to determine the optimum allocation of assets in order to maximize profits. The model was based on constraints which reflected policy and legal bounds on the insurer's activities. A model company with assets of Rs. 100 crores served as the hypothetical insure. Klock and Lee have suggest a Goal Programming approach to the H-D model. This paper is an analysis of the H-D model using Goal Programming and current asset returns. The purpose here is equivalent to the original H-D model.

Keywords: Goal Programming, insurance Management, Resource.

GOAL PROGRAMMING

The concept of Goal Programming is relatively new although it had been postulated some time earlier [6, 24, 25]. A good description of the methodology is presented in a current publication [3, 4]. It is a most important extension of linear programming in the treatment of business models. Briefly, a Goal Programming model is expressed mathematically as follows:

Minimize Z= $\sum_1^m p_i \times \left({d_i}^- + {d_i}^+\right)$

Subject to $a_{11}x_1 + a_{12}x_2 + \cdots \ldots\ldots\ldots\ldots + a_{1n}x_n + d^- - d^+ = g_1$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.$$

$a_{m1}x_1 + a_{m2}x_2 + \cdots \ldots\ldots\ldots\ldots + a_{mn}x_n + {d_m}^- - {d_m}^+ = g_m,$

$$b_{11}x_1 + b_{12}x_2 + \cdots \ldots\ldots\ldots\ldots + b_{1n}x_n \le c_1$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.$$

$$b_{r1}x_1 + b_{r2}x_2 + \cdots \ldots\ldots\ldots\ldots + b_{rn}x_n \le c_r,$$

with $x_j \ge 0$, ${d_i}^- \ge 0, {d_i}^+ \ge 0,\ i = 1,2,3 \ldots . m; j = 1,2,3 \ldots . n.$

The x_1 represent the variables, a_1 the constantans, g_1 the goals with the first m equations expressing the relationships those which the model must satisfy all times. The variables ${d_1}^-$ and ${d_1}^+$ are called deviational variables and represent possible deviations from the respective goals. The former represent under- achievement and the latter overachievement for the respective goals. For any goal equation, at most one of these variables can be non-zero. If both are zero then the goal has been exactly achieved. If overachievement is allowed, then ${d_1}^+$need to appear in the objective function; if under-underachievement is permissible, then ${d_1}^-$ need not appear there.

The objective function minimizes the deviations from the goals, based on a predetermined priority scheme. High priority goals are satisfied before low priority goals, according to a given order. Deviational variable associate with different goals can have the same or different priorities. A modified linear programming computer code was used to solve the model.

THE H-D MODEL IN A GOAL PROGRAMMING CONTEXT

For reference the set of variable and constraints of the H-D linear programming model are presented in Table 1 and Table 2.

TABLE-1: VARIABLES IN THE H-D MODEL

Assets		Liabilities	
A(1)	Bonds	L(1)	Unpaid Claims or LossReserves
A(2)	Common Stocks	L(2)	Unearned premium Reserves
A(3)	Preferred Stocks	L(3)	Miscellaneous Liabilities

A(4)	Mortgages	L(4)	Policyholders Surplus
A(5)	Real Estate	L(5)	Total Liabilities
A(6)	Cash		
A(7)	Premium Balances		
A(8)	Total Assets		

OTHERS VARIABLES
Y Premiums Written

An insurer with total assets of Rs. 100 crores is assumed.

The goal and their relationship to the original constraints together with assumed reasonable priorities are as follows:

Priority	Goal Description	Corresponding Constraints from Table 2
1	Liquidity	7
2	Stability	2,4,5,6
3	Profit	Profit Function

COMMUNICATIONS

TABLE-2: CONSTRAINTS OF THE H-D MODEL

Constraints No.	Constraint	Explanation
(1)	L(4)≥3.0	Policyholder's Surplus (net equity) must equal or exceed 83 crores
(2)	A(1) ≥L1	The bond portfolio must equal or exceed the reserve for unpaid claims.
(3)	Y≤4L(4)	Premium volume must be equal to or less than 4 times policy holders surplus.(General rule regulation in several state)
(4)	A(8)/Y≥1.23	Ratio of assets to premium volume must exceed 1.23 (this is the English cover ratio)
(5)	A(1)+A(4)+A(8) ≥ L(4)+.5[L(1)+L(2)]	Bonds, mortgages and each asset exceed capital plus one half the sum of the unearned premium reserve and the loss reserve.
(6)	A(1)+A(4)+A(8) ≥ L(1)+L(2)	Bonds, mortgages and cash must equal or exceed the unearned premium and the loss reserve.
(7)	A(6) ≥.10L(1)	Cash on hand should be equal to or exceed 10% or unpaid claims (general liquidity rule)
(8)	A(7)-.20Y	Premium balances on an average are assumed to be 20% of premium volume.
(9)	A(4)+A(5) ≤.03A(8)	Mortgage plus real estate should be less than or equal to 5% of total assets.
(10)	.07L(5) ≤L(3) ≤.09L(5)	Miscellaneous liabilities should be 7 and 9 percent of total liabilities
(11)	1.(1)-60Y	Loss reserves are assumed to be 60% of premium volume.
(12)	L(2)-.70Y	Unearned premium reserves are assumed to be 70% of premium volume.
(13)	A(8)-100+1.1Y	Total assets are equal to Rs. 100 + 110% of premium volume.

The (H.D) model in Coal Programming, terms with priorities assigned to the deviational variable is now as follows.

Minimize Z= $p_1 d_1^- + p_2(d_2^- + d_3^- + d_4^- + d_5^-) = p_3 d_6^-$

Subject to the constraints:

Liquidity (Priority $1-p_1$)

$$A(6) - .10L(1) + d_1^- - d_1^+ = 0$$

Stability (Priority $2-p_2$):

$A(1) - L(1) + d_2^- - d_2^+ = 0,$

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$A(8) - 1.25x + d_3^- - d_3^+ = 0,$

$$A(1) + A(4) + A(6) - L(1) - .5L(1) - .5L(2) + d_4^- - d_4^+ = 0,$$

$$A(1) + A(4) + A(6) - L(1) - L(2) + d_5^- - d_5^+ = 0,$$

Profit Goal (Priority 3—p_3):

$$\sum_{1}^{5}\big(R(1) \times A(i)\big) + R \times Y + d_6^- - d_6^+ = p_0$$

Where R (i)1 represents the after tax return on investment A(i); R is the return on premiums written Y and Po is a given profit goal. The model is also subset to the remaining contents of the table 2. The following assets return where used:

TABLE-3

Assets		Return(in percent)
A(1)	Bonds	4,55%
A(2)	Common stocks	1.82%
A(3)	Preferred stocks	3.92%
A(4)	Mortgages	4.50%
A(5)	Real Estate	9.00%

Sources Bond, common and preferred stock where based on a board survey of Standard and Poor's 1975 Trade and Securities, Indian Financial Data, Reserve Bank of India and current market as reflected in leading Financial publications. The returns of mortgages and Real Estate are based on current market conditions. All returns were adjusted for taxes in a 50% tax bracket.

RESULT AND DISCUSSIONS

A numbers of runs were made for R= -0.05, -0.025, 0.0, +0.025 and +0.05. In order to determine the maximum profit possible for each run, avail high profile goal of Po = Rs. 50 crore was set. This goal of course was never achieved, as it served only as an upper bound to the maximum Profit possible for each run. Table 4 shows the allocations for these runs.

TABLE-4: ALLOCATIONS FOR RANGES OF R

Assets		- 0.05	-0.025R	0.0	+0.025	+0.05
A(1)	Bonds	Rs. 95.0	233.5	*	*	*
A(2)	Common Stocks	0.0	0.0	*	*	*
A(3)	Preferred Stocks	0.0	0.0	*	*	*
A(4)	Mortgages	0.0	0.0	*	*	*
A(5)	Real Estate	5.0	14.7	*	*	*
A(6)	Cash	0.0	10.6	*	*	*
A(7)	Premium Balance	0.0	35.3	*	*	*
A(8)	Total Assets	Rs. 100.0	294.1	*	*	*
Liabilities						
L1	Unpaid Claims	0.0	105.9	*	*	*
L2	Unearned Prem. Res.	0.0	123.5	*	*	*
L3	Misc. Liabilities	97.0	20.6	*	*	*
L4	Policyholders surplus	3.0	44.1	*	*	*
L5	Total Liabilities	Rs. 100.0	294.1	*	*	*
Y	Premium Written	0.0	176.5	*	*	*
	Profit	4.8	7.5	11.9	16.4	20.8

*Indicates that the value is the same as that value_____________ to _____ left.

From Table 4 it is seen that since A(6) = .10L(1), then $d_1^- = d_1^+ = 0$.

Therefore the Liquidity goal is satisfied, as in the (H-D) model.

The deviational variables _______ $=d_3^- = d_4^- = d_5^- = 0$ in the Stability constraint equations. Since these appear in the objective function alone associated with Priority 2, the Stability constraint equations have all been overachieved as in the H-D model.

The Profit goal was not achieved and was not expected to do so. It is seen that they ___________ are the same for R ≥ - 0.025, only Y and Profit changing in this range. For R = -0.05, underwriting ceases and the insurer operates as an investment house. This analysis holds actually for R ≤ -0.05, with profit = Rs. 4.8 holding constant for this range. Because of the manner in which the model was specified, it is possible to construct a linear function relating Profit to return on premium R ≤ Ro, we have Y=O and Profit =Rs. 4.8, giving rice to what may be carried

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SOME TOPOLOGICAL INDICES OF TURMERIC

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ABSTRACT

Graph theory has provided chemists with a variety of useful tools, such as topological indices. A topological index Top(G) of a graph G is a number with the property that for every graph H isomorphic to G, Top(H) = Top(G). In this paper, we compute ABC index, ABC_4 index, Randic index, Sum connectivity index, GA index, GA_5 index, First Zagreb index, Second Zagreb index, First Multiple Zagreb index, Second Multiple Zagreb index, Augmented Zagreb index, Harmonic index and Hyper Zagreb index, First Zagreb polynomials, Second Zagreb polynomials, Third Zagreb polynomials, Forgotten topological index, Forgotten polynomials and Symmetric division index of turmeric.

Keywords: ABC index, ABC_4 index, Randic index, Sum connectivity index, GA index , GA_5 index, First Zagreb index, Second Zagreb index, First Multiple Zagreb index, Second Multiple Zagreb index, Augmented Zagreb index, Harmonic index, Hyper Zagreb index, First Zagreb polynomials, Second Zagreb polynomials, Third Zagreb polynomials, Forgotten topological index, Forgotten polynomials, Symmetric division index and turmeric.

1. INTRODUCTION

Turmeric has been used in Asia for thousands of years and is a major part of Ayurveda, Siddha medicine, traditional Chinese medicine, Unani, and the animistic rituals of Austronesian peoples. It was first used as a dye, and then later for its supposed properties in folk medicine.its molecular formula is $C_{21}H_{20}O_6$. Its structure is shown in following figure -1.

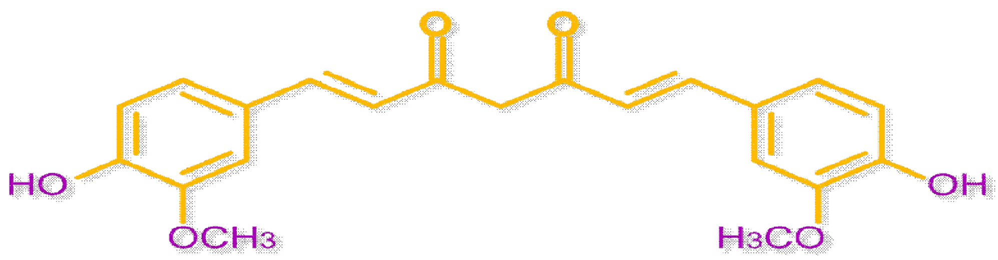

Figure-1

Topological indices are the molecular descriptors that describe the structures of chemical compounds and they help us to predict certain physic-chemical properties like boiling point, enthalpy of vaporization, stability, etc. Molecules and molecular compounds are often modeled by molecular graph. A molecular graph is a representation of the structural formula of a chemical compound in terms of graph theory, whose vertices correspond to the atoms of the compound and edges correspond to chemical bonds. Note that hydrogen atoms are often omitted. All molecular graphs considered in this paper are finite, connected, loop less and without multiple edges. Let G = (V, E) be a graph with vertex set V and edge set E. The degree of a vertex u∈E(G) is denoted by d_u and is the number of vertices that are adjacent to u. The edge connecting the vertices u and v is denoted by uv.

The Atom-bond connectivity index, ABC index is one of the degree based molecular descriptor, which was introduced by Estrada et al. [7] in late 1990's and it can be used for modeling thermodynamic properties of organic chemical compounds, it is also used as a tool for explaining the stability of branched alkanes [8].

Some upper bounds for the atom-bond connectivity index of graphs can be found in [3].The atom-bond connectivity index of chemical bicyclic graphs, connected graphs can be seen in [4, 30]. For further results on ABC index of trees see the papers [11, 21, 29, 31] and the references cited there in.

Definition.1.1. Let G = (V, E) be a molecular graph and d_u is the degree of the vertex u, then ABC index ofG is defined as,

$$ABC(G) = \sum_{uv \in E} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}}$$

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The fourth atom bond connectivity index, ABC_4 (G) index was introduced by M.Ghorbani et al. [15] in 2010. Further studies on ABC_4(G) index can be found in [9, 10].

Definition.1.2. Let G be a graph, then its fourth ABC index is defined as,

$$ABC_4(G) = \sum_{uv\in E(G)} \sqrt{\frac{s_u + s_v - 2}{s_u s_v}}.$$

Where S_u is sum of degrees of all neighbors of vertex u in G.In other words$s_u = \sum_{uv\in E(G)} d_v$, similarly S_V.

The first and oldest degree based topological index is Randic index [23] denoted byχ(G) and was introduced by Milan Randic in 1975. It provides a quantitative assessment of branching of molecules.

Definition.1.3. For the graph G Randic index is defined as,

$$\chi(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d_u d_v}}.$$

Sum connectivity index belongs to a family of Randic like indices and it was introduced by Zhou and N. Trinajstic[33]. Further studies on Sum connectivity index can be found in [34, 35].

Definition.1.4. For a simple connected graph G, its sum connectivity index S(G) is defined as,

$$S(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d_u + d_v}}.$$

The Geometric-arithmetic index, GA(G) index of a graph G was introduced by D. Vukicevi'c et.al[27]. Further studies on GA index can be found in [2, 5, 32].

Definition.1.5. Let G be a graph and e = uv be an edge of G then.

$$GA(G) = \sum_{e=uv\in E(G)} \frac{2\sqrt{d_u d_v}}{d_u + d_v}.$$

The fifth Geometric-arithmetic index, GA_5(G) was introduced by A.Graovac et al [16] in 2011.

Definition.1.6. For a Graph G, the fifth Geometric-arithmetic index is defined as,

$$GA_5(G) = \sum_{uv\in E(G)} \frac{2\sqrt{s_u s_v}}{s_u + s_v}.$$

Where S_u is the sum of the degrees of all neighbors of the vertex uin G, similarly S_v.

A pair of molecular descriptors (or topological index), known as the First Zagreb index Z_1(G) and Second Zagreb index Z_2(G), first appeared in the topological formula for the total π-energy of conjugated molecules that has been derived in 1972 by I. Gutman and N.Trinajsti'c[17]. Soon after these indices have been used as branching indices. Later the Zagreb indices found applications in QSPR and QSAR studies. Zagreb indices are included in a number of programs used for the routine computation of topological indices POLLY, DRAGON, CERIUS, TAM, DISSI. Z_1(G) and Z_2(G) were recognize as measures of the branching of the carbon atom molecular skeleton [20], and since then these are frequently used for structure property modeling. Details on the chemical applications of the two Zagreb indices can be found in the books [25, 26]. Further studies on Zagreb indices can be found in [1, 18, 33, 34, 35].

Definition.1.7. For a simple connected graph G, the first and second Zagreb indices were defined as follows,

$$Z_1(G) = \sum_{e=uv\in E(G)} (d_u + d_v).$$

$$Z_2(G) = \sum_{e=uv\in E(G)} (d_u d_v).$$

Where d_v denotes the degree (number of first neighbors) of vertex v in G.

In 2012, M. Ghorbani and N. Azimi [14] defined the Multiple Zagreb topological indices of a graph G, based on degree of vertices ofG.

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Definition.1.8. For a simple connected graph G, the first and second multiple Zagreb indices were defined as follows

$$PM_1(G) = \prod_{e=uv\in E(G)} (d_u + d_v)$$

$$PM_2(G) = \prod_{e=uv\in E(G)} (d_u d_v).$$

Properties of the first and second Multiple Zagreb indices may be found in [6, 19].

The Augmented Zagreb index was introduced by Furtula et al [12]. This graph invariant has proven to be a valuable predictive index in the study of the heat of formation in octanes andheptanes, is a novel topological index in chemical graph theory, whose prediction power is better than atom-bond connectivity index. Some basic investigation implied that AZI index has better correlation properties and structural sensitivity among the very well established degree based topological indices.

Definition.1.9. Let G = (V,E) be a graph and du be the degree of a vertex u, then augmented Zagreb index is denoted by AZI(G) and is defined as,

$$AZI(G) = \sum_{uv\in E} \left[\frac{d_u d_v}{d_u + d_v - 2}\right]^3.$$

Further studies can be found in [22] and the references cited there in.

The Harmonic index was introduced by Zhong [36]. It has been found that the harmonic index correlates well with the Randic index and with the π-electron energy ofbenzenoid hydrocarbons.

Definition.1.10.Let G = (V, E) be a graph and du be the degree of a vertex u then Harmonic index is defined as,

$$H(G) = \sum_{e=uv\in E(G)} \frac{2}{d_u + d_v}.$$

Further studies onH(G) can be found in [28, 34].

G.H. Shirdel et.al [24] introduced a new distance-based of Zagreb indices of a graph G named Hyper-Zagreb Index.

Definition.1.11. The hyper Zagreb index is defined as,

$$HM(G) = \sum_{e=uv\in E(G)} (d_u + d_v)^2.$$

Fath-Tabar [37] introduced the Third Zagreb index in 2011.Which is defined by.

Definition.1.12.For a simple connected graph G, the third Zagreb index is defined as,

$$ZG_3(G) = \sum_{e=uv\in E(G)} |d_u - d_v|.$$

Again in 2011Fath-Tabar [37] introduced the First, Second and Third Zagreb Polynomials which is defined by,

Definition.1.13. The First, Second and Third Zagreb Polynomials for a simple connected graph G is defined as,

$$ZG_1(G, x) = \sum_{e=uv\in E(G)} x^{d_u + d_v}.$$

$$ZG_2(G, x) = \sum_{e=uv\in E(G)} x^{d_u d_v}.$$

$$ZG_3(G, x) = \sum_{e=uv\in E(G)} x^{|d_u - d_v|}.$$

Definition.1.14.The forgotten topological index is also a degree based topological index, denoted by F(G) for simple graph G .It was encountered in [13], defined as,

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$$F(G) = \sum_{e=uv\in E(G)} [(d_u)^2 + (d_v)^2]$$

Definition.1.15.The forgotten topological polynomials for a graph G defined as,

$$F(G,x) = \sum_{e=uv\in E(G)} x^{[(d_u)^2+(d_v)^2]}$$

Definition.1.16. There are some new degrees based graph invariants, which plays an important role in chemical graph theory. These topological indices are quite useful for determining total surface area and heat formation of some chemical compounds. These graphs invariants are as follow Symmetric division index,

$$SDD(G) = \sum_{e=uv} \left\{\frac{min(d_u,d_v)}{max(d_u,d_v)} + \frac{max(d_u,d_v)}{min(d_u,d_v)}\right\}.$$

2. Main results

Theorem.2.1. The Atom bond connectivity index of Turmeric is 18.96023.

Proof: Consider Turmeric ($C_{21}H_{20}O_6$). Let m_{ij} denotes edges connecting the vertices of degrees d_i and d_j. Two-dimensional structure of Turmeric (as shown in the Figure-1) contains edges of the type $m_{1,3}$, $m_{2,2}$, $m_{2;3}$ and $m_{3;3}$. From the figure-1, the number edges of these types are $|m_{1;3}|=6$, $|m_{2;2}|=4$, $|m_{2;3}|=14$ and $|m_{3;3}|=2$.

∴ The atom-bond connectivity index of Turmeric= ABC($C_{21}H_{20}O_6$)

$$= \sum_{uv\in E} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}}.$$

$$= |m_{1,3}|\sqrt{\frac{1+3-2}{1.3}} + |m_{2,2}|\sqrt{\frac{2+2-2}{2.2}} + |m_{2,3}|\sqrt{\frac{2+3-2}{2.3}} + |m_{3,3}|\sqrt{\frac{3+3-2}{3.3}}.$$

$$= 6 \times \sqrt{\frac{2}{3}} + 4 \times \frac{\sqrt{2}}{2} + 14 \times \frac{1}{\sqrt{2}} + 2 \times \frac{2}{3}.$$

∴ ABC($C_{21}H_{20}O_6$) = 18.96023.

Theorem.2.2. The fourth atom bond connectivity index of Turmeric is14.69752.

Proof: Let $e_{i,j}$denotes the edges of Caffeine with i = S_u and j = S_v. It is easy to see that the summation of degrees of edge endpoints of Turmeric have five edge types $e_{3,5}$, $e_{3,6}$, $e_{5,5}$, $e_{5,6}$,and $e_{6,6}$ as shown in the following figure-1.clearly from the figure -1, $|e_{3,5}| = 2$, $|e_{3,6}| = 4$, $|e_{5,5}| = 6$, $|e_{5,6}| = 1$,$|e_{5,6}|= 8$ and $|e_{6,6}| = 6$.

The fourth atom-bond connectivity index of Turmeric= ABC$_4$($C_{21}H_{20}O_6$).

$$= \sum_{uv\in E(G)} \sqrt{\frac{s_u + s_v - 2}{s_u s_v}}.$$

$$= |m_{3,5}|\left(\sqrt{\frac{3+5-2}{3.5}}\right) + |m_{3,6}|\left(\sqrt{\frac{3+6-2}{3.6}}\right) + |m_{5,5}|\left(\sqrt{\frac{5+5-2}{5.5}}\right) + |m_{5,6}|\left(\sqrt{\frac{5+6-2}{5.6}}\right) + |m_{6,6}|\left(\sqrt{\frac{6+6-2}{6.6}}\right).$$

$$= 2 \times \sqrt{\frac{6}{15}} + 4 \times \sqrt{\frac{7}{18}} + 6 \times \sqrt{\frac{8}{25}} + 8 \times \sqrt{\frac{9}{30}} + 6 \times \sqrt{\frac{10}{36}}.$$

∴ABC$_4$($C_{21}H_{20}O_6$) = 14.69752.

Theorem.2.3.The Randic connectivity index of Turmeric 11.84624.

Proof: Consider Randic connectivity index of Turmeric = χ($C_{21}H_{20}O_6$)

$$= \sum_{uv\in E(G)} \frac{1}{\sqrt{d_u d_v}}.$$

$$= |m_{1,3}|\left(\frac{1}{\sqrt{1.3}}\right) + |m_{2,2}|\left(\frac{1}{\sqrt{2.2}}\right) + |m_{2,3}|\left(\frac{1}{\sqrt{2.3}}\right) + |m_{3,3}|\left(\frac{1}{\sqrt{3.3}}\right).$$

$= 6 \times \left(\frac{1}{\sqrt{3}}\right) + 4 \times \left(\frac{1}{2}\right) + 14 \times \left(\frac{1}{\sqrt{6}}\right) + 2 \times \left(\frac{1}{3}\right).$

$\therefore \chi(C_{21}H_{20}O_6)$=11.84624

Theorem.2.4. The sum connectivity index of Turmeric is 12.0775

Proof: Consider the sum connectivity index of Turmeric = $S(C_{21}H_{20}O_6)$

$$= \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u + d_v}}.$$

$= |m_{1,3}| \left(\frac{1}{\sqrt{1+3}}\right) + |m_{2,2}| \left(\frac{1}{\sqrt{2+2}}\right) + |m_{2,3}| \left(\frac{1}{\sqrt{2+3}}\right) + |m_{3,3}| \left(\frac{1}{\sqrt{3+3}}\right).$

$= 6 \times \left(\frac{1}{2}\right) + 4 \times \left(\frac{1}{2}\right) + 14 \times \left(\frac{1}{\sqrt{5}}\right) + 2 \times \left(\frac{1}{\sqrt{6}}\right).$

$\therefore S(C_{21}H_{20}O_6)$=12.0775.

Theorem.2.5.The Geometric-Arithmetic index of Turmeric is 24.9133.

Proof: Consider the Geometric-Arithmetic index of Turmeric = $GA(C_{21}H_{20}O_6)$

$$= \sum_{e=uv \in E(G)} \frac{2\sqrt{d_u d_v}}{d_u + d_v}.$$

$= |m_{1,3}| \left(\frac{2\sqrt{1.3}}{1+3}\right) + |m_{2,2}| \left(\frac{2\sqrt{2.2}}{2+2}\right) + |m_{2,3}| \left(\frac{2\sqrt{2.3}}{2+3}\right) + |m_{3,3}| \left(\frac{2\sqrt{3.3}}{3+3}\right).$

$= 6 \times \left(\frac{2\sqrt{3}}{4}\right) + 4 \times \left(\frac{2\sqrt{4}}{4}\right) + 14 \times \left(\frac{2\sqrt{6}}{5}\right) + 2 \times \left(\frac{2\sqrt{9}}{6}\right).$

$\therefore GA(C_{21}H_{20}O_6)$= 24.9133.

Theorem.2.6. The fifth Geometric-Arithmetic index of Turmeric is 25.67460.

Proof: Consider the fifth Geometric-Arithmetic index of Turmeric = $GA_5(C_{21}H_{20}O_6)$

$$= \sum_{uv \in E(G)} \frac{2\sqrt{s_u s_v}}{s_u + s_v}.$$

$= |e_{3,5}| \left(\frac{2\sqrt{3.5}}{3+5}\right) + |e_{3,6}| \left(\frac{2\sqrt{3.6}}{3+6}\right) + |e_{5,5}| \left(\frac{2\sqrt{5.5}}{5+5}\right) + |e_{5,6}| \left(\frac{2\sqrt{5.6}}{5+6}\right) + |e_{6,6}| \left(\frac{2\sqrt{6.6}}{6+6}\right).$

$= 2 \times \left(\frac{2\sqrt{15}}{8}\right) + 4 \times \left(\frac{2\sqrt{18}}{9}\right) + 6 \times \left(\frac{2\sqrt{25}}{10}\right) + 8 \times \left(\frac{2\sqrt{30}}{11}\right) + 6 \times \left(\frac{2\sqrt{36}}{12}\right).$

$\therefore GA_5(C_{21}H_{20}O_6)$ = 25.67460.

Theorem.2.7.The First Zagreb index of Turmeric is 122.

Proof: Consider First Zagreb index of Turmeric = $Z_1(C_{21}H_{20}O_6)$

$$= \sum_{e=uv \in E(G)} (d_u + d_v).$$

$= |m_{1,3}|(1 + 3) + |m_{2,2}|(2 + 2) + |m_{2,3}|(2 + 3) + |m_{3,3}|(3 + 3).$

$= 6 \times (1 + 3) + 4 \times (2 + 2) + 14 \times (2 + 3) + 2 \times (3 + 3).$

$= 6 \times 4 + 4 \times 4 + 14 \times 5 + 2 \times 6.$

$\therefore Z_1(C_{21}H_{20}O_6)$= 122.

Theorem.2.8. The Second Zagreb index of Turmeric is 136.

Proof: The Second Zagreb index of Turmeric = $Z_2(C_{21}H_{20}O_6)$

$$= \sum_{e=uv \in E(G)} (d_u . d_v)$$

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$= |m_{1,3}|(1.3) + |m_{2,2}|(2.2) + |m_{2,3}|(2.3) + |m_{3,3}|(3.3).$

$= 6(3) + 4(4) + 14(6) + 2(9).$

$\therefore\ Z_2(C_{21}H_{20}O_6) = 136.$

Theorem.2.9. The First multiple Zagreb index of Turmericis $2.30400000 \times 10^{17}$.

Proof: The First multiple Zagreb index of Turmeric = $PM_1(C_{21}H_{20}O_6)$

$$= \prod_{e=uv\in E(G)} (d_u + d_v)$$

$$= \prod_{e=uv\in 1,3} (d_u + d_v) \prod_{e=uv\in 2,2} (d_u + d_v) \prod_{e=uv\in 2,3} (d_u + d_v) \prod_{e=uv\in 3,3} (d_u + d_v)$$

$= 4^6 \times 4^4 \times 5^{14} \times 6^2.$

$\therefore PM_1(C_{21}H_{20}O_6) = 2.30400000 \times 10^{17}.$

Theorem.2.10. The second multiple Zagreb index of Turmericis $2.63243408 \times 10^{17}$.

Proof: The second multiple Zagreb index of Turmeric = $PM_1(C_{21}H_{20}O_6)$

$$= \prod_{e=uv\in E(G)} (d_u . d_v)$$

$$= \prod_{e=uv\in 1,3} (d_u . d_v) \prod_{e=uv\in 2,2} (d_u . d_v) \prod_{e=uv\in 2,3} (d_u . d_v) \prod_{e=uv\in 3,3} (d_u . d_v).$$

$= 3^6 \times 4^4 \times 6^{14} \times 9^2.$

$\therefore PM_1(C_{21}H_{20}O_6) = 2.63243408 \times 10^{17}.$

Theorem.2.11.The Augmented Zagreb index of Turmeric is 187.03125.

Proof: The augmented Zagreb index of Turmeric = AZI(G) $(C_{21}H_{20}O_6)$

$$= \sum_{uv\in E} \left[\frac{d_u d_v}{d_u + d_v - 2}\right]^3.$$

$= |m_{1,3}|\left[\frac{1.3}{1+3-2}\right]^3 + |m_{2,2}|\left[\frac{2.2}{2+2-2}\right]^3 + |m_{2,3}|\left[\frac{2.3}{2+3-2}\right]^3 + |m_{3,3}|\left[\frac{3.3}{3+3-2}\right]^3.$

$= 6 \times \left(\frac{3}{2}\right)^3 + 4 \times \left(\frac{4}{2}\right)^3 + 14 \times \left(\frac{6}{3}\right)^3 + 2 \times \left(\frac{9}{4}\right)^3.$

$\therefore AZI(C_{21}H_{20}O_{62}) = 187.03125.$

Theorem.2.12. The harmonic index of Turmeric is 11.26667.

Proof: The harmonic index of Turmeric = $H(C_{21}H_{20}O_6)$

$$= \sum_{e=uv\in E(G)} \frac{2}{d_u + d_v}.$$

$= |m_{1,3}|\left(\frac{2}{1+3}\right) + |m_{2,2}|\left(\frac{2}{2+2}\right) + |m_{2,3}|\left(\frac{2}{2+3}\right) + |m_{3,3}|\left(\frac{2}{3+3}\right).$

$= 6 \times \left(\frac{2}{4}\right) + 4 \times \left(\frac{2}{4}\right) + 14 \times \left(\frac{2}{5}\right) + 2 \times \left(\frac{2}{6}\right).$

$H(C_{21}H_{20}O_6) = 11.26667.$

Theorem.2.13.The hyper Zagreb index of Turmeric is632.

Proof: The hyper Zagreb index of Turmeric = $HM(C_{21}H_{20}O_6)$

$$= \sum_{e=uv\in E(G)} (d_u + d_v)^2.$$

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$= |m_{1,3}|(1 + 3)^2 + |m_{2,2}|(2 + 2)^2 + |m_{2,3}|(2 + 3)^2 + |m_{3,3}|(3 + 3)^2.$

$= 6 \times 4^2 + 4 \times 4^2 + 16 \times 5^2 + 2 \times 6^2.$

$\therefore HM(C_{21}H_{20}O_6) = 632.$

Theorem.2.14. The First Zagreb polynomials ofTurmericis$2x^6 + 14x^5 + 10x^4$.

Proof: Consider First Zagreb polynomials of Turmeric = $ZG_1(C_{21}H_{20}O_6, x)$

$$= \sum_{e=uv\in E(G)} x^{d_u+d_v}.$$

$= |m_{1,3}|x^{(1+3)} + |m_{2,2}|x^{(2+2)} + |m_{2,3}|x^{(2+3)} + |m_{3,3}|x^{(3+3)}.$

$= 6 \times x^4 + 4 \times x^4 + 14 \times x^5 + 2 \times x^6.$

$\therefore ZG_1(C_8H_{10}N_4O_2, x) = 2x^6 + 14x^5 + 10x^4.$

Theorem.2.15. The Second Zagreb polynomial of Turmeric is$2x^9 + 14x^6 + 4x^4 + 6x^3$.

Proof: Consider Second Zagreb polynomials of Turmeric = $ZG_2(C_{21}H_{20}O_6, x)$

$$= \sum_{e=uv\in E(G)} x^{d_u d_v}.$$

$= |m_{1,3}|x^{(1.3)} + |m_{2,2}|x^{(2.2)} + |m_{2,3}|x^{(2.3)} + |m_{3,3}|x^{(3.3)}.$

$= 6 \times x^3 + 4 \times x^4 + 14 \times x^6 + 2 \times x^9.$

$\therefore ZG_2(C_{21}H_{20}O_6, x) = 2x^9 + 14x^6 + 4x^4 + 6x^3.$

Theorem.2.16. The Third Zagreb polynomials of Turmericis $6x^2 + 14x + 6$.

Proof: Consider Third Zagreb polynomials of Turmeric = $ZG_3(C_{21}H_{20}O_6, x)$

$$= \sum_{e=uv\in E(G)} x^{|d_u-d_v|}.$$

$= |m_{1,3}|x^{|1-3|} + |m_{2,2}|x^{|2-2|} + |m_{2,3}|x^{|2-3|} + |m_{3,3}|x^{|3-3|}.$

$= 6 \times x^2 + 4 \times x^0 + 14 \times x^1 + 2 \times x^0.$

$\therefore ZG_3(C_{21}H_{20}O_6, x) = 6x^2 + 14x + 6.$

Theorem.2.17.The Forgotten topological index of Turmeric is 310.

Proof: Consider Forgotten topological index of Turmeric = $F(C_{21}H_{20}O_6)$

$$= \sum_{e=uv\in E(G)} [(d_u)^2 + (d_v)^2]$$

$= |m_{1,3}|(1^2 + 3^2) + |m_{2,2}|(2^2 + 2^2) + |m_{2,3}|(2^2 + 3^2) + |m_{3,3}|(3^2 + 3^2).$

$= 6 \times 10 + 4 \times 8 + 14 \times 13 + 2 \times 18.$

$\therefore F(C_{21}H_{20}O_6) = 310.$

Theorem.2.18.The Forgotten polynomials of Turmericis $2x^{18} + 14x^{13} + 6x^{10} + 4x^8$.

Proof: Consider Forgotten polynomials of Turmeric = $F(C_{21}H_{20}O_6, x)$

$$= \sum_{e=uv\in E(G)} x^{[(d_u)^2+(d_v)^2]}.$$

$= |m_{1,3}|x^{(1^2+3^2)} + |m_{2,2}|x^{(2^2+2^2)} + |m_{2,3}|x^{(2^2+3^2)} + |m_{3,3}|x^{(3^2+3^2)}.$

$= 6 \times x^{10} + 4 \times x^8 + 14 \times x^{13} + 2 \times x^{18}.$

$\therefore F(C_{21}H_{20}O_6, x) = 2x^{18} + 14x^{13} + 6x^{10} + 4x^8.$

Theorem.2.19.The Symmetric division index of Turmeric is 56.333333.

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Proof: Consider Symmetric division index of Turmeric = SDD($C_{21}H_{20}O_6$)

$$= \sum_{e=uv\in E(G)} \left\{\frac{min(d_u, d_v)}{max(d_u, d_v)} + \frac{max(d_u, d_v)}{min(d_u, d_v)}\right\}.$$

$$= |m_{1,3}| \left\{\frac{min(1.3)}{max(1.3)} + \frac{max(1.3)}{min(1.3)}\right\} + |m_{2,2}| \left\{\frac{min(2.2)}{max(2.2)} + \frac{max(2.2)}{min(2.2)}\right\} + |m_{2,3}| \left\{\frac{min(2.3)}{max(2.3)} + \frac{max(2.3)}{min(2.3)}\right\} +$$

$$|m_{3,3}| \left\{\frac{min(3.3)}{max(3.3)} + \frac{max(3.3)}{min(3.3)}\right\}.$$

$$= 6 \times \frac{10}{3} + 4 \times 1 + 14 \times \frac{13}{6} + 2 \times 1.$$

∴ SDD($C_8H_{10}N_4O_2$)= 56.333333.

3. CONCLUSION

ABC index, ABC_4 index, Randic connectivity index, Sum connectivity index, GA index , GA_5 index, First Zagreb index, Second Zagreb index, First Multiple Zagreb index, Second Multiple Zagreb index, Augmented Zagreb index, Harmonic index and Hyper Zagreb index, First Zagreb polynomials, Second Zagreb polynomials, Third Zagreb polynomials, Forgotten polynomials, Forgotten topological index and Symmetric division index of Turmeric was computed.

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REVAN INDICES, F-REVAN INDEX AND F-REVAN POLYNOMIAL OF CARBON NANOTUBES

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1. ABSTRACT
Topological indices have a profound significance in chemical graph theory. Recently Kulli [15] has defined a new degree based topological index called Revan index and the concept of Revan vertex degree. In this paper, we compute Revan indices, F-Revan Index and F-Revan polynomial of certain nanotubes and nanotorus.

Keywords: Revan vertex degree, Revan indices, F-Revan index, F-Revan polynomial, first Revan vertex index, nanotubes, nanotorus.

2. INTRODUCTION
A topological index is graph invariant mathematically derived from the graph structure. The advantage of topological indices is in that they may be used directly as simple numerical structural descriptors in a comparison with physical, chemical or biological parameters used in Quantitative Structure Activity **(QSAR)** and Quantitative Structure Property **(QSPR)** study, see [4,5,7].Topological indices could be broadly classified as degree based topological indices or distance based topological indices. Degree based topological indices are based on the vertex adjacency relationship. The Zagreb, the Banhatti and the Gourava indices are the most studied degree based topological indices in chemical graph theory.

To derive the topological index for a chemical compound first their topological representation called molecular graph must be derived. A molecular graph is a collection of points representing the atoms in the molecule and set of lines representing the covalent bonds. These points are named vertices and the lines are named edges in graph theory language. Once the molecular graph is defined the edge partition table of the graph (sometimes degree partition may also be required) should be computed. The edge partition is nothing, but the classification of the edges based on the degree of the vertices they connect. The edge partition table gives the number of edges corresponding to each type of edge. The index is then computed based on the formula defined for that particular index.

Kulli defined a novel degree concept in graph theory: the Revan vertex degree and determined exact formulae for oxide and honeycomb networks. For more information and recent results about Revan indices, see [1,8,9,10,11,12,13,14,15,16].

Consider a graph G which is a finite, simple and connected with vertex set V(G) and edge set E(G). The degree deg(v) of a vertex v is the number of vertices adjacent to v. Let $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum degree among the vertices of G. The Revan vertex degree of a vertex v in G is defined as $r_G(v) = \Delta(G) + \delta(G) - \deg(v)$. The edge connecting the Revan vertices u and v will be denoted by uv. For additional definitions and notations, the reader may refer to [3].

The first, second and third Revan indices were defined as

$$R_1(G) = \sum_{uv\in E(G)} [r_G(u) + r_G(v)] \tag{1}$$

$$R_2(G) = \sum_{uv\in E(G)} r_G(u).r_G(v) \tag{2}$$

$$R_3(G) = \sum_{uv\in E(G)} |r_G(u) - r_G(v)| \tag{3}$$

The first Revan vertex index of a graph G is defined as

$$R_{01}(G) = \sum_{u\in V(G)} \left(r_G(u)\right)^2 \tag{4}$$

In [14], Kulli et. al. introduced F-Revan index and F-Revan polynomial.

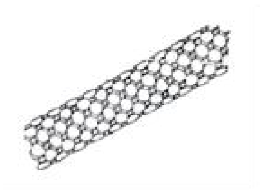
The F-Revan index of a graph G is defined as

$$F_R(G) = \sum_{uv\in E(G)} [r_G(u)^2 + r_G(v)^2] \tag{5}$$

The F-Revan polynomial of a graph G is defined as

$$F_R(G,x) = \sum_{uv\in E(G)} x^{(r_G(u)^2+r_G(v)^2)} \tag{6}$$

The carbon nanotubes are allotropes of carbon belonging to the fullerene family. Fullerenes are allotropes of carbon whose molecules consists of carbon atoms that are connected through a single or double bond forming a closed or partially closed mesh. The carbon nanotube is a class of fullerene where the closed structure is a hollow cylinder. The carbon nanotube themselves can form toroid, coils etc. The toroidal carbon nanotube is called nanotorus or nanoring. The carbon nanotubes are said to possess high mechanical stiffness, tensile strength, electrical conductivity and thermal conductivity [2,6]. This has resulted in numerous applications of nanotubes in the field of electronics, optics, nanotechnology, material sciences etc. In this paper Revan indices of **TUC_4C_8**nanotubes and nonotorus are computed. The figure 1 shows the structure of **TUC_4C_8** nanotubes and nanotorus.



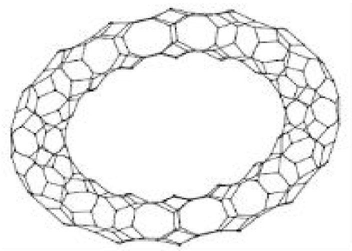


Figure-1: TUC4C8 Nanotubes (left) and Nonotorus(right)

3. $KTUC_4C_8$[M,N] NANOTUBES

In this section we consider **$KTUC_4C_8$[m,n]**nanotubes. The two dimensional lattice of **$KTUC_4C_8$[m,n]** is denoted by K=**$KTUC_4C_8$[m,n]** where n is the number of rows and m is the number of columns, see Figure 1.

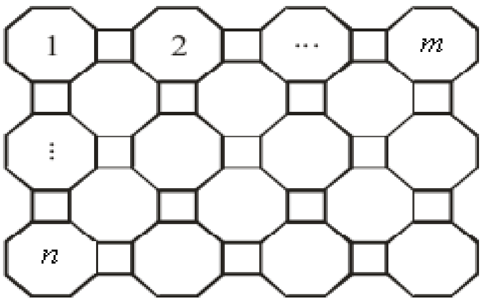


Figure-2: Two dimensional lattice of KTUC4C8[m,n] nanotube

The **$KTUC_4C_8$[m,n]** molecular graph has vertices which has either a degree of two or three. Therefore, according to Revan vertex degree definition, the vertex with degree two will have Revan vertex degree three and that with the degree three will have Revan vertex degree two.

The graph K has three types of edges depending on the Revan vertex degree of end vertices. The edge partition of graph shown in the Figure 1 is given in Table 1.

(deg(u),deg(v))	*$r_G(u)$*	*$r_G(v)$*	*Number of Edges*
(3,3)	2	2	$12mn - 8(m+n) + 4$
(2,3)	3	2	$4m + 4n - 8$
(2,2)	3	3	$2m + 2n + 4$

Table-1: Division of edgesfor K

The computations of Revan Indices and first Revan vertex index of **$KTUC_4C_8$[m,n]**are given in the following theorem

Theorem 1: The Revan indices and first Revan vertex index of K=**$KTUC_4C_8$[m,n]** are

i.$R_1(K) = 48mn$ **ii.**$R_2(K) = 48mn + 10(m+n) + 4$

iii.$R_3(K) = 4m + 4n - 8$ **iv.**$R_{01}(K) = 32mn + 20(m+n)$

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Proof

i. Using (1) and Table (1)

$$R_1(K) = 4[12mn - 8(m+n) + 4] + 5(4m + 4n - 8) + 6(2m + 2n + 4)$$

$R_1(K) = 48mn.$

ii. Using (2) and Table (1)

$$R_2(K) = 4[12mn - 8(m+n) + 4] + 6(4m + 4n - 8) + 9(2m + 2n + 4)$$

$R_2(K) = 48mn + 10(m+n) + 4.$

iii. Using (3) and Table (1)

$$R_3(K) = 0[12mn - 8(m+n) + 4] + 1(4m + 4n - 8) + 0(2m + 2n + 4)$$

$R_3(K) = 4m + 4n - 8.$

iv. The number of vertices with $\boldsymbol{r_K(u) = 3}$ are $4m + 4n$

The number of vertices with $\boldsymbol{r_K(u) = 2}$ are$8mn - 4(m+n)$

Using (4)

$$R_{01}(K) = 9(4m + 4n) + 4[8mn - 4(m+n)]$$

$R_{01}(K) = 32mn + 20(m+n).$

The computations of F-Revan Index and F-Revan polynomial of **KTUC$_4$C$_8$[m,n]**given in the following theorem

Theorem 2: The F-Revan index and F-Revan polynomial of **KTUC$_4$C$_8$[m,n]** are

i. $\boldsymbol{F_R(K) = 96mn + 24(m+n)}$

ii. $\boldsymbol{F_R(K,x) = [12mn - 8(m+n) + 4]x^8 + (4m + 4n - 8)x^{13} + (2m + 2n + 4)x^{18}}$

Proof

i. Using (5) and Table 1

$$F_R(K) = 8[12mn - 8(m+n) + 4] + 13(4m + 4n - 8) + 18(2m + 2n + 4)$$

$$F_R(K) = 96mn + 24(m+n)$$

ii. Using (6) and Table 1

$$F_R(K,x) = [12mn - 8(m+n) + 4]x^{(4+4)} + (4m + 4n - 8)x^{(4+9)} + (2m + 2n + 4)x^{(9+9)}$$

$$F_R(K,x) = [12mn - 8(m+n) + 4]x^8 + (4m + 4n - 8)x^{13} + (2m + 2n + 4)x^{18}$$

4. GTUC$_4$C$_8$[m,n] Nanotubes

In this section we consider **GTUC$_4$C$_8$[m,n]**nanotubes. The two dimensional lattice of **GTUC$_4$C$_8$[m,n]**is denoted by G= **GTUC$_4$C$_8$**[m,n] where n is the number of rows and m is the number of columns, see Figure 2.

The G=**GTUC$_4$C$_8$[m,n]**molecular graph has vertices which has either a degree of two or three. Therefore, according to Revan vertex degree definition, the vertex with degree two will have Revan vertex degree three and that with the degree three will have Revan vertex degree two.

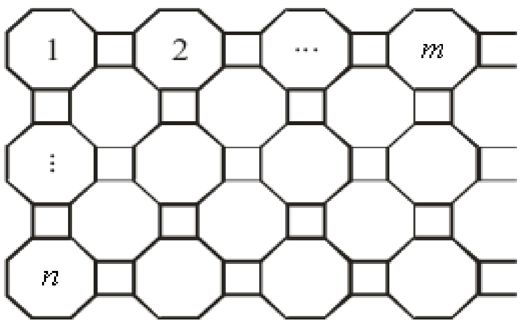

Figure-3: Two dimensional lattice of GTUC4C8[m,n] nanotube

The graph G has three types of edges depending on the Revan vertex degree of end vertices. The edge partition of graph shown in the Figure 2 is given in Table 2.

(deg(u),deg(v))	$r_G(u)$	$r_G(v)$	***Number of Edges***
(3,3)	2	2	$12mn - 8m$
(2,3)	3	2	$4m$
(2,2)	3	3	$2m$

Table-2: Division of edgesfor G

Theorem 3: The Revan indices and first Revan vertex index of G= **GTUC$_4$C$_8$[m,n]** are

i.$R_1(G) = 48mn$ **ii.**$R_2(G) = 48mn + 10m$

iii.$R_3(G) = 4m$ **iv.**$R_{01}(G) = 32mn + 20m$

Proof

i. Using (1) and Table (2)

$$R_1(G) = 4(12mn - 8m) + 5(4m) + 6(2m)$$

$R_1(G) = 48mn.$

ii. Using (2) and Table (2)

$$R_2(G) = 4(12mn - 8m) + 6(4m) + 9(2m)$$

$R_2(G) = 48mn + 10m.$

iii. Using (3) and Table (2)

$$R_3(G) = 0(12mn - 8m) + 1(4m) + 0(2m)$$

$R_3(G) = 4m.$

iv. The number of vertices with $\boldsymbol{r_G(u) = 3}$ are $\boldsymbol{4m}$

The number of vertices with $\boldsymbol{r_G(u) = 2}$ are $\boldsymbol{8mn - 4m}$

Using (4)

$$R_{01}(G) = 9(4m) + 4[8mn - 4m]$$

$R_{01}(G) = 32mn + 20m.$

Theorem 4: The F-Revan index and F-Revan polynomial of **GTUC$_4$C$_8$[m,n]** are

i. $\boldsymbol{F_R(G) = 96mn + 24m}.$

ii. $\boldsymbol{F_R(G,x) = (12mn - 8m)x^8 + (4m)x^{13} + (2m)x^{18}}$

Proof

i. Using (5) and Table 2

$$F_R(G) = 8(12mn - 8m) + 13(4m) + 18(2m)$$

$$F_R(G) = 96mn + 24m$$

ii. Using (6) and Table 2

$$F_R(G,x) = (12mn - 8m)x^{(4+4)} + (4m)x^{(4+9)} + (2m)x^{(9+9)}$$

$$F_R(G,x) = (12mn - 8m)x^8 + (4m)x^{13} + (2m)x^{18}$$

5. HTUC$_4$C$_8$[m,n]Nanotorus

In this section we consider **HTUC$_4$C$_8$**nanotorus. The two dimensional lattice of **HTUC$_4$C$_8$[m,n]** is denoted by H=**HTUC$_4$C$_8$[m,n]** where n is the number of rows and m is the number of columns, see Figure 3.

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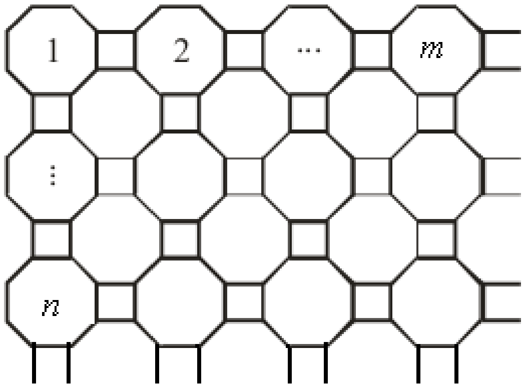

Figure-4: Two dimensional lattice of HTUC4C8[m,n] nanotorus

The H=**HTUC$_4$C$_8$[m,n]** molecular graph has vertices has degree three. Therefore, according to Revan vertex degree definition, the vertices will have Revan vertex degree three. The graph H has edges with end vertices having Revan vertex degree two. The edge partition of graph shown in the Figure 3 is given in Table 3.

(deg(u),deg(v))	*$r_G(u)$*	*$r_G(v)$*	*Number of Edges*
(3,3)	3	3	$12mn$

Table-3: Division of edgesfor H

Theorem 5: The Revan indices and first Revan vertex index of H= **HTUC$_4$C$_8$[m,n]** are

i.$R_1(H) = 72mn$ **ii.**$R_2(H) = 108mn$

iii.$R_3(H) = 0$ **iv.**$R_{01}(H) = 72mn$

Proof

i. Using (1) and Table (3)

$$R_1(H) = 6(12mn)$$

$R_1(H) = 72mn.$

ii. Using (2) and Table (3)

$$R_2(H) = 9(12mn)$$

$R_2(H) = 108mn.$

iii. Using (3) and Table (3)

$$R_3(H) = 0(12mn)$$

$R_3(H) = 0.$

iv. The number of vertices with $\boldsymbol{r_H(u) = 3}$ are $8mn$

Using (4)

$$R_{01}(H) = 9(8mn)$$

$$R_{01}(H) = 72mn$$

Theorem 6: The F-Revan index and F-Revan polynomial of H= **HTUC$_4$C$_8$[m,n]** are

i. $\boldsymbol{F_R(H) = 216mn}$.

ii. $\boldsymbol{F_R(H, x) = 12mnx^{18}}$

Proof

i. Using (5) and Table 3

$$F_R(H) = 18(12mn)$$

$$F_R(H) = 216mn$$

ii. Using (6) and Table 3

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$$F_R(H,x) = (12mn)x^{(9+9)}$$

$$F_R(H,x) = 12mnx^{18}$$

6. CONCLUSION

In this paper the Revan indices, F-Revan index and F-Revan polynomial were computed for **$KTUC_4C_8$, $GTUC_4C_8$** nanotubesand **$HTUC_4C_8$**nanotorus. The study of effectiveness of the computed indices in QSPR and QSAR needs to be taken up as a future work.

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EFFECT OF COUPLE STRESSES ON THE CHARACTERISTICS OF SQUEEZE FILM LUBRICATION BETWEEN CURVED CIRCULAR PLATES

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ABSTRACT
In this paper efforts have been directed to study the effect of couple stresses on the characteristics of squeeze film lubrication between curved circular plates. The modified Reynolds type equation is derived on the basis of Stokes couple stress fluid theory. The closed form expressions for the squeeze film pressure, load carrying capacity and the time height relation are obtained. The results are presented for both concave and convex curved circular plates. The results are compared with the corresponding Newtonian case. It is found that the effect of couple stresses is to increase the load carrying capacity and to lengthen the squeeze film time as compared to the corresponding Newtonian fluid case.

Keywords: Couple stress; Squeeze film, Curved circular plates.

1. INTRODUCTION

The squeeze film phenomena are widely observed in machine tools, gears, bearings, rolling elements and biolubrication. In view of their wide-ranging applications, numerous theoretical and experimental studies have been made [1-7]. Recently Perkins and Wollom [8] made an experimental study of behaviour of an oscillating oil squeeze film. Most of the theoretical studies on squeeze film lubrication between the plane parallel plates or between the curved circular plates are based on the Newtonian constitutive approximation for the lubricants. However this approximation is not a satisfactory engineering approach for most of the practical problems in lubrication. Hence the non-Newtonian property of the lubricant must be taken into account in any realistic study of these bearings. The modern lubricants exhibiting non-Newtonian behaviour are the fluids containing long chained polymer additives. The microcontinuum theory derived by Stokes [9] is the simplest generalization of the classical theory of fluids, which allows for polar effects such as the presence of couple stresses and body couples. Many investigators have used the couple stress fluid theory to analyze the performance of various bearing systems [10-14]. These studies have led to the perditions such as larger load carrying capacity, lower coefficient of friction and delayed time of approach in comparison with the Newtonian case. However the effect of couple stresses on the squeeze film lubrication between curved circular plates has not been studied so for. Hence in the present paper, a theoretical investigation is made to study the effect of couple stresses on the squeeze film characteristics of curved (concave/convex) circular plates.

2. MATHEMATICAL FORMULATION AND SOLUTION OF THE PROBLEM

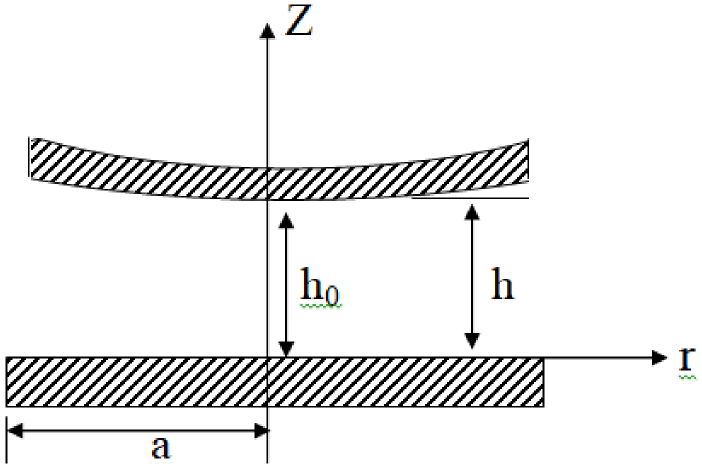

Fig-1: Geometry and configuration of curved circular plates

Figure 1 shows the configuration of the squeeze film geometry under consideration. The upper curved circular plate approaching the lower flat circular plate with a uniform squeezing velocity $\frac{dh_0}{dt}$. The lubricant in the film region is assumed to be a Stokes [9] couple stress fluid. It is also assumed that, the body forces and body couples are absent. Under the usual assumption of hydrodynamic lubrication applicable to thin film [15], the equation of motion for couple stress fluid take the form

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$$\frac{\partial p}{\partial r} = \mu \frac{\partial^2 u}{\partial z^2} - \eta \frac{\partial^4 u}{\partial z^4}, \qquad (1)$$

$$\frac{\partial p}{\partial z} = 0, \qquad (2)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0, \qquad (3)$$

where p is the presser, μ is the Newtonian viscosity, η is the material constant characterizing the couple stress and is of dimension of momentum. The ratio $(\frac{\eta}{\mu})$ has the dimension of length Square and hence characterizes the material length of the fluid.

The film thickness h (r) is defined as;

$$h = h_0 e^{-\beta r^2}. \qquad (4)$$

It is to be noted that h is axisymmetric about $r = 0$; also an amazing array of film shapes can be generated by merely choosing an appropriate value for the constant β. Convex film can be generated for β <0 and concave ones for β >0.

The relevant boundary conditions for the velocity components are;

$$u = 0, \quad \frac{\partial^2 u}{\partial z^2} = 0, \qquad (5)$$

$$w = 0, \qquad (6)$$

at z = 0 and

$$u = 0, \frac{\partial^2 u}{\partial z^2} = 0, \qquad (7)$$

$$w = \frac{\partial h}{\partial t}, \qquad (8)$$

at z = h .

Since p is independent of z because of equation (2). The solution of equation (1) subject to boundary conditions (5) and (7) is

$$u = \frac{1}{2\mu}\frac{dp}{dr}\left[z^2 - zh + \frac{2}{l^2}\left(1 - \frac{\cosh(\frac{1}{2}(2z-h)l)}{\cosh(\frac{lh}{2})} \right) \right], \qquad (9)$$

where $l = \left(\frac{\eta}{\mu}\right)^{1/2}$ is the couple stress parameter.

Integration of equation (3) across the fluid film and the use of boundary conditions (6) and (8) and an expression (9) for u, gives the modified Reynolds equation in the form

$$\frac{1}{r}\frac{d}{dr}\left[r f(h,l) \frac{dp}{dr} \right] = 2\mu \frac{dh}{dt}, \qquad (10)$$

where

$$\boldsymbol{f}(\boldsymbol{h},\boldsymbol{l}) = \frac{\boldsymbol{h}^3}{6} - \frac{2}{\boldsymbol{l}^2}\left[\boldsymbol{h} - \frac{2}{\boldsymbol{l}}\tanh\left(\frac{\boldsymbol{lh}}{2}\right) \right].$$

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The relevant boundary conditions for the pressure are

$$p = 0 \quad \text{at} \quad r = a \tag{11}$$

$$\frac{\partial p}{\partial r} = 0 \quad \text{at} \quad r = 0 \tag{12}$$

Introducing the non – dimensional variables and parameters

$$\bar{\boldsymbol{h}} = \frac{\boldsymbol{h}}{\boldsymbol{h}_0} = \boldsymbol{e}^{-\bar{\beta}\mathrm{R}^2} \;,\; \mathrm{R} = \frac{\mathrm{r}}{\mathrm{a}} \;,\; \bar{\mathrm{p}} = \frac{\mathrm{p}\mathrm{h}_0^3}{\mathrm{a}^2\,\mu\,\dot{\mathrm{h}}_0} \;,\; \tau = \frac{\boldsymbol{l}}{\mathrm{h}_0} \;,\; \bar{\beta} = \beta\,\mathrm{a}^2 \tag{13}$$

in to (10), (11), (12) and once integration of (10) with respect to r and the use of boundary condition (12) yields

$$\frac{d\bar{p}}{dR} = \frac{R}{f(\bar{h},\tau)} \;, \tag{14}$$

where

$$\boldsymbol{f}(\bar{\boldsymbol{h}},\tau) = \frac{\bar{\boldsymbol{h}}^3}{6} - \frac{2}{\tau^2}\left\{\bar{\boldsymbol{h}} - \frac{2}{\tau}\tanh\left(\frac{\tau\,\bar{\mathrm{h}}}{2}\right)\right\}.$$

The non-dimensional fluid film pressure $\bar{p}$ is obtained by solving equation (14) numerically with the condition (11).

Load capacity

The load carrying capacity of the squeeze film is obtained by integrating the pressure over the plates

$$W = 2\pi\int_0^a prdr \;, \tag{15}$$

which in non-dimensional form is given by

$$\bar{W} = \frac{Wh_0^3}{2\pi\mu\left|\dot{h}_0\right|a^4} = -\int_0^1 R\bar{p}dR \;. \tag{16}$$

Time-Height Relationship

The time taken to attain the film thickness $\boldsymbol{h}_{02}$ from an initial film thickness h_{01} under a constant load $\bar{W}$ is obtained from (16) as

$$\Delta t = \left(\frac{1}{h_{02}^2 - h_{01}^2}\right)\frac{\bar{W}}{W}2\pi\mu\,\mathrm{a}^4 \;, \tag{17}$$

which in non-dimensional form is given by

$$\Delta T = \frac{\Delta t W h_0^2}{2\pi\mu\,\mathrm{a}^4} = \left(\frac{1}{\bar{h}_{02}^2} - \frac{1}{\bar{h}_{01}^2}\right)\bar{W} \tag{18}$$

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3. RESULTS AND DISCUSSIONS

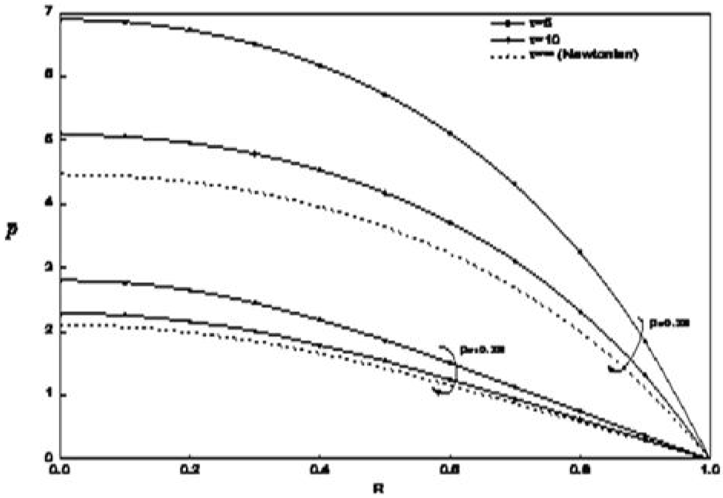

Fig. 2 Variation of non-dimensional pressure $\bar{p}$ as a function of R for different values of τ with β=0.25 and -0.25

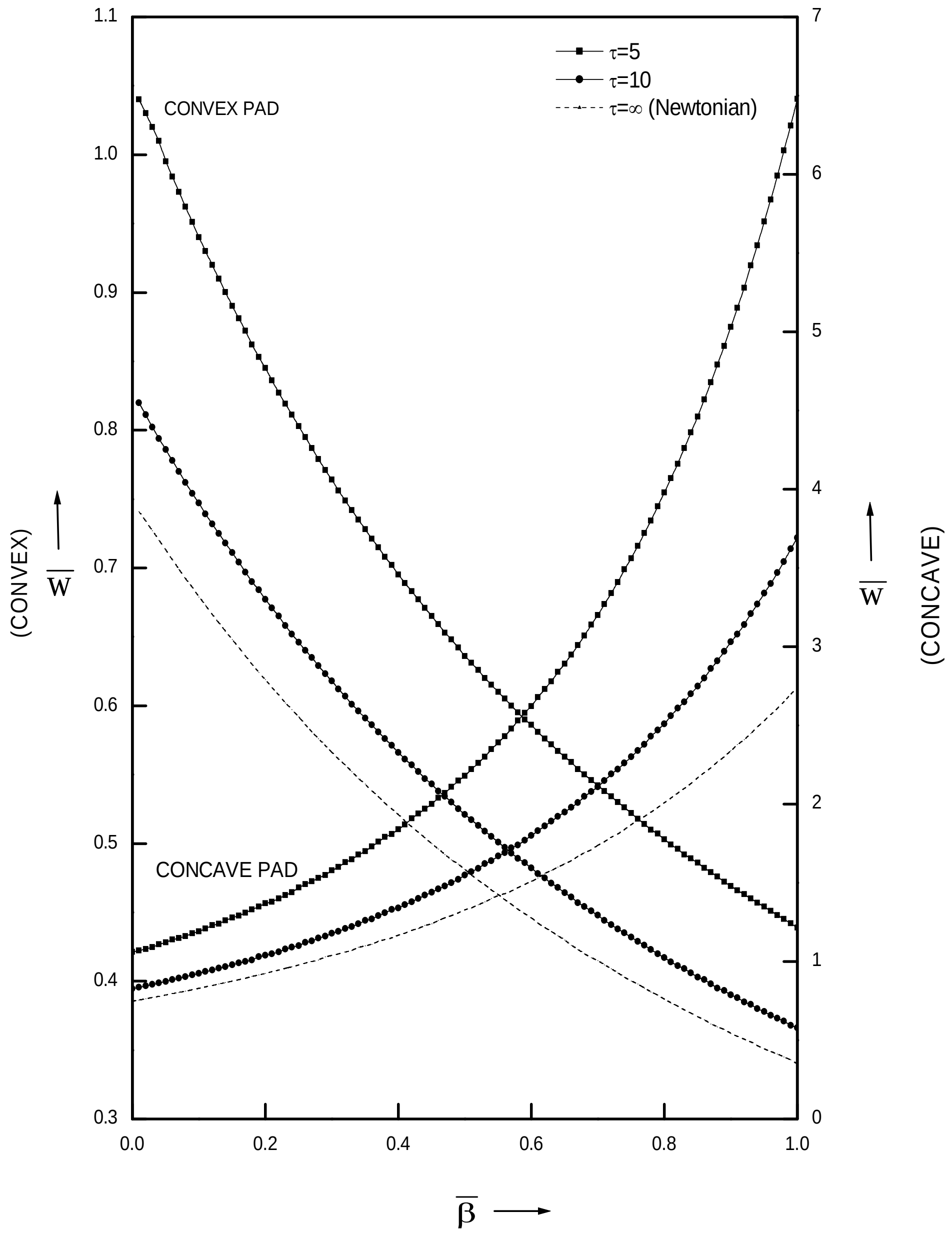

Fig. 3 Non-dimensioanl load $\overline{W}$ as a function of normalized curvature parameter $\overline{\beta}$ for different couplestress parameter τ

In the present paper, the effect of couple stresses on the characteristics of squeeze film lubrication between a curved circular plate and a flat plate is analyzed. The squeeze film characteristics are the functions of the couple stress parameter $\tau (= \frac{1}{h_0}\left(\frac{\eta}{\mu}\right)^{\frac{1}{2}})$ and the non-dimensional curvature parameter $\overline{\beta}$. The negative values of $\overline{\beta}$ ($\overline{\beta}$ <0) produce the convex pad and positive values of $\overline{\beta}$ generates the concave pad. Since the ratio $\frac{\eta}{\mu}$ has the dimension of length squared and h_0 being the minimum film thickness, the non-dimensional couple stress parameter τ gives a mechanism of interaction of the fluid with the bearing geometry. The numerical value of τ depends both on the chain length of the polar additives, $\left(\frac{\eta}{\mu}\right)^{\frac{1}{2}}$ and the minimum film thickness h_0. It is expected that the polar effects would be prominent either when the microstructure size of the polar additives is large or when the minimum film thickness is small. i.e. when τ is small.

The variation of non-dimensional pressure $\overline{P}$ as a function of the non-dimensional radial co-ordinate R for different values of τ is depicted in Fig.2 for both concave ($\overline{\beta}$ =0.25) and convex ($\overline{\beta}$ = -0.25) pad geometries. The dotted curves in the graph corresponds to the Newtonian case $(\tau \rightarrow \infty)$. It is found that the effect of couple stress is to increase the squeeze film pressure as compared to the corresponding Newtonian case. Further, the increase in $\overline{p}$ is more pronounced in the case of concave pad ($\overline{\beta}$ =0.25) as compared to the convex pad ($\overline{\beta}$ = -0.25).

Figure 3 shows the variation of non-dimensional load $\overline{W}$ with the nondimensional curvature parameter $\overline{\beta}$ for different values of τ for both concave and convex pads. The dotted curve in the graph corresponds to the Newtonian case. An increased $\overline{W}$ is observed for the couple stress fluid as compared to the corresponding Newtonian case for both concave and convex pad geometries. As the non-dimensional curvature parameter $\overline{\beta}$ increases numerically the non-dimensional load $\overline{W}$ decreases for convex pads, whereas it increases for concave pads. The relative increase in $\overline{W}$ as compared to the Newtonian case is observed through the relative difference coefficient $\boldsymbol{R}_{\overline{W}}\left(= \frac{\overline{\boldsymbol{W}}_{coplestress} - \overline{\boldsymbol{W}}_{Newtonian}}{\overline{\boldsymbol{W}}_{Newtonian}} \times 100\right)$. The values of $R_{\overline{W}}$ for various values of $\overline{\beta}$ and for two values of τ is presented in Table 1. It is found that an increase of nearly 73 % in $\overline{W}$ for $\overline{\beta}$ =0.6 and τ =5.

The squeeze film time for any curved plate can be easily calculated from equation (18) by reading appropriate value of $\overline{W}$ from Figure 3 and substituting it in equation (18) for the given value of couple stress parameter τ.

Table-1: Relative load $R_{\overline{W}}$

$\overline{\beta}$	$R_{\overline{W}}$	
	τ=5	τ=10
-0.6	31.390	8.071
-0.4	33.397	8.637
-0.2	36.510	9.369
0.0	40.720	10.440
0.2	48.108	12.432
0.4	57.264	14.529
0.6	73.509	19.205

4. CONCLUSIONS

The squeeze film lubrication between a curved circular plate and a plane circular plate with couple stress fluid is presented on the basis of Stokes microcontinuum theory for couple stress fluids. As the couple stress parameter $\tau \rightarrow \infty$ the squeeze film characteristics presented in this paper agree with those of Newtonian case studied by Murti [6]. It is found that the effect of couple stresses is to increase the squeeze film pressure, the load carrying capacity and the squeeze film time as compared to the corresponding Newtonian case. These results are more pronounced for concave pads. Hence the squeeze film bearings with couple stress fluid sustain larger load for a longer time as compared to the Newtonian case, by which improves the bearing performance.

$$\boldsymbol{f}(\overline{\boldsymbol{h}},\tau)=\frac{\overline{\boldsymbol{h}}^{3}}{6}-\frac{2}{\tau^{2}}\left\{\overline{\boldsymbol{h}}-\frac{2}{\tau}\tanh\left(\frac{\tau\,\overline{\mathrm{h}}}{2}\right)\right\}$$

NOMENCLATURE

a	outer radius of the plate
h	fluid film thickness ($=h_0 e^{-\beta r^2}$)
h_0	minimum film thickness
h_{01}	initial central film thickness
h_{02}	final central film thickness
$\overline{h}$	non- dimensional film height $(=h/h_0)$
$\overline{h}_{01}$	non-dimensional initial central film thickness $\left(=\frac{\boldsymbol{h}_{01}}{\boldsymbol{h}_0}\right)$
$\overline{h}_{02}$	non-dimensional final central film thickness $\left(=\frac{\boldsymbol{h}_{02}}{\boldsymbol{h}_0}\right)$
l	couple stress parameter $(=\sqrt{\eta/\mu})$
p	pressure in the film region
$\overline{p}$	non-dimensional pressure in the fluid region $\left(=\frac{ph_0^3}{a^2\mu\,\dot{h}_0}\right)$
r	radial co ordinate
Δt	time taken for a reduction in central film thickness from $\boldsymbol{h}_{01}$ to $\boldsymbol{h}_{02}$
ΔT	non-dimensional squeeze film time $\left(=\frac{\Delta t W h_0^2}{2\pi\mu\,\mathrm{a}^4}\right)$
W	load carrying capacity
$\overline{W}$	non-dimensional load carrying capacity $\left(=\frac{\boldsymbol{W}\,\boldsymbol{h}^3}{2\pi\mu\lvert\dot{\boldsymbol{h}}_0\rvert\boldsymbol{a}^4}\right)$
β	curvature parameter
$\overline{\beta}$	non-dimensional curvature parameter (= $\beta\ a^2$)
μ	isotropic viscosity
τ	non-dimensional couple stress parameter $(=l/h_0)$

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OHMIC HEATING EFFECT ON PERISTALTIC FLOW WITH DOUBLE-DIFFUSIVE CONVECTION OF NANOFLUID

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ABSTRACT

In the present investigation, we have studied the impact of ohmic heating on the peristaltic flow of nanofluid and double-diffusive convection through the channel under long wavelength and low Reynolds number assumptions. There are many practical uses of ohmic heating such as conductors in electronics, electronic stoves, and other electric heaters. The heat and mass transfer occurs concurrently with the complicity of the fluid motion is known as double diffusion. Analytical solutions have been calculated by using Homotopy Analysis Method for velocity, pressure rise, temperature, solutul (species) concentration and nanoparticle volume fraction. The transport equations involves the combined effects of Brownian motion and thermophoresis diffusion of nanoparticles. Physical analysis has been carried out through graphs.

Keywords: Peristaltic Transport, Nanofluid, Ohmic heating, Double diffusion.

1. INTRODUCTION

In rheology, the fluids can easily transport from one place to another place with supporting of peristaltic pumping. This type of pumping is known as peristalsis. Peristalsis helps in the transporting physiological fluids in the human body such as swallowing of food through the esophagus and the vasomotion of small blood vessels. Latham [1] was first initiated the concept of peristaltic mechanism in 1966. After the work of Latham, Jaffrin et al. [2] explore the peristaltic pumping system. Many researchers and scientist diverted their research interest towards the study of peristaltic transport by considering viscous and non-viscous fluids with different geometries. Some of these investigations have been reported in the references list [3-6].

The word “nanofluid” was first formulated by Choi in 1995 [7]. Nanofluid is a liquid containing nanoparticles with a representative length of 1-100nm [8]. The study of nanotechnology based on nanofluids has received considerable attention due to its applications in engineering and biomedical. In biomedical, magnetite nanoparticles are targeted for magnetic resonance imaging (MRI) and during drug delivery. Researchers studied the connection of peristalsis and nanofluid problems with different geometries. Some of these investigations have been reported in the references list [9-10]. Joule heating is also known as ohmic heating and resistive heating. Ohmic heating is the process by which the passage of an electric current through a conductor produces heat. There are many practical uses of joule heating such as conductors in electronics, electric stoves, and other electric heaters, power lines, fuses. The influence of Joule heating on peristaltic flow investigated is mentioned by many researchers [11-13].

In recent years, researchers and scientists have been focused on double-diffusive convection on peristaltic transport. Peristaltic pumping with double-diffusive natural convective nanofluid studied by Noreen et al. [14]. The heat and mass transfer occurs concurrently and it leads to a complex fluid motion is called double diffusion. Double diffusion has important applications in solid-state physics, chemical engineering, geophysics, oceanography, astrophysics and biology as well as in many engineering applications such as natural gas storage tanks, solar ponds, metal solidification processes, and crystal manufacturing. The research on double diffusion was continued by Nield et al. [15] by considering the onset of double diffusion-convection over a nanofluid layer.

The problem considered in the present research work has a potential industrial application, biomedical and engineering. The investigation of double-diffusive convection on peristaltic transport is the innovative idea of research. The detailed literature survey points out that many researchers have worked on peristaltic transport, independently. However, no research is carried out to exhibit the effect of ohmic heating over a channel in the presence of nanofluid. Moreover, the concept of adding Double-diffusive problems has so many practical applications in chemical engineering, geophysics, oceanography, astrophysics, and biology. Also, the consideration of ohmic heating. Further, the non-linear partial differential equations modeled in the paper are solved by utilizing non-dimensional quantities. Those equations are solved by utilizing the semi-analytical technique is Homotopy Analysis Method [16] and Physical analysis has been carried out through graphs.

2. MODELING

Let us consider the peristaltic transport of nanofluid through the channel with the sinusoidal wave propagating towards down its walls. Here we consider the Cartesian coordinate system ($\tilde{X}, \tilde{Y}$) such that $\tilde{X}$-axis is considered

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along the center line in the direction of wave propagation and $\tilde{Y}$-axis is transverse. The geometry of the wall surface can be written as

$$\left.\begin{aligned} \tilde{Y} = \tilde{h}_1 &= d_1 + a_1 \cos\left[\frac{2\pi}{\lambda}\left(\tilde{X} - c\tilde{t}\right)\right], \\ \tilde{Y} = \tilde{h}_2 &= -d_2 - b_1 \cos\left[\frac{2\pi}{\lambda}\left(\tilde{X} - c\tilde{t}\right) + \phi\right]. \end{aligned}\right\} \tag{1}$$

Here a_1 and b_1 are the amplitudes of the waves, λ is the wavelength, ϕ is the phase difference with d_1, d_1 are satisfies the below condition.

$$a_1^2 + b_1^2 + 2a_1 b_1 \cos\phi \le \left(d_1^2 + d_2^2\right).$$

The velocity components $\tilde{U}$and $\tilde{V}$ along with the $\tilde{X}$ and $\tilde{Y}$ directions, respectively, in the fixed frame, the velocity field V is taken as

$$V = \left[\tilde{U}\left(\tilde{X}, \tilde{Y}, \tilde{t}\right), \tilde{V}\left(\tilde{X}, \tilde{Y}, \tilde{t}\right), 0\right] \tag{2}$$

The non-dimensional governing equations are

The continuity equation:

$$\frac{\partial \tilde{U}}{\partial \tilde{X}} + \frac{\partial \tilde{U}}{\partial \tilde{Y}} = 0 \tag{3}$$

The momentum equation:

$$\begin{aligned} \rho_f \left(\frac{\partial \tilde{U}}{\partial \tilde{t}} + \tilde{U}\frac{\partial \tilde{U}}{\partial \tilde{X}} + \tilde{V}\frac{\partial \tilde{U}}{\partial \tilde{Y}}\right) &= -\frac{\partial \tilde{p}}{\partial \tilde{X}} + \mu\left(\frac{\partial^2 \tilde{U}}{\partial \tilde{X}^2} + \frac{\partial^2 \tilde{U}}{\partial \tilde{Y}^2}\right) + \left(1 - \phi_1\right)\rho_f g\beta\left(\tilde{T} - \tilde{T}_0\right) \\ &\quad -\left(\rho_p - \rho_f\right) g\left(\tilde{C} - \tilde{C}_0\right) - g\left(\rho_p - \rho_{f0}\right)\left(\tilde{F} - \tilde{F}_0\right), \end{aligned} \tag{4}$$

$$\rho_f \left(\frac{\partial \tilde{V}}{\partial \tilde{t}} + \tilde{U}\frac{\partial \tilde{V}}{\partial \tilde{X}} + \tilde{V}\frac{\partial \tilde{V}}{\partial \tilde{Y}}\right) = -\frac{\partial \tilde{p}}{\partial \tilde{Y}} + \mu\left(\frac{\partial^2 \tilde{V}}{\partial \tilde{X}^2} + \frac{\partial^2 \tilde{V}}{\partial \tilde{Y}^2}\right). \tag{5}$$

The thermal energy equation:

$$\begin{aligned} (\rho c)_f \left(\frac{\partial \tilde{T}}{\partial \tilde{t}} + \tilde{U}\frac{\partial \tilde{T}}{\partial \tilde{X}} + \tilde{V}\frac{\partial \tilde{T}}{\partial \tilde{Y}}\right) &= k^*\left(\frac{\partial^2 \tilde{T}}{\partial \tilde{X}^2} + \frac{\partial^2 \tilde{T}}{\partial \tilde{Y}^2}\right) + (\rho c)_p D_B\left(\frac{\partial \tilde{C}}{\partial \tilde{X}} + \frac{\partial \tilde{C}}{\partial \tilde{Y}}\right)\left(\frac{\partial \tilde{T}}{\partial \tilde{X}} + \frac{\partial \tilde{T}}{\partial \tilde{Y}}\right) \\ &\quad + (\rho c)_p \frac{D_T}{T_m}\left(\frac{\partial^2 \tilde{T}}{\partial \tilde{X}^2} + \frac{\partial^2 \tilde{T}}{\partial \tilde{Y}^2}\right)^2 + \Phi \end{aligned} \tag{6}$$

The solutal concentration equation:

$$\frac{\partial \tilde{C}}{\partial \tilde{t}} + \tilde{U}\frac{\partial \tilde{C}}{\partial \tilde{X}} + \tilde{V}\frac{\partial \tilde{C}}{\partial \tilde{Y}} = D_S\left(\frac{\partial^2 \tilde{C}}{\partial \tilde{X}^2} + \frac{\partial^2 \tilde{C}}{\partial \tilde{Y}^2}\right) + D_{CT}\left(\frac{\partial^2 \tilde{T}}{\partial \tilde{X}^2} + \frac{\partial^2 \tilde{T}}{\partial \tilde{Y}^2}\right) \tag{7}$$

The nanoparticle volume fraction equation:

$$\frac{\partial \tilde{F}}{\partial \tilde{t}} + \tilde{U}\frac{\partial \tilde{F}}{\partial \tilde{X}} + \tilde{V}\frac{\partial \tilde{F}}{\partial \tilde{Y}} = D_B\left(\frac{\partial^2 \tilde{F}}{\partial \tilde{X}^2} + \frac{\partial^2 \tilde{F}}{\partial \tilde{Y}^2}\right) + \frac{D_T}{T_m}\left(\frac{\partial^2 \tilde{T}}{\partial \tilde{X}^2} + \frac{\partial^2 \tilde{T}}{\partial \tilde{Y}^2}\right). \tag{8}$$

Where $\tilde{p}$ is the pressure, ρ_f is the density of the fluid, Φ is the constant heat addition/absorption, g is the acceleration due to gravity, κ is the volume expansion coefficient, $\tilde{T}$ is the temperature of the fluid, $\tilde{C}$ is the solutal concentration, $\tilde{F}$ is nanoparticle volume fraction, $(\rho c)_f$ is the heat capacity of the fluid, k^* is the thermal conductivity, $(\rho c)_p$ is the effective heat capacity of the nanoparticle material, D_B is the Brownian diffusion coefficient and D_T is Thermophoresis diffusion, T_m is the fluid mean temperature, D_S is the solutal diffusivity, D_{TC} and D_{CT} are Dufour diffusivity and Soret diffusivity.

The relations between the laboratory and wave frame are introduced through

$$\tilde{x} = \tilde{X} - c\tilde{t}, \qquad \tilde{y} = \tilde{Y}$$
$$\tilde{u}(\tilde{x},\tilde{y}) = \tilde{U} - c, \qquad \tilde{v}(\tilde{x},\tilde{y}) = \tilde{V} \tag{9}$$

Where $(\tilde{u},\tilde{v})$ and $(\tilde{x},\tilde{y})$ indicate the velocity components and coordinates in the wave frame.

The corresponding boundary conditions for the above problem

$$\left.\begin{aligned} &\tilde{u} = -c,\ \tilde{T} = \tilde{T}_0,\ \tilde{C} = \tilde{C}_0,\ \tilde{F} = \tilde{F}_0 \text{ at } \tilde{y} = \tilde{h}_1 = d_1 + a_1 \cos\left(\frac{2\pi}{\lambda}\left(\tilde{x} - c\tilde{t}\right)\right), \\ &\tilde{u} = -c,\ \tilde{T} = \tilde{T}_1, \tilde{C} = \tilde{C}_1,\ \tilde{F} = \tilde{F}_1 \text{ at } \tilde{y} = \tilde{h}_2 = -d_2 - b_1 \cos\left(\frac{2\pi}{\lambda}\left(\tilde{x} - c\tilde{t}\right) + \phi\right). \end{aligned}\right\} \tag{10}$$

Introducing the following dimensionless variables

$$\left.\begin{aligned} &x = \frac{\tilde{x}}{\lambda},\ y = \frac{\tilde{y}}{b},\ t = \frac{c\tilde{t}}{\lambda},\ p = \frac{b^2\tilde{p}}{c\lambda\mu}, v = \frac{\tilde{v}}{c},\ \delta = \frac{b}{\lambda},\ u = \frac{\tilde{u}}{c}, \mathrm{Re} = \frac{\rho_f cb}{\mu},\ \beta^* = \frac{k^*}{(\rho c)_f}, \mathrm{Pr} = \frac{\nu}{\beta^*}, \\ &Gr_T = \frac{\left(1-\phi_1\right)\rho_f g\beta b^2\left(\tilde{T}_1 - \tilde{T}_0\right)}{c\mu},\ Gr_C = \frac{\left(\rho_p - \rho_f\right)\left(\tilde{C}_1 - \tilde{C}_0\right)}{\left(1-\phi_1\right)\beta\left(\tilde{T}_1 - \tilde{T}_0\right)\rho_f},\ Q_0 = \frac{a^2\Phi}{\left(\tilde{T}_1 - \tilde{T}_0\right)(\rho c)_f}, \\ &Nb = \frac{(\rho c)_p D_B\left(\tilde{C}_1 - \tilde{C}_0\right)}{(\rho c)_f \nu},\ Nt = \frac{(\rho c)_p D_T\left(\tilde{T}_1 - \tilde{T}_0\right)}{(\rho c)_f T_m \nu}, \varphi = \frac{\tilde{C} - \tilde{C}_0}{\tilde{C}_1 - \tilde{C}_0}\ \theta = \frac{\tilde{T} - \tilde{T}_0}{\tilde{T}_1 - \tilde{T}_0}, \gamma = \frac{\tilde{F} - \tilde{F}_0}{\tilde{F}_1 - \tilde{F}_0} \\ &N_{TC} = \frac{(\rho c)_f D_{TC}\left(\tilde{C}_1 - \tilde{C}_0\right)}{k^*\left(\tilde{T}_1 - \tilde{T}_0\right)}\ N_{CT} = \frac{D_{CT}\left(\tilde{T}_1 - \tilde{T}_0\right)}{D_S\left(\tilde{T}_1 - \tilde{T}_0\right)}, Gr_F = \frac{\left(\rho_p - \rho_{f0}\right) g b^2\left(\tilde{F}_1 - \tilde{F}_0\right)}{c\mu} \end{aligned}\right\}$$

(11)

where Re is Reynolds number, Pr is Prandtl number, Q_0 is the heat source or sink parameter, Gr_c is the solutal Grashof number, Gr_T is the thermal Grashof number, Gr_F is the nanoparticle Grashof number, Nb is Brownian motion parameter, Nt is thermophoresis parameter, N_{TC} is Dufour parameter and N_{CT} is the Soret parameter.

The above non dimensional equations using non-dimensional quantities can be written

$$\frac{\partial p}{\partial x} = \frac{\partial^2 u}{\partial y^2} + Gr_T\theta + Gr_C\varphi - Gr_F\gamma, \tag{12}$$

$$\frac{\partial p}{\partial y} = 0, \tag{13}$$

$$\frac{\partial^4\psi}{\partial y^4} + Gr_T\frac{\partial\theta}{\partial y} + Gr_C\frac{\partial\varphi}{\partial y} - Gr_F\frac{\partial\gamma}{\partial y} = 0, \tag{14}$$

$$\frac{\partial^2\theta}{\partial y^2} + N_{TC}\,\mathrm{Pr}\frac{\partial^2\varphi}{\partial y^2} + Nb\,\mathrm{Pr}\frac{\partial\theta}{\partial y}\frac{\partial\gamma}{\partial y} + Nt\,\mathrm{Pr}\left(\frac{\partial\theta}{\partial y}\right)^2 + \mathrm{Pr}\,Q_0 = 0, \tag{15}$$

$$\frac{\partial^2\varphi}{\partial y^2} + N_{CT}\frac{\partial^2\theta}{\partial y^2} = 0, \tag{16}$$

$$\frac{\partial^2\gamma}{\partial y^2} + \frac{Nt}{Nb}\frac{\partial^2\theta}{\partial y^2} = 0. \tag{17}$$

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Corresponding boundary conditions are

$$\left.\begin{aligned}
&\psi=\frac{F^*}{2}, \quad \frac{\partial\psi}{\partial y}=-1, \quad \theta=0, \quad \varphi=0, \quad \gamma=0 \quad \text{at} \quad y=h_1=1+a\cos x,\\
&\psi=-\frac{F^*}{2}, \quad \frac{\partial\psi}{\partial y}=-1, \quad \theta=1, \quad \varphi=1 \quad \gamma=0 \quad \text{at} \quad y=h_2=-d-b\cos(x+\phi).
\end{aligned}\right\} \tag{18}$$

3. RESULTS AND DISCUSSION

From the above mentioned description of the considered problem the system of non linear coupled partial differential equations is obtained. Such system is difficult to solve explicitly to get exact solutions. However the advancement of techniques during last few decades has provided more efficient ways to solve the complex non-linear models. Thus the problem in hand is approximated semi-analytically using Homotopy Analysis Method in Mathematica software and graphical results are drawn in Origin. Therefore this section comprises the development of velocity, pressure rise, temperature, solutal(species) concentration and nanoparticle volume fraction corresponding to variation of Q_0 is the heat source or sink parameter, Gr_c is the solutal Grashof number, Gr_T is the thermal Grashof number, Gr_F is the nanoparticle Grashof number, Nb is Brownian motion parameter, Nt is thermophoresis parameter, N_{TC} is Dufour parameter and N_{CT} is the Soret parameter.

3.1. Velocity distribution

The effects of Gr_C , Gr_F and Gr_T on velocity profile $u(y)$ are presented through the figures 1 to 3. It is observed from Fig. 1 that velocity in all the regions of the peristaltic pumping decreases with an increase in the values of Gr_C. Fig. 2 demonstrate the behaviour of velocity profile for various values of nanoparticle Grashof number Gr_F. It is depicted from Fig. 2 that the velocity profile increases with an increasing values of Gr_F. The variation of velocity for different values of thermal Grashof number Gr_T is shown in Fig. 3. It is observed from Fig. 3 that velocity in all the regions of the peristaltic pumping decreases with an increasing values of Gr_T.

3.2. Pressure rise

Figures 4-6 are prepared to analyze the effects of Gr_C , Gr_F, Gr_T and λ_1 on pressure rise. It is observed from Fig. 4 that pressure rise in all the regions of the peristaltic pumping increases with an increase in the values of Gr_C. Fig. 5 demonstrate the behaviour of pressure rise for various values of nanoparticle Grashof number Gr_F. It is depicted from Fig. 5 that the pressure rise decreases with an increasing values of Gr_F. The variation of pressure rise for different values of thermal Grashof number Gr_T is shown in Fig. 6. It is observed from Fig. 8 that pressure rise in all the regions of the peristaltic pumping increases with an increasing values of Gr_T.

3.3. Temperature

Figures 7 to 11 are prepared to examine the temperature via Nb, Nt, N_{TC}, N_{CT} and Q_0. Brownian motion parameter Nb has an increasing effect on temperature (see fig. 7). Substantial increase in temperature is seen by enhancing thermophoresis parameter Nt (see fig. 8). Ratio between wall properties of surface and reference temperature enhances due to an increment in thermophoresis parameter and so temperature increases. Figures 9 and 10 impact the influence of Soret parameter N_{CT} and Dufour parameter N_{TC} on temperature profile. With increasing N_{TC} and N_{CT}, the profiles for temperature are similar to those in figures 9 and 10. However, they ascend more smoothly. The temperature is found to be enhanced both with N_{TC} and N_{CT}. Effect of thermal radiation parameter Q_0 on temperature is depicted in fig. 11. It is noticed that temperature increases for increasing values of Q_0.

3.4. Solutal (Species) Concentration

Figures 12 to 15 shows the solutal (species) concentration profile are presented under the effects of Nb, Nt, N_{TC} and N_{CT}. Solutal Species contration values are significantly reduced

3.5. Nanoparticle Volume Fraction

The behavior of nanoparticle volume fraction for different values of Nb, Nt, N_{TC} and N_{CT} are shown in figures 16 to 19. It is observed from figures 16 to 17 that nanoparticle fraction increase in the values of Nb and decreases with an increase in the values of Nt. The effects of Soret parameter and Dufour parameter on nanoparticle fraction are presented in Figure 18 and 19.

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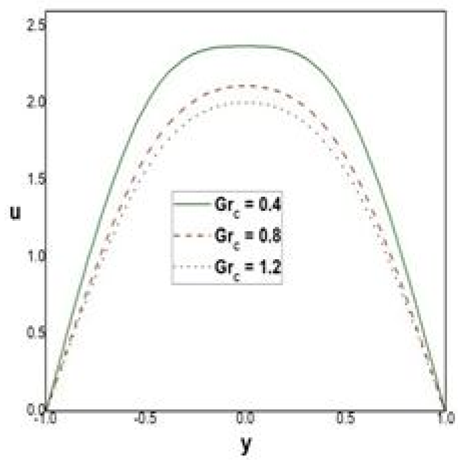

Fig-1: Influence of Gr_C on velocity

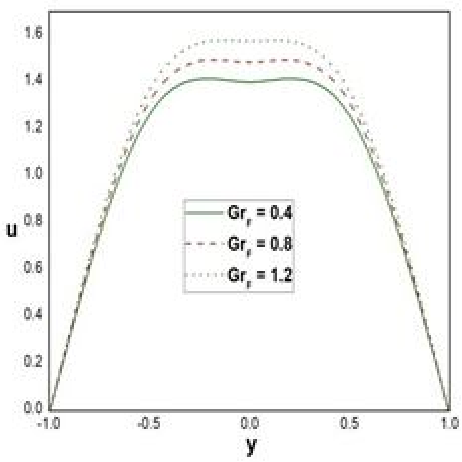

Fig-2: Influence of Gr_F on velocity

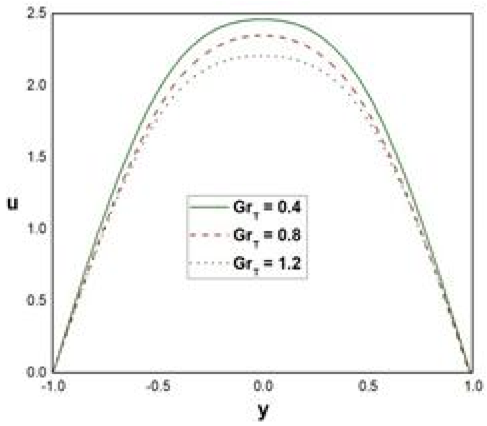

Fig-3: Influence of Gr_T on velocity

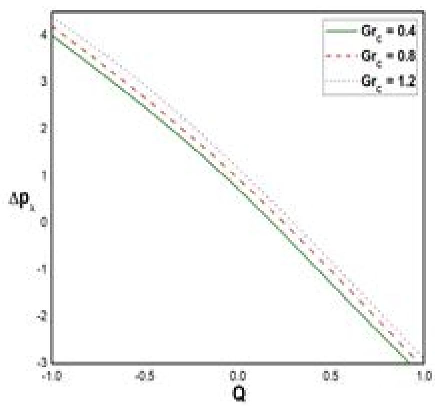

Fig-4: Influence of Gr_C on pressure rise

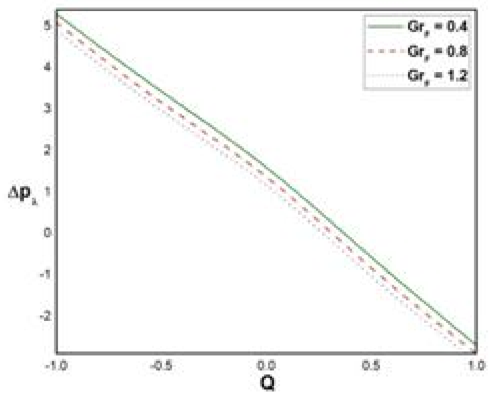

Fig-5: Influence of Gr_F pressure rise

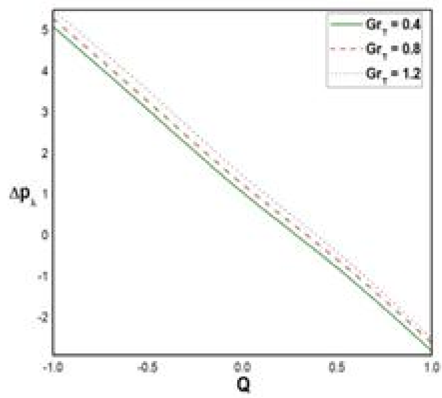

Fig-6: Influence of Gr_T on pressure rise

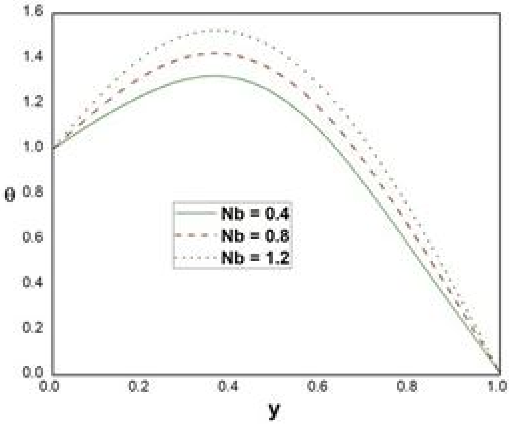

Fig-7: Influence of Nb on temperature

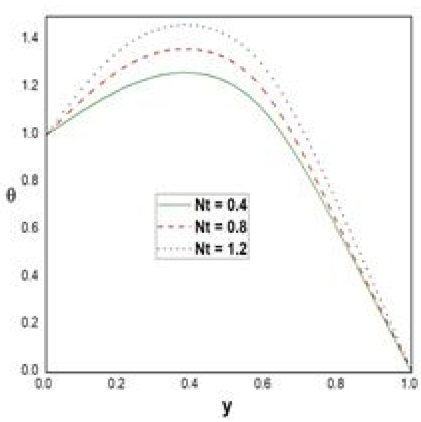

Fig-8: Influence of Nt on temperature

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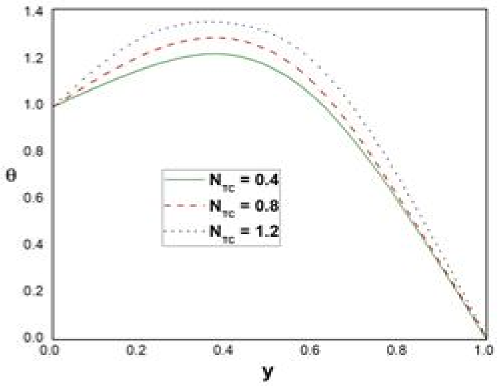

Fig-9: Influence of N_{TC} on temperature

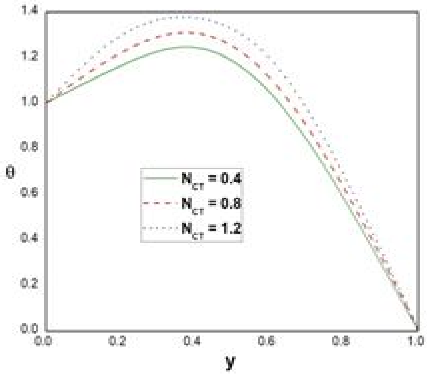

Fig-10: Influence of N_{CT} on temperature

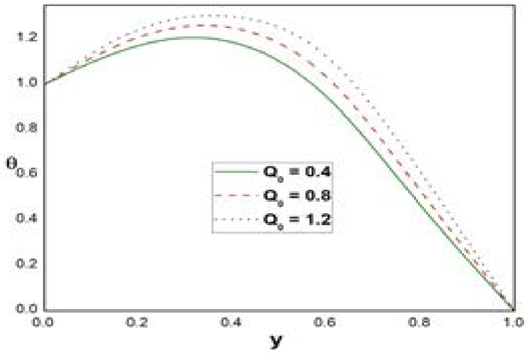

Fig-11: Influence of Q_0 on temperature

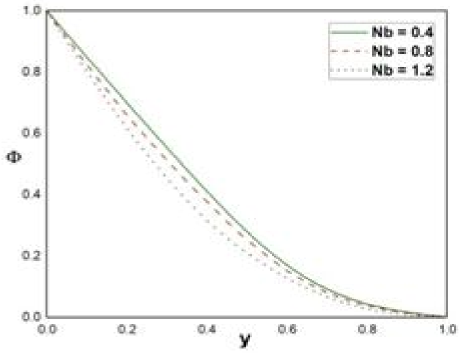

Fig-12

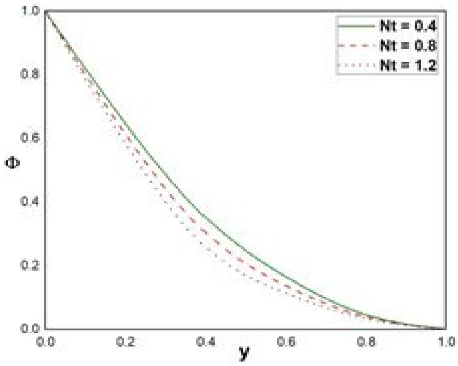

Fig-13

Fig-12: Influence of Nb on solutal (species) concentration.

Fig-13: Influence of Nt on solutal (species) concentration.

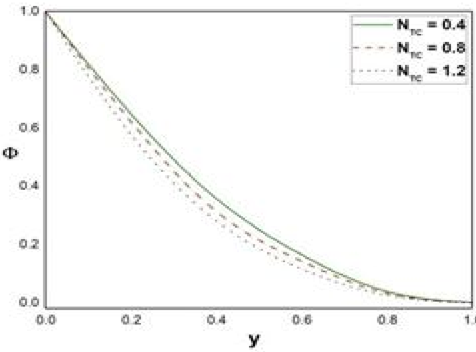

Fig-14

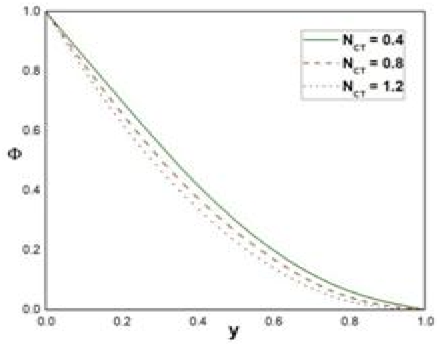

Fig-15

Fig-14: Influence of N_{TC} on solutal (species) concentration.

Fig-15: Influence of N_{CT} on solutal (species) concentration.

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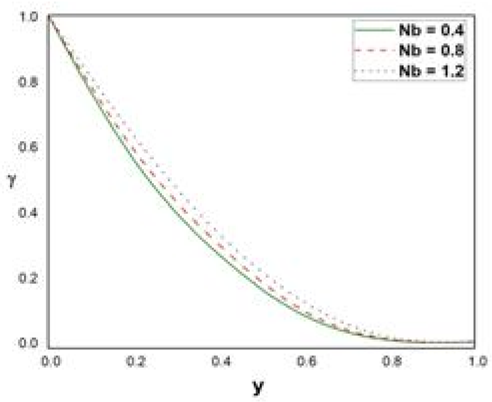

Fig-16

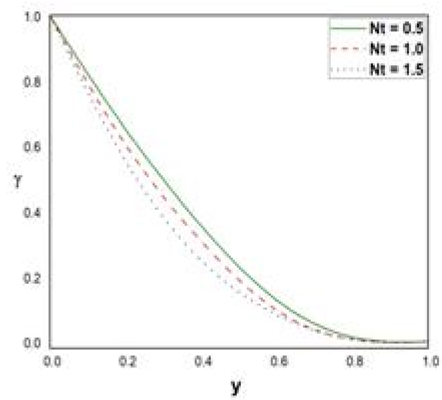

Fig-17

Fig-16: Influence of Nb on nanoparticle volume fraction.

Fig-17: Influence of Nt on nanoparticle volume fraction.

When heat and mass transfer occur simultaneously in fluid motion, an energy flux can be generated, not only by temperature gradients and energy flux can be generated composition gradients also. The energy flux caused by a composition gradient is termed the Dufour or diffusion-thermo effect. Onther way, mass fluxes can also be created by temperature gradients and this embodies the Soret or thermal diffusion effect. Such effects are significant when density differences exist in the flow regime. These parameters arise in the energy and species conservation equations. Inspection of Figure 18 shows that a small decrease induced in nanoparticle fraction caused by increasing of Soret parameter. almost identical response is sustained by the nanoparticle volume fraction profiles with an increase in the Dufour number (see fig. 19).

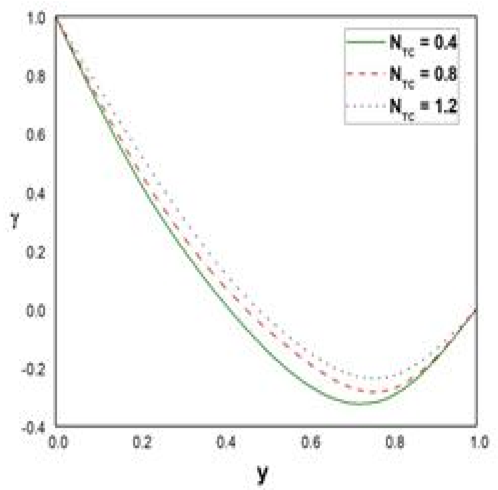

Fig-18

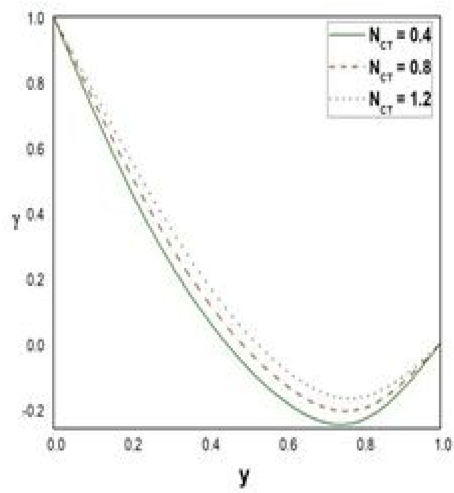

Fig-19

Fig-18: Influence of N_{TC} on nanoparticle volume fraction.

Fig-19: Influence of N_{CT} on nanoparticle volume fraction.

4. CONCLUSION

In this study we discuss the double-diffusive natural convective peristaltic transport of a nanofluid with effect of ohmic heating in a channel is discussed. The main findings of the presented study are listed below.

- It is observed that the velocity field decreases whole domain of channel when we increase the thermal Grashof number.
- It is observed that there is increasing trend of pressure rise as solutal Grashof number increases.
- It is observed that solutal concentration will be decreasing as Brownian motion parameter increases whole domain of channel.
- It is observed that temperature is increasing with an increase in thermophoresis parameter.
- It is observed that nanoparticle volume fraction increasing by increases of Soret and Dufour effects.

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NEW FORMS OF IRRESOLUTE MAPS IN FUZZY TOPOLOGY

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ABSTRACT
In this paper new forms of rgfb-mappings are considered, applying the concept of rgfb-closed sets which are known as rgfab-irresolute maps,rgfab-closed maps, contra rgfab-irresolute maps and contra rgfab-closed maps. With the help of these maps we have obtained a characterization of rgfb$T_{1/2}$ -space .Their properties and characterizations are discussed

Keywords: rgfab irresolute, rgfab-ciosed map, contra rgfab irresolute, contra rgfab-ciosed and fuzzy topological spaces (in short fts)

1. INTRODUCTION

Zadeh in [7] introduced the fundamental concept of fuzzy .Chang [4] was introduced by the study of fuzzy topology. The theory of fuzzy topological spaces was afterwards developed by more than few authors. The concept of fuzzy b –open sets and b-generalized closed sets were introduced in [1,2].In this paper, We introduce the concept of irresolute maps called rgfab-irresolute and rgfab closed by using rgfb closed sets and some of their properties.A new form called as contra rgfab irresolute and contra rgfab closed maps are also defined.By using the help of these a characterization of rgfb$T_{1/2}$ –space was obtained.

2. PRELIMINARY

All through this paper (X,τ) ,(Y, σ) and (Z, ρ) (or simply X,Y and Z) always denotefuzzy topological spaces.

Defination 2.1:A fuzzy set A in fts X is

(a) Fuzzy b-open if and only if $A \leq (Int(Cl(A)))v(Cl(Int(A)))$ [1] (The family of b-open is denoted by bo(X).

(b) Fuzzy b-closed if and only if $A \geq (Int(Cl(A))) \wedge(Cl(Int(A)))$[1].

(c) Fuzzy regular open if and only if $A = Int(Cl(A))$ [1].

(d) Fuzzy regular closed if and only if $A = Cl(Int(A))$ [1].

Remark 2.2: (a) A subset A is fuzzy b-closed if and only if $bCl(A)=A$ [1]

(b) $bCl(A)=A \text{ v } (Int(Cl(A)))\wedge(Cl(Int(A)))$ [1]

Remark 2.3: From [1],we have the followings:

(a) Every fuzzy regular open set is fuzzy open.

(b) Every fuzzy open set is fuzzy b- open.

Defination 2.4: A map $f : X \rightarrow Y$ is said to be a

(a) Fuzzy continuous if $f^{-1}(A)$ is fuzzy open in X,for every fuzzy open A in Y.

(b) Fuzzy b-continuous if $f^{-1}(A)$ is fuzzy b-open in X,for every fuzzy open A in Y.

(c) Fuzzy b^*-continuous if $f^{-1}(A)$ is fuzzy b-open in X,for every fuzzy b-open A in Y.

(d) Fuzzy b^*-open if $f(A)$ is fuzzy b-open in Y,for every fuzzy b-open A in X.

(e) rgfb-continuous if $f^{-1}(A)$ is rgfb closed in X ,for every fuzzy closed A in Y.

(f) rgfb-irresolute if $f^{-1}(A)$ is rgfb closed in X ,for every rgfb closed A in Y.

Defination 2.5: Ina fts X, afuzzy set A is called fuzzy generalized b-closed (fgb-closed) if $bCl(A) \leq B$, at any time $A \leq B$ and B is fuzzy open.

Defination 2.6: In fts let A be fuzzy set which is called regular generalized fuzzy b-closed (rgfb- closed) if $bCl(A) \leq B$, at any time $A \leq B$ and B is fuzzy regular open.

Remark2.7:A fuzzy set A in fts X is called rgfb-open if and only if 1-A is rgfb-closed in X.

Defination 2.8: A fuzzy topological space X is regular generalized fuzzy b$T_{1/2}$ space (rgfb$T_{1/2}$) if each rgfb-closed in X is fuzzy b-closed in X.

3. RGFAB-IRRESOLUTE AND RGFAB- CLOSED MAP

Defination 3.1: A map $f : X \to Y$ is called regular generalized fuzzy approximately b- irresolute (rgfab-irresolute) if $bCl(A) \leq f^{-1}(B)$, whenever B is fuzzy b-open of Y and A is rgfb closed of X such that $A \leq f^{-1}(B)$.

Defination 3.2: A map $f : X \to Y$ is called regular generalized fuzzy approximately b- closed (rgfab-closed) if $f(B) \leq bInt(A)$, whenever B is fuzzy b-closed in X and A is rgfb-open in Y such that $f(B) \leq A$.

Theorem3.3:A map $f : X \to Y$ is called rgfab-irresolute if $f^{-1}(B)$ is fuzzy b-closed in X for all fuzzy b-open B $\in Y$.

Proof: Let A be rgfb- closed in X and $B \in bo(Y)$ from the above definition we have $A \leq f^{-1}(B)$.If $f^{-1}(B)$ is fuzzy b-closed in X then $bCl(A) \leq bCl\, f^{-1}(B) = f^{-1}(B)$.That is $bCl(A) \leq f^{-1}(B)$. Hence f is rgfab-irresolute.

In general converse is not true.

We have the following theorem.

Theorem 3.4: A map $f : X \to Y$ if fuzzy b-open and b-closed in X coinsides,then f is rgfab-irresolute if and only iff $f^{-1}(B)$ is fuzzy b-closed in X for all $B \in bo(Y)$.

Proof: Suppose $f : X \to Y$ be rgfab-irresolute. Let $B \leq A$,where $A \in bo(X)$.Since everyfuzzy set in X be both fuzzy b-open and b-closed **[ ]**. Therefore every fuzzy setin X be both rgfb-open and rgfb-closed. Since $B \in bo(Y)$, $f^{-1}(B)$ is rgfb closed in X. So for $f : X \to Y$ be rgfab-irresolute, Since $bCl\, f^{-1}(B) \leq f^{-1}(B)$ but $f^{-1}(B) \leq bCl\, f^{-1}(B)$.Hence $bCl\, f^{-1}(B) = f^{-1}(B)$.This proves $f^{-1}(B)$ is fuzzy b-closed in X.

Converse follows from Theorem 3.3

Theorem 3.5: A map $f : X \to Y$ is called rgfab-closed if $f(A) \in bO(Y)$ for all fuzzy b- closed A in X.

Proof: Let A be fuzzy b-closed in X ,B be rgfb open in Y such that $f(A) \leq B$. Therefore $bInt(f(A)) \leq bInt(B)$ that implies $f(A) \leq bInt(B)$.Hence $f : X \to Y$ is rgfab-closed.

The converse is true under certain conditions

Theorem 3.6: A map $f : X \to Y$ if fuzzy b-open and b-closed in Y coinsides, then f is rgfab-closed if and only if $f(B) \in bo(Y)$ for every fuzzy b-closed B in X .

Proof: Assume $f : X \to Y$ be rgfab-closed. Let every fuzzy set in X be both fuzzy b- open and b-closed. Therefore every fuzzy set of Yare both rgfb-open and rgfb-closed. Let B be fuzzy b-closed of X.Then f(B) be rgfb-open in Y.So far $f : X \to Y$ be rgfab-closed $f(B) \leq bInt(f(B))$. But $bInt(f(B)) \leq f(B)$Therefore $bInt(f(B)) = f(B)$.Hence f(B) is fuzzy b- open set in X.

Converse follows from Theorem3.5.

Theorem 3.7: If a surjective map $f : X \to Y$ is fb^*continuous and rgfab-closed,then the

$f^{-1}(B)$ is rgfb-closed in X(rgfb-open) in X for all rgfb-closed (rgfb-open) B in Y.

Proof: Let B be rgfb-closed in Y and $f^{-1}(B) \leq A$, where A is fuzzy b-open in X.Then $1-A \leq 1- f^{-1}(B)$ that implies $1-f(A) \leq 1-B$.Since f is rgfab-closed,$1-f(A) \leq bInt(1- B)$that implies $1-f(A) \leq 1-bCl(B)$. Thus$1-A \leq 1- f^{-1}(bCl(B))$ that implies $f^{-1}(bCl(B)) \leq A$.Since f is fb^*continuous $f^{-1}(bCl(B))$ is rgfb-closed of X. As we have $bCl(f^{-1}(B)) \leq bCl(f^{-1}(bCl(B))) = (f^{-1}(bCl(B)) \leq A$.Hence$f^{-1}(B)$ is rgfb-closed in X.

Theorem 3.8: If $f : X \to Y$ is fb^*continuous and rgfab-closed then f(A) is rgfb-closed in Y for all rgfb closed $A \in X$.

Proof: Let $f(A) \leq B$ where A is rgfb-closed in X and B is any fuzzy b-open in Y. $f^{-1}(B)$ is fuzzy b-open in X and $A \leq f^{-1}(B)$ by definition of fb^*continuous.We have $bCl(A) \leq f^{-1}(B)$ since A be rgfb-closed in X.Therefore $f(bCl(A)) \leq B$. f(bCl(A)) is rgfb-closed in Y because f is rgfabclosed.Hence$bCl(f(A)) \leq bCl(f(bCl(A))) = f(bCl(A)) \leq B$.Thus f(A) is rgfb-closed in Y.

Theorem 3.9: If $f : X \to Y$ is rgfab irresolute and rgfab-closed then f(A) is rgfb-closed of Y for all rgfb-closed A of X.

Proof: Let $f(A) \leq B$ where rgfb-closed $A \in X$ and $B \in bo(Y)$,that implies $A \leq f^{-1}(B)$.

$bCl(A) \leq f^{-1}(B)$ because f is rgfab-irresolute.Therefore $f(bCl(A)) \leq B$.Thus $bCl(f(A)) \leq bCl(f(bCl(A))) = f(bCl(A)) \leq B$. Hence f(A) is rgfb-closed in Y.

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Theorem 3.10: If f : X → Y and g : Y → Z are two maps such that gof:X→ Z then,

(a) gof is rgfab-closed, where f is fuzzy b-closed and g is rgfab-closed.

(b) gof is rgfab-closed, where f is rgfab-closed and g is fuzzy b-open and g^{-1} is rgfb-open.

(c) gof is rgfab-irresolute, where f is rgfab-irresolute and g is fb^*continuous.

Proof

(a) Let A be fuzzy b-closed in X and B be rgfb-open in Z such that (gof) (A) ≤ B. Since f is fuzzy b-closed ,f(A) is fuzzy b-closed in Yand g is rgfab-closed that implies g(f(A)) ≤bInt(B). So (gof)(A) ≤bInt(B).Hence gof is rgfab-closed.

(b) Let A be fuzzy b-closed in X and B be rgfb-open in Z such that (gof) (A) ≤ B . Thus f(A) ≤ g^{-1}(B). Then f(A)≤bInt(g^{-1}(B)),since g is fuzzy b-open and g^{-1}rgfb-open.Therefore (gof)(A)=g(f(A))≤ g(bInt(g^{-1}(B)))≤bInt(g(g^{-1}(B))≤ bInt(B).There fore gof is rgfab-closed.

(c) Let A be rgfb-closed in X and B ∈ Z such that A∈$(gof)^{-1}$(B). g^{-1}(B) is fuzzy b-open in Y because g is fb^*continuous. bCl(A) ≤ f^{-1}(g^{-1}(B))=$(gof)^{-1}$(B).Therefore gof is rgfab-irresolute.

Defination 3.11: A map f : X → Y is said to be contra rgfab-irresolute if f^{-1}(B) is fuzzy b-closed in X for all fuzzy b-open B∈ Y.

Defination 3.12: A map f : X → Y is said to be contra rgfab-closed if f(A) is fuzzy b-open in Y for all fuzzy b-closed A∈X.

Theorem 3.13: A map f : X → Y the following statements are holds good:

(a) f is contra rgfab- irresolute.

(b) f^{-1}(B) is fuzzy b-closed in X for all fuzzy b-open B∈ Y.

Proof: Follows from the definition**[ ]**

Theorem 3.14: If f : X → Y and g : Y → Z are two maps then,

(a) gof is contrargfab-irresolute,where f is contrargfab-irresolute and g is fb^*continuous.

(b) gof is contrargfab-irresolute,where g is contrargfab-irresolute and f is fb^*continuous.

Proof

(a) Given g is fb^*continuous, Let B be fuzzy b-open in Z therefore g^{-1}(B) is fuzzy b-open in Y.Since f is contrargfab-irresolute , f^{-1}(g^{-1}(B)) is fuzzy b-closed in X where g^{-1}(B) is fuzzy b-open in Y. Hence $(gof)^{-1}$(B) is fuzzy b-closed in X. Since inverse image of fuzzy b-open set in Z is fuzzy b-closed in X ,Thus gof is contra rgfab-irresolute.

(b) Given g is contrargfab-irresolute, Let A be fuzzy b-open in Z.Such that g^{-1}(B) is fuzzy b-closed in Y. We have f is fb^*continuous and 1- g^{-1}(B) is fuzzy b-open in Y.Hence f^{-1}(1- g^{-1}(B))=1-f^{-1}(g^{-1}(B)) is fuzzy b-open in X.That implies

f^{-1}(g^{-1}(B))=$(gof)^{-1}$(B) is fyzzy b-closed in X. Since by definition gofis contrargfab-irresolute.

Theorem 3.15: Let X be a fts the following statements are holds good:

(a) X is rgfb$T_{1/2}$ –space,

(b) A map f : X → Y is rgfab-irresolute for every fuzzy topological space Y.

Proof

(a) ⇒(b) Let A be rgfb-closed subset of X and B∈bO(Y) such that A≤f^{-1}(B).Since X is rgfb$T_{1/2}$ –space, therefore bCl(A) ≤f^{-1}(B).Hence f is rgfab-irresolute.

(b) ⇒(a) Let A be rgfb-closed subset of X and f : X → Y be the identity map ,where σ = {0,A,1}. Therefore A is rgfb-closed in X and fuzzy b-open in Y. A≤f^{-1}(A) since f is rgfab irresolute .It follows that bCl(A) ≤ A. Thus A is fuzzy b-closed. Since every rgfb-closed is fuzzy b- closed in X .Therefore X is rgfb$T_{1/2}$ –space.

Theorem 3.16: Let Y be a fts the following statements are holds good:

(a) Y is rgfb$T_{1/2}$ –space,

(b) A map f : X → Y is rgfab-closed , for every fuzzy topological space X.

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Proof: Similar to Theorem 3.15.

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THEOREMSONRAMANUJAN'SREMARKABLEPRODUCTOFTHETA-FUNCTION AND THEIR EXPLICITEVALUATIONS

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ABSTRACT

In this article, Ramanujan defined a m,n[3], Dharmendra. B. N defined m,n[5] for any positive real numbers mandn involving Ramanujan's produc to ftheta-functions. We established new relation between am, nanddm, n and explicit evaluations of dm,n.

2000 Mathematics Subject Classification. 11B65, 11A55, 33D10, 11F20, 11F27 Secondary 11F27.

Keywords and phrases. Class invariant, Modular equation, Theta-function, Cubic continued fraction.

1. INTRODUCTION

Ramanujan's general theta-function [14] $f(a, b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} \quad , \quad |ab| < 1,$$

$= (-a; ab)\infty(-b; ab)\infty(ab; ab)\infty.$ (1.1)

where $(a,q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), |q| < 1.$

Three special cases of $f(a, b)$ are as follows:

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} a^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty} \tag{1.2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \tag{1.3}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2} = (q;\ q)_\infty \tag{1.4}$$

Ramanujan defines[[14], p.338]

$$a_{m,n} = \frac{ne^{\frac{-(n-1)\pi}{4}\sqrt{\frac{m}{n}}}\psi^2\left(e^{-\pi\sqrt{mn}}\right)\varphi^2\left(-e^{-2\pi\sqrt{mn}}\right)}{\psi^2\left(e^{-\pi\sqrt{\frac{m}{n}}}\right)\varphi^2\left(-e^{-2\pi\sqrt{\frac{m}{n}}}\right)} \tag{1.5}$$

Hethen,offersalistofeighteenparticularvalues,whicharelistedinpages337and338 of[3].AlltheseeighteenvalueshavebeenestablishedbyBerndt,ChanandZhang[4]. M. S.MahadevaNaikaandB.N.Dharmendra[7],alsoestablishedsomegeneraltheorems forexplicitevaluationsoftheproductofam,nandfoundsomenewexplicitvaluesfrom it.Furtherresultsonam,ncanbefoundbyM. S.MahadevaNaika,B. N. DharmendraandK.Shivashankar[11],andM. S.MahadevaNaikaandM.C.MaheshKumar[9].RecentlyNipenSaikia[12] established new properties of am,n.

In [11], M. S.MahadevaNaika et al. defined the product

$$b_{m,n} = \frac{ne^{\frac{-(n-1)\pi}{4}\sqrt{\frac{m}{n}}}\psi^2\left(-e^{-\pi\sqrt{mn}}\right)\varphi^2\left(-e^{-2\pi\sqrt{mn}}\right)}{\psi^2\left(-e^{-\pi\sqrt{\frac{m}{n}}}\right)\varphi^2\left(-e^{-2\pi\sqrt{\frac{m}{n}}}\right)}. \tag{1.6}$$

Theyestablishedgeneraltheoremsforexplicitevaluationofbm,nandobtainedsome particular values. MahadevaNaika et al. [10] established general formulas for explicit valuesofRamanujan'scubiccontinuedfraction$V(q)$intermsoftheproducts$am,n$and bm,ndefined above,where

$$V(q) := \frac{q^{1/3}}{1} \underset{+}{} \frac{q+q^2}{1} \underset{+}{} \frac{q^2+q^4}{1} \underset{+}{} \frac{q^3+q^6}{1} \underset{+}{} \cdots, |q| < 1, \tag{1.7}$$

and found some particular values of $V(q)$.

In [5], B. N. Dharmendra defined the Ramanujan theta function product

$$d_{m,n} = \frac{f\left(-e^{-2\pi\sqrt{\frac{n}{m}}}\right)\varphi(e^{-\pi\sqrt{mn}})}{e^{\frac{-(m-1)\pi}{12}\sqrt{\frac{n}{m}}} f\left(-e^{-2\pi\sqrt{mn}}\right)\varphi\left(e^{-\pi\sqrt{\frac{n}{m}}}\right)}, \tag{1.8}$$

where$m$and$n$arepositiverealnumbers.Heestablishseveralpropertiesoftheproduct $dm,n$and prove general formulas for explicit evaluations of $dm,n$and find its explicit values.

The main purpose of this paper is to establish new relation between $am,n$and $dm,n$and also some new explicit evaluations of dm,n.

2. PRELIMINARYRESULTS

In this section, we collect several identities which are useful in proving our main results.

Lemma 2.1.[7] *If m is any positive rational,*

$$a_{m,3} := \frac{3q^{1/2}\psi^2(-q^3)\varphi^2(q^3)}{\psi^2(-q)\varphi^2(q)}, \tag{2.1}$$

$$P := \frac{\psi(-q)}{q^{1/4}\psi(-q^3)} \, and Q := \frac{\varphi(q)}{\varphi(q^{1/3})}. \tag{2.2}$$

Then

$$a_{m,3}^2 := \frac{9(1+P^4)}{P^4(9+P^4)} = \frac{9(1-Q^4)}{Q^4(Q^4-9)}, \; Q^4 \neq 9. \tag{2.3}$$

Lemma 2.2.[5] *If* $d_{3,n} := \frac{f(-q^2)\varphi(q^3)}{q^{1/6}f(-q^6)\varphi(q)}; \; q := e^{-\pi\sqrt{\frac{n}{3}}}$ (2.4)

$$P := \frac{\varphi(q)}{\varphi(q^3)} \, and Q := \frac{f(-q^2)}{q^{1/6}f(-q^6)}. \tag{2.5}$$

Then $d_{3,n}^6 := \frac{P^4-9}{P^4(1-P^4)}, \; if P^4 \neq 1$. (2.6)

Lemma 2.3.[7]*Let*

$$a_{m,5} := \frac{5q\psi^2(-q^5)\varphi^2(q^5)}{\psi^2(-q)\varphi^2(q)}, \tag{2.7}$$

$$P := \frac{\psi(-q)}{q^{1/2}\psi(-q^5)} \, and Q := \frac{\varphi(q)}{\varphi(q^5)}. \tag{2.8}$$

Then $a_{m,5} := \frac{5(1+P^2)}{P^2(5+P^2)} = \frac{5(1-Q^2)}{Q^2(Q^2-5)}, \; Q \neq \sqrt{5}$. (2.9)

Lemma 2.4.[5] *If*

$$d_{5,n} := \frac{f(-q^2)\varphi(q^5)}{q^{1/3}f(-q^{10})\varphi(q)}; \; q := e^{-\pi\sqrt{\frac{n}{5}}} \tag{2.10}$$

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$P := \frac{\varphi(q)}{\varphi(q^5)}$ and $Q := \frac{f(-q^2)}{q^{1/3} f(-q^{10})}$. (2.11)

Then $d_{5,n}^3 := \frac{5-P^2}{P^2(P^2-1)}$, if $P^2 \neq 1$. (2.12)

3. MODULAR RELATION BETWEEN $a_{m,n}$ AND $d_{m,n}$

Theorem 3.1. *If* $u := d_{m,3}$ *and* $v := a_{m,3}$ *then*

$$u^3 - \frac{1}{u^3} = 3\left(\frac{1}{v} - v\right). \quad (3.1)$$

Proof. From Lemma (2.2), we obtain

$$P^4 := \frac{u^6-1-\sqrt{u^{12}+34u^6+1}}{2u^6}. \quad (3.2)$$

Employing the above equation (3.2) in Lemma (2.1), we get

$$(3u^3v^2-v-3u^3+vu^6)(3u^3v^2+v-3u^3-vu^6)=0 \quad (3.3)$$

Byexaminingthebehavioroftheabovefactorsnear $q=0$,wecanfindaneighborhood abouttheorigin,wherethefirstfactoriszero;whereasanotherfactorisnotzerointhis neighborhood.BytheIdentityTheoremfirstfactorvanishesidentically.Thiscompletes theproof.

Theorem 3.2. *If* $u := d_{m,5}$ *and* $v := a_{m,5}$ *then*

$$\left(u^3 + \frac{1}{u^3}\right) + 8 = 5\left(v + \frac{1}{v}\right) \quad (3.4)$$

Proof. From Lemma (2.4), we obtain

$$P^2 := \frac{u^3-1-\sqrt{u^6+18u^2+1}}{2u^2} \quad (3.5)$$

Employing the above equation (3.5) in Lemma (2.3), we get

$$(5v^2u^3-8vu^3-v-vu^6+5u^3)=0 \quad (3.6)$$

Byexaminingthebehavioroftheabovetermnear $q=0$.Thiscompletestheproof.

4. EXPLICIT EVALUATION OF $d_{m,n}$

In this section our goal is to establish several explicit evaluation of $d_{3,n}$ and $d_{5,n}$ for some values of n.

Letting $n=3$, then $a_{3,3} = \frac{1}{\sqrt{3}}$[3], substituting this value in (3.3) we obtain with an equation,

$-2u^3 + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}u^6 = 0$ and solving for u wegetthedesiredresult.i.e., $d_{3,3} = (2+\sqrt{3})^{\frac{1}{3}}$.

Similarly we can obtain for remaining values of n which is mentioned in the table1.

Sr. No	$a_{m,n}$	$d_{m,n}$
1	$a_{3,2} = \frac{\sqrt{\sqrt{3}-1}(1-\sqrt{3}+\sqrt{6})}{2}$	$d_{3,2} = \frac{1}{2}\left((12\sqrt{3}+16\sqrt{2}-12-4\sqrt{6})\sqrt{\sqrt{3}-1}\right)^{\frac{1}{3}}$
2	$a_{3,3} = \frac{1}{\sqrt{3}}$	$d_{3,3} = (2+\sqrt{3})^{\frac{1}{3}}$
3	$a_{3,5} = \frac{3-\sqrt{5}}{2}$	$d_{3,5} = \frac{(28+12\sqrt{5})^{\frac{1}{3}}}{2}$
4	$a_{3,7} = 2-\sqrt{3}$	$d_{3,7} = \frac{\sqrt{3}+\sqrt{7}}{2}$
5	$a_{3,9} = \frac{1}{(2^{1/3}+1)^2}$	$d_{3,9} = (5+(4)2^{\frac{1}{3}}+(3)2^{\frac{2}{3}})^{\frac{1}{3}}$
6	$a_{3,11} = 2\sqrt{3}-\sqrt{11}$	$d_{3,11} = (10+3\sqrt{11})^{\frac{1}{3}}$
7	$a_{3,13} = \left(\frac{\sqrt{10+2\sqrt{13}}-\sqrt{2\sqrt{13}-6}}{4}\right)^8$	$d_{3,13} = \frac{12+4\sqrt{13}+(1+\sqrt{13})\sqrt{10+2\sqrt{13}}\sqrt{2\sqrt{13}-6}}{16}$
8	$a_{3,15} = \frac{2-\sqrt{3}}{3}$	$d_{3,15} = ((2+\sqrt{5})(\sqrt{15}+4))^{\frac{1}{3}}$
9	$a_{3,19} = 2\sqrt{19}-5\sqrt{3}$	$d_{3,19} = 2+\sqrt{3}$
10	$a_{3,31} = \sqrt{2-\sqrt{3}}3$	$d_{3,31} = \frac{\sqrt{31}+3\sqrt{3}}{2}$
11	$a_{3,35} = 4\sqrt{21}+10\sqrt{3}-8\sqrt{5}-3\sqrt{35}$	$d_{3,35} = ((4\sqrt{5}+9)(\sqrt{35}+6))^{\frac{1}{3}}$
12	$a_{3,55} = 3\sqrt{11}-104\sqrt{3}-7$	$d_{3,55} = \frac{31(\sqrt{5}+3)(\sqrt{11}+\sqrt{15})}{124}$
13	$a_{3,59} = 102\sqrt{3}-23\sqrt{59}$	$d_{3,59} = (530+69\sqrt{59})^{\frac{1}{3}}$
14	$a_{3,71} = \left(\frac{\sqrt{10+2\sqrt{13}}-\sqrt{2\sqrt{13}-6}}{4}\right)^8$	$d_{3,71} = \frac{\left((\sqrt{3}+2)\sqrt{2}\sqrt{346+200\sqrt{3}}+16\sqrt{135619+78300\sqrt{3}}\right)^{\frac{1}{3}}}{2}$

TABLE 1.

If $n = 5$, then $a_{5,5} = \frac{1}{5}$[3], substituting this value in (3.6) we obtain with an equation $\frac{18}{5}u^3 - \frac{1}{5}u^6 - \frac{1}{5} = 0$andsolvingfor$u$wegetthedesiredresult.

i.e., $d_{5,5} = \frac{(3+\sqrt{5})}{2}$.

Similarly we obtain for remaining values of n which is mentioned in the table2.

Sr. No	$a_{m,n}$	$d_{m,n}$
1	$a_{5,2} = (\sqrt{2+1})(\sqrt{5}-2)$	$d_{5,2} = \frac{(\sqrt{5}-1)(\sqrt{2}+1)}{2}$
2	$a_{5,5} = \frac{1}{5}$	$d_{5,5} = \frac{3+\sqrt{5}}{2}$
3	$a_{5,9} = (2-\sqrt{3})^2$	$d_{5,9} = (31+8\sqrt{15})^{\frac{1}{3}}$
4	$a_{5,11} = \left(\frac{\sqrt{7+\sqrt{5}}-\sqrt{\sqrt{5}-1}}{8}\right)^8$	$d_{5,11} = \frac{(1+\sqrt{5})\left(12+\sqrt{14+2\sqrt{5}}\sqrt{-2+2\sqrt{5}}\right)}{16}$
5	$a_{5,13} = \left(\frac{\sqrt{9+\sqrt{65}}}{2} - \sqrt{7+\sqrt{652}}\right)^2$	$d_{5,13} = \frac{(\sqrt{5}+1)(\sqrt{13}+3)}{4}$
6	$a_{5,21} = 32+3\sqrt{105}-4\sqrt{123+12\sqrt{105}}$	$d_{5,21} = \left((\sqrt{35}+6)(15\sqrt{3}+26)\right)^{\frac{1}{3}}$
7	$a_{5,29} = \left(\sqrt{49+4\sqrt{145}}-\sqrt{48+4\sqrt{145}}\right)^2$	$d_{5,29} = \frac{13+\sqrt{145}+(\sqrt{145}-7)\sqrt{12+\sqrt{145}}}{4}$
8	$a_{5,33} = (2-\sqrt{3})^2(2\sqrt{3}-\sqrt{11})^2$	$d_{5,33} = \left((9+4\sqrt{5})(89+12\sqrt{55})\right)^{\frac{1}{3}}$
9	$a_{5,69} = \frac{(5-\sqrt{23})^2(7\sqrt{5}-\sqrt{11})^2}{4}$	$d_{5,69} = \left((1126+105\sqrt{115})(26+15\sqrt{3})\right)^{\frac{1}{3}}$
10	$a_{5,77} = 11303+576\sqrt{385}-1524\sqrt{55}-4272\sqrt{7}$	$d_{5,77} = \frac{63}{4}+\frac{3}{4}\sqrt{55}\sqrt{7}$ $+\left(\frac{743}{608}-\frac{189}{3040}\sqrt{55}\sqrt{7}\right)\sqrt{99794330+5085990\sqrt{55}\sqrt{7}}$

TABLE 2.

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EVALUATION OF QUEUEING SYSTEMS FOR C- SERVERS UNDER MULTIPLE DIFFERENTIATED VACATIONS AND WITH PARTIAL VACATION INTERRUPTIONS

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ABSTRACT
In this paper we considered M/M/C queueing systems with two types of vacations strategies namely type 1 and type 2 vacations, where the distributions followed by the duration of each type of vacation are independent of each other. Type 1 vacations can be taken after serving at least one customer where as type 2 vacations can be taken after a zero busy period. Also it is assumed that type 2 vacations can be interrupted when the number of customers in the system reaches a predefined threshold. We discussed about the steady state probabilities of various states and the average waiting time generated in the system under vacation interruption policy. We also analysed the influence of the number of servers and the value of threshold on the average waiting time generated in the system. Further we examined the relation between the number of servers in the system and the value of the threshold chosen for the vacation interruption.

Keywords-Multi server queueing systems; Vacation queueing systems; Di erentiated vacations; Average waiting time; partial vacation interruption.

1. INTRODUCTION

Queueing systems with server vacations is one in which a server may become unavailable for a certain period of time from the primary service center and the time that the server is away from the service area is termed as a vacation. There are many reasons for a server to take a vacation which include machine break downs, insufficient work load in human behaviour, power saving mode, secondary task assigned for the servers, refreshment time etc.

Server vacation queueing systems can be applied in those situations where the servers wish to utilize his idle time for different purposes. Hence these types of queueing systems have many applications while modelling variety of real world situations. The idea of vacation queueing system was introduced by Levi and Yenchiali in [8]. An exhaustive survey on vacation models was done by Doshi [3, 4] and a detailed interpretation of the model was discussed in the book of Takagi [15]. Several researchers have attracted on the vacation queueing models among them Tian and Zhang [16] elaborated this concept a while. Single working vacation and multiple working vacations are two major schemes of vacations offered for the server in vacation queueing systems and in which there may be situations like the server have to work with a different rate rather than completely stopping the service during a vacation. This concept was first discussed in the paper of Servi and Finn [12] in the name of working vacations and Wu and Takagi [18] generalized the model in [12] to an M/G/1 queue with general working vacations. Further lot of fruitful theories related to multiple working vacation queueing systems was emerged. Tian et al. [17] considered the discrete time Geo/Geo/1 queue with multiple working vacations. Baba [2] studied a batch arrival $M^X/M/1$ queue with multiple working vacations. Ibe and Isijola [5] introduced the differentiated vacation policies in order to distinguish the durations of successive vacations offered for the server in a multiple vacation queueing system. Recently we proposed queueing systems with multiple servers under differentiated working vacations [14]. Ibe and Isijola [6] extend the model described in [5] to a model with vacation interruptions. The vacation interruption policies was introduced by Li and Tian in [19] and also they analysed the discrete time GI/Geo/1 queue with working vacation and vacation interruptions in [10]. Li, Tian, and Ma studied the GI/M/1queue with working vacations and vacation interruptions in [11]. Zhang and Hou analysed the M/G/1 queue with working vacations and vacations interruptions in [19]. By using Erlang –k type distribution Ayyappan, Sekar and Ganapathi [1] studied about M/M/1 retrial queueing system with vacation interruption. Krishnamoorthy and Sreenivasan [7] discussed about M/M/2 queueing system with heterogeneous servers including one with working vacation. Sreenivasan, Chakravarthy and Krishnamoorthy [13] have introduced a threshold N, where $1 \leq N < \infty$, such that the server offering services during a vacation will have the vacation interruption when the queue size reaches N.

In this paper we introduced the vacation interruption policies into a multiple server queueing system with differentiated vacations. Thus the proposed model of vacation queueing system is assumed to be consists of C servers and for each server in the system has the opportunity to take two types vacations having different durations namely type 1 and type 2 vacations. The type 1 vacation is taken after a non zero busy period of each of the servers and type 2 vacation is taken when the servers returned from a vacation and find all C-queues are empty. Since we considered vacation interruption policies in recommended model the servers can come back to

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the normal working level before the vacation ends. In other words under vacation interruption policies the servers are forced to returned from a vacation when the number of customers in the system reaches some predefined thresholds. By analysing the characteristics of type 1 and type 2 vacations it's quite trivial that interrupting a vacation in a multiple server differentiated vacation queueing system makes sense to interrupt type 2 vacations instead of type 1 vacations. Thus we assumed that the type 2 vacations of the servers will be interrupted when the number of customers reaches a threshold value and we refer to this as partial vacation interruption in a multiple server queueing system.

The rest of the paper is organized as follows. The model is described in section 2. The steady state analysis of the model is provided in section 3. Numerical results are shown in section 4 and the obtained results are concluded in section 5.

2. SYSTEM MODEL

We consider a multiple vacation queueing system with C severs, where the customers arrive according to a Poisson process with rate λ and the service rate of customers are assumed to be distributed exponentially with mean $\frac{1}{\mu}(\mu > \lambda)$. Each of the C servers of the respective system are allowed to take two types of vacations namely type 1 and type 2 vacations depending on their working periods. The type 1 vacation can be taken when each servers of the system have treated at least one customer waiting in the queue where as type 2 vacations can be taken if no customers are waiting in any of the C queues for services when they returned from a vacation. In other words type 1 vacation can be started after completing a working period of each of the servers with a nonzero duration, but type 2 vacations can be started after a busy period of zero duration. The type 2 vacations can be repeated as long as the system is empty. The durations of both type 1 and type 2 vacations are assumed to be distributed exponentially with means $\frac{1}{\gamma_1}$ and $\frac{1}{\gamma_2}$ respectively. In order to provide the information about the nature of the vacation in which the servers of the system are currently occupied we define the state of the system as (k, m) where k denote the number of customers of the system and m is defined as follows,

$$m = \begin{cases} 0, & if\ all\ the\ servers\ are\ in\ active\ mode \\ 1, & if\ all\ the\ servers\ are\ in\ type\ 1\ vacation \\ 2, & if\ all\ the\ servers\ are\ in\ type\ 2\ vacation. \end{cases}$$

We extend the model described above by introducing vacation interruption strategies of partial type among the servers of the system. In partial vacation interruption policy the servers cannot be interrupted when they are engaged in a type 1 vacation. They can only be interrupted when they are in type 2 vacations. We assumed that under this policy the servers are interrupted from type 2 vacations when the number of customers in the system reaches a predefined threshold say k_2. Since we considered a queueing system with C servers it is also assumed that $k_2 > C$. The state transition diagram for the proposed model is given in figure 1. While look into the diagram it is obvious that when the system is in state$(k_2 - 1,2)$, the next customer arrival forces the vacation to end and a transition to the state $(k_2, 0)$ take place.

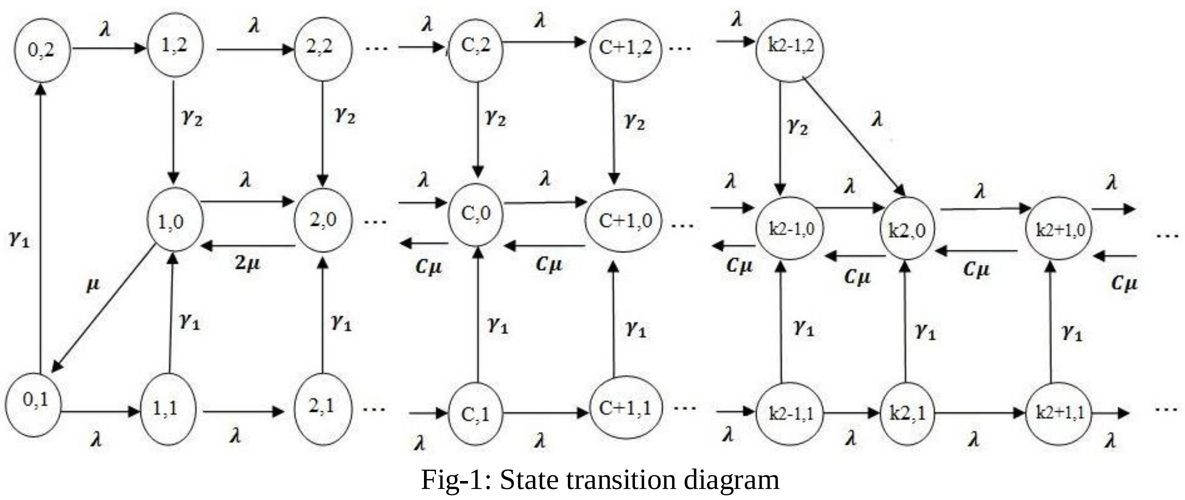

Fig-1: State transition diagram

3. STEADY STATE ANALYSIS

Let $P_{k,m}$ denote the steady state probability that the system is in the state (k, m) where $k \in N$ and $m = 0,1,2$. By analyzing the state transition diagram given in figure 1 we can derive the following theorem regarding the steady state probabilities and the average queue length of the recommended model.

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Theorem

Under partial vacation interruption policy the steady state probabilities $P_{k,m}$ for different values of m are given by

(i) $$P_{k,0} = \begin{cases} \left(\frac{\rho^{k-1}}{k!} + \sum_{i=0}^{k-2} \frac{\rho^{k-i-1}(i+1)!}{k!}\left[\alpha_1\beta_1^{i+1} + \alpha_2\beta_2^{i+1}\right]\right)P_{1,0}, & k \le C \\ \left(\left(\frac{\rho}{C}\right)^{k-C} A(C,0) + \sum_{i=1}^{k-C}\left(\frac{\rho}{C}\right)^{i}\left[\alpha_1\beta_1^{k-i} + \alpha_2\beta_2^{k-i}\right]\right)P_{1,0}, & C+1 \le k \le k_2 \\ \left(\left(\frac{\rho}{C}\right)^{k-k_2} A(k_2,0) + \frac{\rho\alpha_1\beta_1^{k_2}}{C\beta_1-\rho}\left[\beta_1^{k-k_2} - \left(\frac{\rho}{C}\right)^{k-k_2}\right]\right)P_{1,0}, & k > k_2 \end{cases}$$

(ii) $P_{k,1} = \alpha_1{\beta_1}^k P_{1,0}$, $k = 0,1,2 \ldots$, and

(iii) $P_{k,2} = \alpha_2{\beta_2}^k P_{1,0}$, $k = 0,1,\ldots,k_2-1$.

Also the average queue length of the system can be calculated by $E(m) = (Q_1 + Q_2 + Q_3 + Q_4)P_{1,0}$, where C is the number of servers in the system, $\rho = \frac{\lambda}{\mu}$ is the offered load, $\alpha_1 = \frac{\mu}{(\lambda+\gamma_1)}$, $\alpha_2 = \frac{\gamma_1\mu}{\lambda(\lambda+\gamma_1)}$, $\beta_1 = \frac{\lambda}{(\lambda+\gamma_1)} <1$, $\beta_2 = \frac{\lambda}{(\lambda+\gamma_2)} <1$, $P_{1,0} = \frac{1}{S_1+S_2+S_3+S_4+S_5}$,

$S_1 = \sum_{k=1}^{C}\left(\frac{\rho^{k-1}}{k!} + \sum_{i=0}^{k-2}\frac{\rho^{k-i-1}(i+1)!}{k!}\left[\alpha_1\beta_1^{i+1} + \alpha_2\beta_2^{i+1}\right]\right)$,

$S_2 = \frac{\rho}{C-\rho}\left(1 - \left(\frac{\rho}{C}\right)^{k_2-1-C}\right)A(C,0) + \sum_{n=C+1}^{k_2-1}\left\{\frac{\rho\alpha_1\beta_1^C}{C\beta_1-\rho}\left[\beta_1^{n-C} - \left(\frac{\rho}{C}\right)^{n-C}\right] + \frac{\rho\alpha_2\beta_2^C}{C\beta_2-\rho}\left[\beta_2^{n-C} - \left(\frac{\rho}{C}\right)^{n-C}\right]\right\}$, $S_3 = \frac{CA(k_2,0)}{C-\rho} + \frac{\rho\alpha_1\beta_1^{k_2}}{(1-\beta_1)(C-\rho)}$, $S_4 = \frac{\alpha_1}{1-\beta_1}$, $S_5 = \frac{\alpha_2(1-\beta_2^{k_2})}{1-\beta_2}$,

$$A(C,0) = \frac{\rho^{C-1}}{C!} + \sum_{i=0}^{C-2}\frac{\rho^{C-i-1}(i+1)!}{C!}\left[\alpha_1\beta_1^{i+1} + \alpha_2\beta_2^{i+1}\right]$$

$$A(k_2,0) = \left(\frac{\rho}{C}\right)^{k_2-C} A(C,0) + \sum_{i=1}^{k_2-C}\left(\frac{\rho}{C}\right)^{i}\left[\alpha_1\beta_1^{k_2-i} + \alpha_2\beta_2^{k_2-i}\right]$$

$Q_1 = \frac{\alpha_1\beta_1^{C+1}}{(1-\beta_1)^2}$,

$Q_2 = \frac{C\rho A(C,0)}{(C-\rho)^2}\left((k_2-1-C)\left(\frac{\rho}{C}\right)^{k_2-C} - (k_2-C)\left(\frac{\rho}{C}\right)^{k_2-1-C} + 1\right) +$

$\sum_{n=C+1}^{k_2-1}(n-C)\left\{\frac{\rho\alpha_1\beta_1^C}{C\beta_1-\rho}\left[\beta_1^{n-C} - \left(\frac{\rho}{C}\right)^{n-C}\right] + \frac{\rho\alpha_2\beta_2^C}{C\beta_2-\rho}\left[\beta_2^{n-C} - \left(\frac{\rho}{C}\right)^{n-C}\right]\right\}$,

$Q_3 =$
$A(k_2,0)\left(\frac{k_2C(C-\rho)+C\rho}{(C-\rho)^2}\right) + \frac{\rho\alpha_1\beta_1^{k_2}}{C\beta_1-\rho}\left\{\frac{\beta_1+k_2(1-\beta_1)}{(1-\beta_1)^2} - \left(\frac{C\rho+k_2C(C-\rho)}{(C-\rho)^2}\right)\right\} -$
$C\left\{\frac{CA(k_2,0)}{C-\rho} + \frac{\rho\alpha_1\beta_1^{k_2}}{(1-\beta_1)(C-\rho)}\right\}$,

and $Q_4 = \frac{\alpha_2\beta_2^{C+1}}{(1-\beta_2)^2}\left((k_2-1-C)(\beta_2)^{k_2-C} - (k_2-C)(\beta_2)^{k_2-1-C} + 1\right)$.

Proof:

By considering global balance in figure 1 we obtain

$\mu P_{1,0} = (\lambda + \gamma_1)P_{0,1}$ (1)

$\gamma_1 P_{0,1} = \lambda P_{0,2}$ (2)

$\lambda P_{k-1,2} = (\lambda + \gamma_2)P_{k,2}$, $k = 1,2,3,\ldots k_2-1$ (3)

and

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$\lambda P_{k-1,1} = (\lambda + \gamma_1) P_{k,1}, \qquad k = 1,2,3, \dots$ (4)

Thus

$P_{0,1} = \frac{\mu}{(\lambda+\gamma_1)} P_{1,0} = \alpha_1 P_{1,0}$ (5)

$P_{0,2} = \frac{\gamma_1}{\lambda} P_{0,1} = \alpha_2 P_{1,0}$ (6)

$P_{k,2} = \left(\frac{\lambda}{\lambda+\gamma_2}\right)^k P_{0,2} = \alpha_2 {\beta_2}^k P_{1,0}\,,\ k = 0,1, \dots, k_2 - 1\,,$ (7)

and

$P_{k,1} = \left(\frac{\lambda}{\lambda+\gamma_1}\right)^k P_{0,1} = \alpha_1 {\beta_1}^k P_{1,0}\,,\ k = 0,1,2 \dots,$ (8)

where $\alpha_1 = \frac{\mu}{(\lambda+\gamma_1)}$, $\alpha_2 = \frac{\gamma_1 \mu}{\lambda(\lambda+\gamma_1)}$, $\beta_1 = \frac{\lambda}{(\lambda+\gamma_1)}$ and $\beta_2 = \frac{\lambda}{(\lambda+\gamma_2)}$.

By analysing the local balance for the steady state probabilities of busy periods we obtain

$\lambda P_{k-1,1} + \lambda P_{k-1,2} + \lambda P_{k-1,0} = k\mu P_{k,0}\,, \quad k \le C$ (9)

$\lambda P_{k-1,1} + \lambda P_{k-1,2} + \lambda P_{k-1,0} = C\mu P_{k,0}\,,\ C + 1 \le k \le k_2.$ (10)

Hence by using equations (7) and (8) we can write

$P_{k,0} = \frac{\rho}{k}\left(P_{k-1,0} + \alpha_1 {\beta_1}^{k-1} P_{1,0} + \alpha_2 {\beta_2}^{k-1} P_{1,0}\right), \qquad k \le C$ (11)

and

$P_{k,0} = \frac{\rho}{C}\left(P_{k-1,0} + \alpha_1 {\beta_1}^{k-1} P_{1,0} + \alpha_2 {\beta_2}^{k-1} P_{1,0}\right),\ C + 1 \le k \le k_2,$ (12)

where $\rho = \frac{\lambda}{\mu}$. Solving equations (11) and (12) recursively we get

$P_{k,0} = \left(\frac{\rho^{k-1}}{k!} + \sum_{i=0}^{k-2} \frac{\rho^{k-i-1}(i+1)!}{k!}\left[\alpha_1 \beta_1^{i+1} + \alpha_2 \beta_2^{i+1}\right]\right) P_{1,0},\ k \le C$ (13)

and

$P_{k,0} = \left(\left(\frac{\rho}{C}\right)^{k-C} A(C,0) + \sum_{i=1}^{k-C} \left(\frac{\rho}{C}\right)^i \left[\alpha_1 \beta_1^{k-i} + \alpha_2 \beta_2^{k-i}\right]\right) P_{1,0},\ C + 1 \le k \le k_2,$ (14)

where $A(C,0) = \frac{\rho^{C-1}}{C!} + \sum_{i=0}^{C-2} \frac{\rho^{C-i-1}(i+1)!}{C!}\left[\alpha_1 \beta_1^{i+1} + \alpha_2 \beta_2^{i+1}\right]$. (15)

Similarly from local balance we obtain for $k \ge k_2 + 1$,

$\lambda P_{k-1,0} + \lambda P_{k-1,1} = C\mu P_{k,0}$ (16)

Thus

$P_{k,0} = \frac{\rho}{C}\left(P_{k-1,0} + \alpha_1 {\beta_1}^{k-1} P_{1,0}\right),\ k \ge k_2 + 1$ (17)

From equation (14) we have

$$P_{k_2,0} = \left(\left(\frac{\rho}{C}\right)^{k_2-C} A(C,0) + \sum_{i=1}^{k_2-C} \left(\frac{\rho}{C}\right)^i \left[\alpha_1 \beta_1^{k_2-i} + \alpha_2 \beta_2^{k_2-i}\right]\right) P_{1,0}$$

$= A(k_2,0) P_{1,0}$(say) (18)

By using equation (18) we solve equation (17) recursively and we obtain

$P_{k,0} = \left(\left(\frac{\rho}{C}\right)^{k-k_2} A(k_2,0) + \frac{\rho \alpha_1 \beta_1^{k_2}}{C\beta_1 - \rho}\left[\beta_1^{k-k_2} - \left(\frac{\rho}{C}\right)^{k-k_2}\right]\right) P_{1,0}, \quad k = k_2, k_2 + 1 \dots$ (19)

To find the value of $P_{1,0}$ we consider the law of total probability,

$\sum_{k=1}^{\infty} P_{k,0} + \sum_{k=0}^{\infty} P_{k,1} + \sum_{k=0}^{k_2-1} P_{k,2} = 1$

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That is $1 = \sum_{k=1}^{C} P_{k,0} + \sum_{k=C+1}^{k_2-1} P_{k,0} + \sum_{k=k_2}^{\infty} P_{k,0} + \sum_{k=0}^{\infty} P_{k,1} + \sum_{k=0}^{k_2-1} P_{k,2}$

By substituting the respective probabilities in the above summations and by following some algebraic calculations in each summation we can derive that

$$P_{1,0} = \frac{1}{S_1+S_2+S_3+S_4+S_5}, \tag{20}$$

Where

$$S_1 = \sum_{k=1}^{C} \left(\frac{\rho^{k-1}}{k!} + \sum_{i=0}^{k-2} \frac{\rho^{k-i-1}(i+1)!}{k!} \left[\alpha_1 \beta_1^{i+1} + \alpha_2 \beta_2^{i+1} \right] \right)$$

$$S_2 = \frac{\rho}{C-\rho} \left(1 - \left(\frac{\rho}{C}\right)^{k_2-1-C} \right) A(C,0) + \frac{\rho \alpha_1 \beta_1^C}{C\beta_1 - \rho} \sum_{n=C+1}^{k_2-1} \left[\beta_1^{n-C} - \left(\frac{\rho}{C}\right)^{n-C} \right] + \frac{\rho \alpha_2 \beta_2^C}{C\beta_2 - \rho} \sum_{n=C+1}^{k_2-1} \left[\beta_2^{n-C} - \left(\frac{\rho}{C}\right)^{n-C} \right]$$

$$S_3 = \frac{CA(k_2,0)}{C-\rho} + \frac{\rho \alpha_1 \beta_1^{k_2}}{(1-\beta_1)(C-\rho)}$$

$$S_4 = \frac{\alpha_1}{1-\beta_1}$$

and $S_5 = \frac{\alpha_2 (1-\beta_2^{k_2})}{1-\beta_2}$

The average queue length of a queueing system with C servers is given by

$$E(m) = \sum_{n=C}^{\infty} (n-C) P_n$$

Hence in this case we have $E(m) = \sum_{n=C}^{\infty} (n-C) P_{n,1} + \sum_{n=C}^{k_2-1} (n-C) P_{n,0} +$

$\sum_{n=k_2}^{\infty} (n-C) P_{n,0} + \sum_{n=C}^{k_2-1} (n-C) P_{n,2}$ (21)

Consider the summations in the above equation separately and by evaluating them with some algebraic calculations we obtain the following equations.

$\sum_{n=C}^{\infty} (n-C) P_{n,1} = \left(\frac{\alpha_1 \beta_1^{C+1}}{(1-\beta_1)^2} \right) P_{1,0}$ (22)

$$\sum_{n=C}^{k_2-1} (n-C) P_{n,0} = \left\{ \frac{C\rho A(C,0)}{(C-\rho)^2} \left((k_2-1-C)\left(\frac{\rho}{C}\right)^{k_2-C} - (k_2-C)\left(\frac{\rho}{C}\right)^{k_2-1-C} + 1 \right) + \sum_{n=C+1}^{k_2-1} (n-C) \left\{ \frac{\rho \alpha_1 \beta_1^C}{C\beta_1 - \rho} \left[\beta_1^{n-C} - \left(\frac{\rho}{C}\right)^{n-C} \right] + \frac{\rho \alpha_2 \beta_2^C}{C\beta_2 - \rho} \left[\beta_2^{n-C} - \left(\frac{\rho}{C}\right)^{n-C} \right] \right\} \right\} P_{1,0}$$

(23)

$\sum_{n=k_2}^{\infty} (n-C) P_{n,0} = \Bigg\{ A(k_2,0) \left(\frac{k_2 C(C-\rho) + C\rho}{(C-\rho)^2} \right) + \frac{\rho \alpha_1 \beta_1^{k_2}}{C\beta_1 - \rho} \left\{ \frac{\beta_1 + k_2(1-\beta_1)}{(1-\beta_1)^2} - \left(\frac{C\rho + k_2 C(C-\rho)}{(C-\rho)^2} \right) \right\} - \quad C \Bigg\{ \frac{CA(k_2,0)}{C-\rho} +$

ρα1β1k21−β1(C−ρ)P1,0 (24)

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and

$$\sum_{n=C}^{k_2-1}(n-C)P_{n,2} = \left\{\frac{\alpha_2\beta_2^{C+1}}{(1-\beta_2)^2}\left((k_2-1-C)(\beta_2)^{k_2-C}-(k_2-C)(\beta_2)^{k_2-1-C}+1\right)\right\} P_{1,0}$$

(25)

By substituting the equations (22),(23),(24) and (25) in (21) we get the required formula for $E(m)$. This completes the proof.

4. NUMERICAL ANALYSIS

In this section we investigated about the effects of parameter measures on the average waiting time ($E(v)$) generated in the system under partial vacation interruption policies. We also discussed some strategies for selecting the threshold values for the interruptions. Throughout this section we fixed the service rate μ as 0.4. Since we considered a multiple server queueing system we chosen two values for C, $C = 2$ and $C = 3$. As mentioned earlier under partial interruption policies only type 2 vacations can be interrupted and this occurs only when the number of customers reaches the value k_2. We considered two values for k_2, $k_2 = 6$ and $k_2 = 8$ in three different cases of vacation durations. That is, when type 1 vacation duration is longer than type 2 vacation (ie, $\gamma_1 < \gamma_2$); when type 1 vacation is shorter than type 2 vacation (ie, $\gamma_1 > \gamma_2$) and when both vacations types are of the same durations (ie, $\gamma_1 = \gamma_2$). In each of the respective cases we have separately evaluated the average waiting time generated in both 2 and 3 server queueing systems.

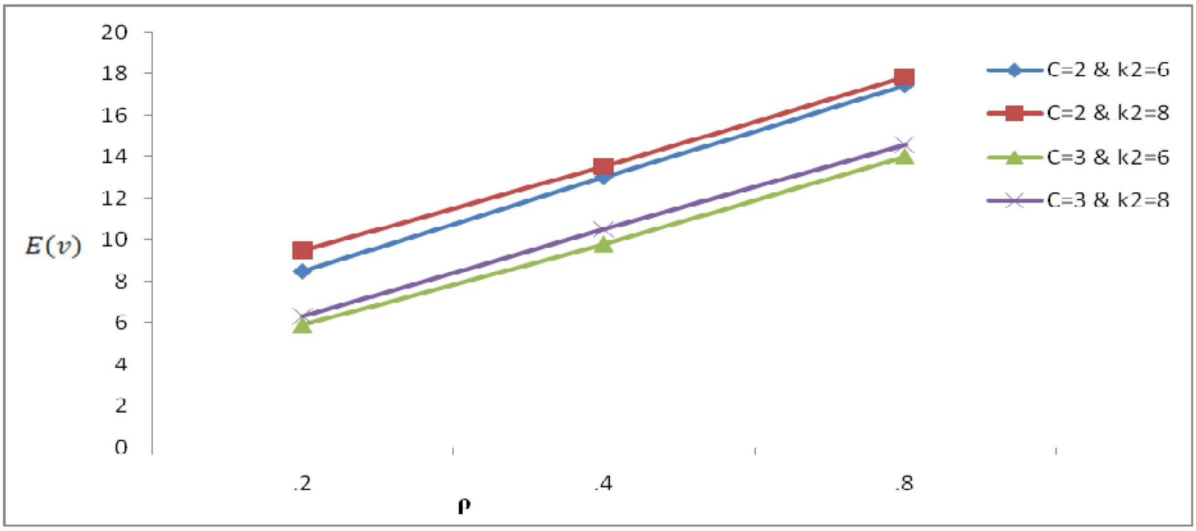

Fig-2: Mean time in the system by varying ρ for $\gamma_1 = 0.05$ and $\gamma_2 = 0.1$

From figure 2 and 3 it can be observed that there is no significant impact to the mean waiting time in the system as k_2 varied. But when the number of servers is increased the mean waiting time is reduced noticeably in the system. More over in these cases we can't derive any proper relation between the number of servers, average waiting time and the value of k_2. Thus when $\gamma_1 \leq \gamma_2$, interrupting vacations of longer or equal duration doesn't have a significant impact on the average waiting time generated in the system.

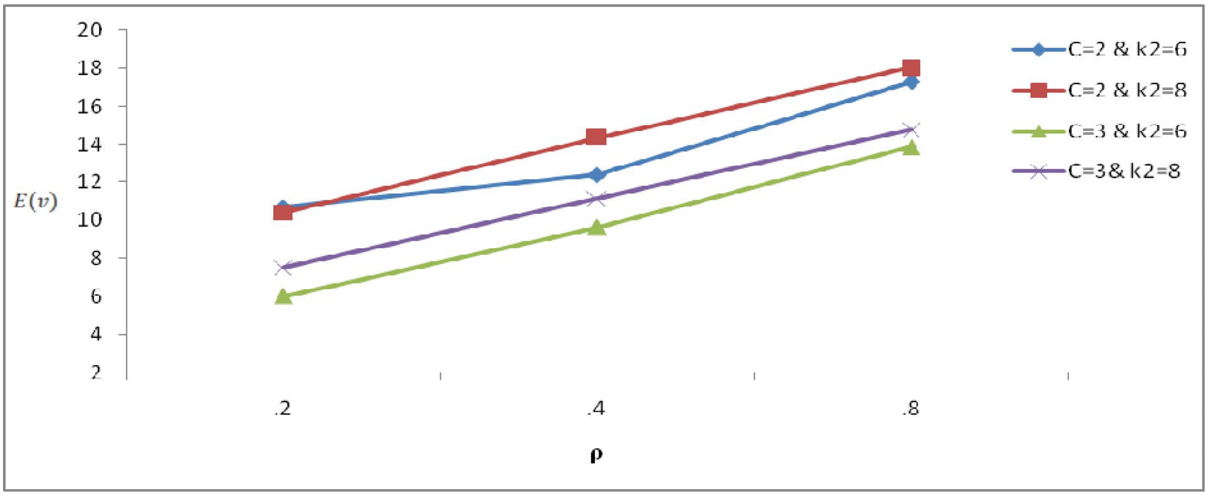

Fig-3: Mean time in the system by varying ρ for $\gamma_1 = 0.05$ and $\gamma_2 = 0.05$

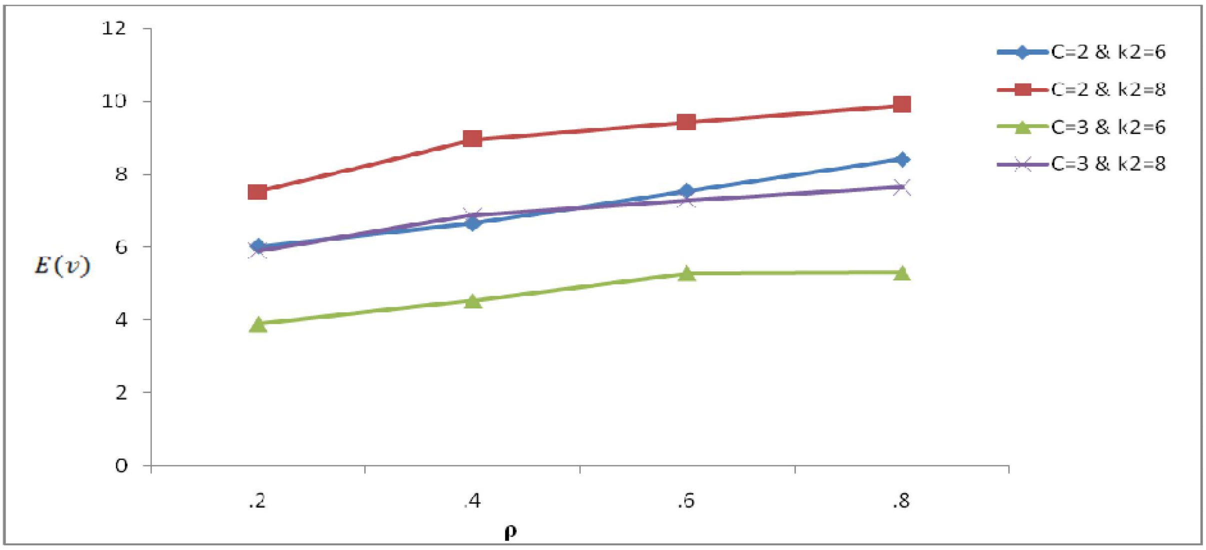

Fig-4: Mean time in the system by varying ρ for $\gamma_1 = 0.1$ and $\gamma_2 = 0.05$

While observing the graphs given in figure 4 for the third case $(ie, \ \gamma_1 > \gamma_2)$, it can be noticed that there is a systematic change in the average waiting time generated in the system according to the changes occur in the value of k_2 and the number of servers (C). In other words when $C = 2$, we can observe that the average waiting time decreases as k_2 decreases. So if we increase the value of k_2 the average waiting time increases gradually in the system. Hence there is no significant role for the partial vacation interruption with higher values of k_2. The case is similar when the number of servers is 3 in the system. But comparatively the average waiting time generated in 3 server queueing system is smaller than that of 2 server queueing system for each value of k_2 . So it is possible to increase the value of k_2 by increasing the number of servers in the system. Also the average waiting time corresponding to each values of C and k_2 in this case are relatively smaller than that of the previous cases.

5. CONCLUSION

In this paper we proposed a multiple server differentiated vacation queueing system with partial vacation interruption policies. Under this policy the servers can be interrupted only during type 2 vacations. We derived the formulas for calculating the steady state probabilities and average waiting time generated in the recommended model. The results indicates that the average waiting time generated in the system is more sensitive to the vacation interruption when the mean duration of type 2 vacation is larger than that of type 1 vacation. The graphical interpretations show that as threshold increases the vacation termination has no significant advantage over when there was no termination. But it is possible to increase the value of threshold by increasing the number of servers in the system. As the number of servers increase the average waiting time will decreased gradually even if we increase the thresholds for partial vacation interruptions.

The proposed queueing system can be widely used to model many real life situations. For example it can be used to design the working principles of the pharmacies in hospitals during rainy seasons because the number of patients who are suffering from viral fevers will increase in count in the time of rainy season. Such patients always have the tendency to lie down and want to take rest properly. Also their presence for a long time in hospitals may affect the normal persons. So it better to avoid the formation of large queues of patients and their relatives in front of pharmacies. Due to the same reasons partial vacation interruption policies can also be applied among doctors to provide better come fast treatment for such patients. The considered model can also be applied in various streams of multinational companies, administrative sections of industries, transaction sections of banks, ticket counters of railway stations, theatre and so on.

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HORADAM POLYNOMIAL COEFFICIENT ESTIMATES FOR A CLASS OF λ-BI-PSEUDO-STARLIKE FUNCTIONS

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ABSTRACT

In this investigation, we propose to make use of the Horadam polynomials, we introduce a class of bi-univalent functions which is associated with sigmoid activation function $\emptyset(s) = \frac{2}{1+e^{-s}}$ where $s \geq 0$. For functions of the form $f_{\emptyset}(z) = z + \sum_{n=2}^{\infty} \frac{2}{1+e^{-s}} a_n z^n$ belonging to this class, the coefficient inequalities are discussed. Some interesting remarks of the results presented here are also investigated.

Keywords and phrases: Analytic functions, bi-univalent functions, bi-pseudo-starlike functions, Sigmoid function, Horadam polynomial, Fekete-Szegö inequality.

1. INTRODUCTION AND DEFINITIONS

Let □ = $(-\infty, \infty)$ be the set of real numbers, □ be the set of complex numbers and

□ := $\{1,2,3, \ldots\ldots\} = \mathbb{N}_0 \backslash \{0\}$

be the set of positive integers. Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1)$$

which are analytic in the open unit disk $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, by S we shall denote the class of all functions in $\mathcal{A}$ which are univalent in Δ.

It is well known that every function $f \in S$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \Delta)$$

and

$$f(f^{-1}(w) = w \qquad \left(|w| < r_0(f); \quad r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (52a_2^3 - 5a_2a_3 + a_4)\, w^3 + \cdots$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in Δ if both the function f and its inverse f^{-1} are univalent in Δ. Let $\sum$ denote the class of bi-univalent functions in Δ given by (1.1). For the detailed study and the various subclasses of bi-univalent functions one could refer Srivastava et al. [16]. Actually, Srivastava et al. [16] essentially revived the study of different subclasses of bi-univalent function class $\sum$ in the recent year. Further, we could refer very recent works [1, 2, 3, 5, 6, 7, 11, 12, 13, 14, 15] including the references therein.

For analytic functions f and g in Δ, f is said to be subordinate to g if there exists an analytic function w such that

$$w(0) = 0, \quad |w(z)| < 1 \text{ and } f(z) = g(w(z)) \quad (z \in \Delta).$$

This subordination will be denoted here by

$$f \prec g \qquad (z \in \Delta)$$

or, conventionally, by

$$f(z) \prec g(z) \qquad (z \in \Delta).$$

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In particular, when g is univalent in Δ,

$$f \prec g \qquad (z \in \Delta) \Leftrightarrow f(0) = g(0) \quad and \quad f(\Delta) \subset g(\Delta).$$

The Horadam polynomials $h_n\ (x,\ a,\ b;\ p,\ q)$, or briefly $h_n\ (x)$ are given by the following recurrence relation (see [9, 10]):

$$h_n\,(x) = pxh_{n-1}\,(x) + qh_{n-2}\,(x) \qquad (n \in \mathbb{N}\backslash\{1,2\}) \tag{1.2}$$

with

$$h_1\,(x) = a, \qquad h_2\,(x) = bx \ \ and \ \ h_3\,(x) = pbx^2 + aq \tag{1.3}$$

where a, b, p and q are real constants.

The generating function of the Horadam polynomials $h_n(x)$ (see [10]) is given by

$$G(x,z) \coloneqq \sum_{n=1}^{\infty} h_n\ (x)z^{n-1} = \frac{a + (b - ap)xz}{1 - pxz - qz^2} \tag{1.4}$$

Here, and in what follows, the argument $x \in \mathbb{R}$ is independent of the argument $z \in \mathbb{C}$ that is, $x \neq \Re(z)$.

It is interesting to note that for particular values of a, b, p and q, the Horadam polynomial

$h_n\,(x)$ leads to various polynomials, among those, we illustrate some for example: taking $a = b = p = q = 1$, we get the Fibonacci polynomials $F_n\,(x)$ for $a = 2$ and $b = p = q = 1$,we get the Lucas polynomials $L_n\,(x)$, putting $a = q = 1$ and $b = p = 2$, we get the Pell polynomials $P_n\,(x)$, taking $a = b = p = 2$ and $q = 1$,we get the Pell-Lucas polynomials $Q_n\,(x)$, when $a = b = 1, p = 2$ and $q = -1$, we get the Chebyshev polynomials $T_n\ (x)$ of the first kind and for $a = 1, b = p = 2$ and $q = -1$ we get the Chebyshev polynomials $U_n\,(x)$of the second kind are few cases (see, [10, 14] for more details).

Let $\mathcal{A}_{\emptyset}$ denoted the class of functions of the form

$$f_{\emptyset}\,(z) = z + \sum_{n=2}^{\infty} \frac{2}{1 + e^{-s}} a_n\ z^n : = \sum_{n=2}^{\infty} \emptyset(s) a_n z^n, \tag{1.5}$$

where $\emptyset(s) = \frac{2}{1+\,e^{-s}}$ is the sigmoid activation function and $s \geq 0$, Also, $\mathcal{A}_1 \coloneqq \mathcal{A}$ (see [8]).

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Definition 1.1. A function $f \in \sum$ of the form (1.1) belongs to the class $\mathcal{G}_{\Sigma}(\lambda$,Ø(s); x), $\lambda \geq 1$ and $\emptyset(s) = \frac{2}{1+\,e^{-s}}, s \geq 0$, if the following conditions are satisfied:

$$\frac{z\,[f'_{\emptyset}(z)]^{\lambda}}{f_{\emptyset}\,(z)} \prec \mathrm{G}(x,z) + 1 - a, \qquad z \in \Delta \tag{1.6}$$

and for $g_{\emptyset}(w) = f_{\emptyset}^{-1}(w)$

$$\frac{w[g'_{\emptyset}\,(w)]^{\lambda}}{g_{\emptyset}(w)} \prec \mathrm{G}(x,w) + 1 - a, \qquad w \in \Delta \tag{1.7}$$

where the real constants a and $b$are as in (1.3).

The geometric properties of the function class $\mathcal{G}_{\Sigma}(\lambda, \emptyset(s); x)$ vary according to the values assigned to the parameters involved. For example, $\mathcal{G}_{\Sigma}(1,\emptyset(0); x) \equiv S_{\Sigma}^{*}(x)$ was introduced and studied by Srivastava et al. [14].

Remark 1.1. In its special case when $a = 1, b = p = 2$ and $q = -1$ and $x \to t$, the generating function in (1.4) reduces to that of the Chebyshev polynomial $U_n\ (t)$ of the second kind, which is given explicitly by

$$U_n\,(t) = (n + 1)_2\,F_1\,\left(-n, n + 2; \frac{3}{2}; \frac{1 - t}{2}\right) = \frac{\sin\,(n + 1)\varphi}{sin\varphi}, \qquad (t = cos\varphi)$$

in terms of the celebrated Gauss hypergeometric function 2F1: In this special case, the bi-univalent function classes $\mathcal{G}_{\Sigma}(\lambda, \emptyset(s); x)$ and $S_{\Sigma}^{*}(\emptyset(s); x)$ would become the classes $\mathcal{G}_{\Sigma}(\lambda, \emptyset(s); t)$ and $S_{\Sigma}^{*}(\emptyset(s); t)$ respectively. The classes $\mathcal{G}_{\Sigma}(\lambda, \emptyset(0); t)$ and $S_{\Sigma}^{*}(\emptyset(0); t)$ were studied earlier by Altmkaya and Yalcin [2] and Magesh and Bulut [13]. Also, $\mathcal{G}_{\Sigma}(\lambda, \emptyset(s); t)$ was studied by [4].

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In this investigation, we find the estimates for the coefficients $|a_2|$ and $|a_3|$ for functions in the subclass $\mathcal{G}_\Sigma(\lambda, \varnothing(s); x)$.

2. COEFFICIENT ESTIMATES

In the following theorem, we obtain coefficient estimates for function f in the class $\mathcal{G}_\Sigma(\lambda, \varnothing(s); x)$.

Theorem 2.1. *Let* $f_\varnothing(z) = z + \sum_{n=2}^{\infty} \varnothing(s) a_n z^n$ *be in the class* $\mathcal{G}_\Sigma(\lambda, \varnothing(s); x)$. Then

$$|a_2| \le \frac{|bx|\sqrt{|bx|}}{\varnothing(s)\sqrt{|Q|}}, \qquad and \qquad |a_3| \le \frac{|bx|}{\varnothing(s)(3\lambda-1)} + \frac{b^2x^2}{\varnothing(s)(2\lambda-1)^2},$$

where

$Q = [\lambda(2\lambda-1)b - p(2\lambda-1)^2]bx^2 - qa(2\lambda-1)^2.$

Proof. Let $f \in \mathcal{G}_\Sigma(\lambda, \varnothing(s); x)$ be given by Taylor-Maclaurin expansion (1.1). Then, from the Definition 1.1, for some analytic functions Ψ and Φ such that

$$\Psi(0) = 0; \quad \Phi(0) = 0, \ |\Psi(z)| < 1 \quad and \quad |\Phi(z)| < 1 \qquad (\forall\, z, w \in \Delta),$$

we can write

$$\frac{z[f'_\phi(z)]^\lambda}{f_\phi(z)} = G\big(x, \Psi(z)\big) + 1 - a \tag{2.1}$$

$$\frac{w[g'_\phi(w)]^\lambda}{g_\phi(w)} = G\big(x, \Phi(z)\big) + 1 - a \tag{2.2}$$

Or, equivalently,

$$\frac{z[f'_\varnothing(z)]^\lambda}{f_\varnothing(z)} = 1 + h_1(x) - a + h_2(x)\Psi(z) + h_3(x)[\Psi(z)]^2 + \cdots \tag{2.3}$$

and

$$\frac{w[g'_\varnothing w]^\lambda}{g_\varnothing(w)} = 1 + h_1(x) - a + h_2(x)\Phi(w) + h_3(x)[\Phi(w)]^2 + \cdots \tag{2.4}$$

From (2.3) and (2.4) and in view of (1.3), we obtain

$$\frac{z[f'_\varnothing(z)]^\lambda}{f_\varnothing(z)} = 1 + h_2(x)\psi_1(z) + [h_2(x)\psi_2 + h_3(x)\psi_1^2]z^2 + \cdots \tag{2.5}$$

and

$$\frac{w[g'_\varnothing w]^\lambda}{g_\varnothing(w)} = 1 + h_2(x)\phi_1 w + [h_2(x)\phi_2 + h_3(x)\phi_1^2]w^2 + \cdots \tag{2.6}$$

It is fairly well known that

$$|\Psi(z)| = |\psi_1 z + \psi_2 z^2 + \cdots| < 1 \quad and \quad |\Phi(w)| = |\phi_1 w + \phi_1 w^2 + \cdots| < 1,$$

and

$$|\psi_k| \le 1 \quad and \quad |\phi_k| \le 1 \quad (k \in \mathbb{N}).$$

Thus upon comparing the corresponding coefficients in (2.5) and (2.6), we have

$$(2\lambda-1)\phi(s)a_2 = h_2(x)\psi_1 \tag{2.7}$$

$$(3\lambda-1)\phi(s)a_3 + (2\lambda^2-4\lambda+1)\phi^2(s)a_2^2 = h_2(x)\psi_2 + h_3(x)\psi_1^2 \tag{2.8}$$

$$-(2\lambda-1)\phi(s)a_2 = h_2(x)\phi_1 \tag{2.9}$$

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$$(2\lambda^2 + 2\lambda - 1)\phi^2(s)a_2^2 - (3\lambda - 1)\phi(s)a_3 = h_2(x)\phi_2 + h_3(x)\phi_1^2 \qquad (2.10)$$

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From (2.7) and (2.9), we can easily see that

$$\psi_1 = -\phi_1 \qquad (2.11)$$

and

$$2(2\lambda - 1)^2\phi^2(s)a_2^2 = [h_2(x)]^2(\psi_1^2 + \phi_1^2)$$

$$a_2^2 = \frac{[h_2(x)]^2(\psi_1^2 + \phi_1^2)}{2(2\lambda - 1)^2\phi^2(s)} \qquad (2.12)$$

If we add (2.8) and (2.10), we get

$$2\lambda(2\lambda - 1)\phi^2(s)a_2^2 = h_2(x)(\psi_2 + \phi_2) + h_3(x)(\psi_1^2 + \phi_1^2) \qquad (2.13)$$

By substituting (2.12) in (2.13), we obtain

$$a_2^2 = \frac{[h_2(x)]^3(\psi_2 + \phi_2)}{[2\lambda(2\lambda - 1)[h_2(x)]^2 - 2h_3(x)(2\lambda - 1)^2]\phi^2(s)} \qquad (2.14)$$

which yields

$$|a_2| \le \frac{|bx|\sqrt{|bx|}}{\phi(s)\sqrt{|[\lambda(2\lambda - 1)b - p(2\lambda - 1)^2]bx^2 - qa(2\lambda - 1)^2|}}. \qquad (2.15)$$

By subtracting(2.10) from (2.8) and in view of (2.11), we obtain

$$2(3\lambda - 1)\phi(s)a_3 - 2(3\lambda - 1)\phi^2(s)\, a_2^2 = h_2(x)(\psi_2 - \phi_2) + h_3(x)(\psi_1^2 - \phi_1^2)$$

$$a_3 = \frac{h_2(x)(\psi_2 - \phi_2)}{2(3\lambda - 1)\phi(s)} + \phi(s)\, a_2^2. \qquad (2.16)$$

Then in view of (2.12), (2.16) becomes

$$a_3 = \frac{h_2(x)(\psi_2 - \phi_2)}{2(3\lambda - 1)\phi(s)} + \frac{[h_2(x)]^2(\psi_1^2 + \phi_1^2)}{2(2\lambda - 1)^2\phi^2(s)}$$

Applying (1.3), we deduce that

$$|a_3| \le \frac{|bx|}{(3\lambda - 1)\phi(s)} + \frac{b^2x^2}{(2\lambda - 1)^2\phi(s)}.$$

Corollary 2.1. Let $f_{\emptyset}(z) = z + \sum_{n=2}^{\infty} \emptyset(s)a_n\, z^n$ *be in the class* $S_{\Sigma}^{*}(\emptyset(s); x)$. *Then*

$$|a_2| \le \frac{|bx|\sqrt{|bx|}}{\phi(s)\sqrt{|[b - p]bx^2 - qa|}}, \quad and \quad |a_3| \le \frac{|bx|}{2\phi(s)} + \frac{b^2x^2}{\phi(s)}.$$

Remark 2.1. The results obtained in Corollary 2.1 coincide with results found in [14, Corollary 1 and Corollary 3] for s = 0. Also, in view of Remark 1.1, Theorem 2.1 coincide with results in [4] and for s = 0 Corollary 2.1 reduce to the results discussed in [2, 13].

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ON CONTRA sgωα –CONTINUOUS FUNCTIONS IN TOPOLOGICAL SPACES

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ABSTRACT
In this paper we introduce weaker forms of continuous functions called contra sgωα − continuous functions in topological spaces and studied some of their properties. We also discuss the relationships between this new class of function with the other class of functions.

Keywords: Topological Spaces, sgωα-closed sets, sgωα-open sets, sgωα-continuous functions, and contra sgωα-continuous functions.

1. INTRODUCTION

The concept of continuity is very important in general topology. Levine [14] introduced the concept of semi-open sets and semi-continuity in topological spaces and investigated some of their properties. The stronger and weaker forms of continuous functions have been introduced and obtained by several mathematicians. Dontchev [9] introduced a new class of functions called contra- continuous functions in topological spaces. Jafari and Noiri [11], [12] studied other new weaker forms of this class of functions called contra – α cntinuous and contra precontinuous functions. A new weaker forms of functions called contra semi- continuous functions were introduced and investigated by Dontchev and Noiri [8]

The main aim of this paper is to introduce and study a new class of continuous functions called contra semi generalized ωα –continuous (briefly contra sgωα- continuous) functions in topological spaces. Also some of their fundamental properties, characterizations are discussed and the relationships with some other functions are discussed.

2 PRELIMINARIES

Definition 2.1: A subset A of a topological space X is called

1) regular open [23] if A =int(cl(A)) and regular closed if A = cl(int(A)).
2) semi-open set [14] if A⊆ cl(int(A)) and semi-closed set if int(cl(A)) ⊆ A.
3) pre-open set [15] if A ⊆int(cl(A)) and pre-closed set if cl(int(A)) ⊆ A.
4) α-open set [17] if A ⊆int(cl(int(A))) and α-closed set if cl(int(cl(A))) ⊆ A.
5) semi-preopen set [2] (= β-open [1]) if A ⊆ cl(int(cl(A))) and semi-pre closed set [2] (= β-closed [1]) if int(cl(int(A))) ⊆ A.

Definition 2.2: A subset A of a topological space X is called

1. ω-closed [25] if cl(A) ⊆U whenever A ⊆U and U is semi-open in X.
2. ωα-closed [4] if αcl(A) ⊆U whenever A ⊆U and U is ω-open in X.
3. generalized ωα-closed (briefly gωα-closed) set[6] if αcl(A) ⊆U whenever A ⊆U and U is ωα-open set in X.
4. semi generalized ωα-closed (briefly sgωα-closed) set [21] if scl(A) ⊆U whenever A ⊆U and U is ωα-open set in X.

We denote the family of all sgωα-closed (resp. sgωα-open) subsets of X by sgωαC(X) (resp.sgωαO(X)).

Definition 2.3 [22]: In a topological space X if every sgωα-closed set is semi closed then X is called $T_{sg\ \omega\alpha}$-space.

Definition 2.4: A function f: (X, τ) → (Y, σ) is called R-map (resp. perfectly continuous[9], sgωα-continuous[20], contra-continuous [11]) if the inverse image of every regular open (resp. open, open, open) set in Y is regular – open (resp. clopen, sgωα-open, closed) in X.

Definition 2.5 [20]: A function f: (X, τ) → (Y, σ) is sgωα irresolute if the if the inverse image of every sgωα-closed set in Y is sgωα- closed in X.

3 CONTRA SGωA- CONTINUOUS FUNCTIONS IN TOPOLOGICAL SPACES

In this section, we introduce the new class of functions called contra semi generalized $\omega\alpha$ – continuous (briefly contra sgωα- continuous) functions in Topological Spaces and obtain some of their properties and characterizations.

Definition 3.1: A function f: X → Y is called contra sgωα – continuous function if $f^{-1}(A)$ is sgωα- -closed in X for every open set A in Y.

Remark 3.1: Every contra- continuous function is contra sgωα- continuous

Converse of the above remark need not be true as seen from the following example

Example 3.1: Let X = {a, b, c}, τ = {X, ϕ, {a}, {b, c}} and σ = {Y, ϕ, {a,b}}. Then the identity function f: X → Y is contra sgωα -continuous but not contra continuous function, since for the open set {a,b} in Y, f^{-1}({a,b}) = {a,b} is not closed in X

Remark 3.2: Every contra sgωα- continuous function defined on $T_{sg\omega\alpha}$.space is contra semi continuous

Remark 3.3: From the following examples , we can observe that both contra sgωα- continuous and sgωα- continuous functions are independent of each other

Example 3.2: Let X = Y ={a, b, c}, τ = {X, ϕ, {a},{b}, {a, b}} and σ = {Y, ϕ, {a,b}}. Define a function f: X → Y to be identity function.. Then the function f is sgωα -continuous but not contra sgωα -continuous function, since for the open set {a,b} in Y, f^{-1}({a,b}) = {a,b} is not sgωα -closed in X but it is sgωα -open set in X.

Example 3.3 : Let X=Y={a,b,c}, τ = {X, ϕ, {c}} and σ = {Y, ϕ, {a,b}} Define a function

f: X → Y to be identity function. Then the function f is contra sgωα -continuous but not sgωα -continuous function, since for the open set {a,b} in Y, f^{-1}({a,b}) = {a,b} is not sgωα –open in X but it is sgωα -closed set in X.

Lemma 3.4: [3] Let A and B be any two subsets of a space X. Then

i) x ∈ ker(A) if and only if A and F are not disjoint and F∈ C (X, x)

ii) if A is open in X then A ⊂ ker(A) and A = ker(A)

iii) if A ⊂ B then ker(A) ⊂ ker(B)

Theorem 3.5 : Let f: X → Y be any function. Then the following statements are equivalent:

i) if is contra sgωα- continuous

ii) For every F∈C(Y) , f^{-1}(F) ∈sgωα O(X)

iii) for each point x ∈ X and F ∈ C(Y,f(x))there exists U ∈sgωα O(X,x) such that f(U) ⊂ F

iv) For each point x ∈ X and each V∈O(Y) such that f(x) does not belong to V ,there exists

K∈sgωα C(X) not containing x such that f^{-1} (V) ⊂ K

v) f(sgωα -cl(A)) ⊂ ker(f(A))

vi) If B⊂ Y then sgωα -cl(f^{-1} (B))⊂f^{-1}(ker(B))

Theorem 3.6 : If a function f: X → Y is contra sgωα- continuous and Y is regular then f is sgωα - continuous .

Proof: Let x ∈ X and V∈O(Y,f(x)). By hypothesis, Y is regular space, there exists W∈ (Y,f(x)) such that cl(W) ⊂ V. By Theorem 3.5 (iii) and f is contra sgωα- continuous there exists U∈ sgωα O(X) such that f(U) ⊂ cl(W). Thus f(U) ⊂ cl(W) ⊂ V. Hence f is sgωα- continuous.

Definition 3.2: In a topological space X if every sgωα -open set is closed in X then X is said to be a locally sgωα - indiscrete space**.**

Theorem 3.7 : Letf: X → Y be contra sgωα – continuous function**.** If X is locally sgωα - indiscrete space then f is continuous.

Proof: Let V ∈ O(Y). Then f^{-1}(V) ∈ O(X) as f is contra sgωα – continuous function and X is locally sgωα - indiscrete space.Hence f is continuous.

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Definition 3.3 :A function f: X $\rightarrow$ Y is called weakly sg$\omega\alpha$ - continuous if for each x $\in$X and each open set V of Y containing f(x), there exists U$\in$ sg$\omega\alpha$O(X) such that f(U) $\subset$cl(V).

Theorem 3.8: If a function f: X $\rightarrow$ Y is contra sg$\omega\alpha$ - continuous then f is weakely sg$\omega\alpha$ - continuous.

Proof : Let V be an open set in Y . Since cl(V) is closed in Y and from Theorem 3.5,f^{-1}(cl(V)) is sg$\omega\alpha$- open in X. Put U =f^{-1}(cl(V)) then f(U)$\subset$ f(f^{-1}(cl(V))) $\subset$ cl(V). This shows that f is weakly sg$\omega\alpha$- continuous

Remark 3.4: Composition of two contra sg$\omega\alpha$ - continuous functions need not be contra sg$\omega\alpha$ - continuous

Example 3.9: X=Y=Z={a,b,c},τ = {X, ϕ, {a}, {a, b}} and σ = {Y, ϕ, {c } and **η**= {Z,ϕ, {a,b}}. Let f: X $\rightarrow$ Y and g: Y$\rightarrow$ Z be the identity maps then f and g are contra sg$\omega\alpha$ - continuos . Let gof: X $\rightarrow$ Z be the identity map. Then gof is not contra sg$\omega\alpha$- continuous, since for the open set A = {a, b} in Z, $(gof)^{-1}$({a,b})= f^{-1} (g^{-1}({a,b}))=f^{-1} ({a,b}) ={a,b} is not sg$\omega\alpha$ - closed in X.

Theorem 3.10 : If a function f: X $\rightarrow$ Y is sg$\omega\alpha$ - irresolute, g : Y$\rightarrow$ Z is contra continuous then gof: X $\rightarrow$ Z contra sg$\omega\alpha$ – continuous

Proof : Let G be an open set in Z. Since g is contra continuous , g^{-1}(G) is closed in Y. Therefore g^{-1}(G)sg$\omega\alpha$- closed in Y. Since f is sg$\omega\alpha$- irresolute f^{-1}(g^{-1}(G)) = $(gof)^{-1}$(G) is sg$\omega\alpha$- closed in X. Therefore gof is contra sg$\omega\alpha$- continuous.

Corollary 3.11: If a function f: X $\rightarrow$ Y is sg$\omega\alpha$ - irresolute, g : Y$\rightarrow$ Z is contra sg$\omega\alpha$ - continuous then gof: X $\rightarrow$ Z contra sg$\omega\alpha$– continuous.

Definition 3.4 :[24] In a topological space X if every open subset of X is closed then the space X is said to be locally indiscrete space

Theorem 3.12: Let f: X $\rightarrow$ Y be a continuous function. If X is locally indiscrete space then f is contra sg$\omega\alpha$–continuous

Proof : Let G $\in$ O(Y). Then f^{-1}(G) $\in$O(X) as f is continuous. From hypothesis X is locally indiscrete space. Then f^{-1}(G) $\in$ C(X) and hence f^{-1}(G) $\in$ sg$\omega\alpha$C(X) by [8] . Hence f is contra sg$\omega\alpha$–continuous

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LEFT GAMMA IDEALS OF TERNARY LA GAMMA SEMI GROUPS

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ABSTRACT

In this paper, we have discussed ternary LA(Left Almost) Γ-semigroups and the structural properties of ternary LA Γ-semi groups have been investigated. We also have discussed left Γ- ideal in ternary LA Γ-semigroups and proved that every left Γ- ideal P of ternary LA-Γ-semi group M with left identity is a quasi-prime ideal if and only if $(x\beta y)\beta z \in P \Rightarrow x \in P$ or $y \in P$ or $z \in P$ $\forall \alpha, \beta \epsilon \Gamma$ and $\forall x, y, z \epsilon M$.Also we proved that If S is a left ideal of M and P is a quasi prime ideal of M then $S \cap P$ is a quasi prime ideal of M.

Keywords: Γ-semi group, ternary Γ-semi group, Left Almost Γ-semi group, left Γ- ideal

1. INTRODUCTION AND PRELIMINARIES:

SIOSON [1] introduced the ideal theory in ternary semigroups. He also introduced the notion of regular ternary semigroup and characterized by using the properties of quasi-ideals. SANTIAGO [2] developed the theory of ternary semigroups. M.Y.Abbasi and Abul Basar[3] have studied some properties of locally associative LA-Γ-semigroups,anti-rectangular LA-Γ-semi groups. etc. And also proved some important properties of bi-Γ ideals of an LA-Γ-semi group. SHABIR and BASHIR [4] launched prime ideals in ternary semigroups. They also defined the concepts of weakly pure ideal and purely prime ideal in a ternary semigroup without order. Pairote Yiarayong [5] established the notion of ternary LA-semigroups. We introduce concept of left Γ-ideal of ternary LA Γ- semi group and it is proved that every left Γ- ideal of ternary LA-Γ-semi group with left identity of M is a quasi-prime ideal if and only if $(x\beta y)\beta z \in P \Rightarrow$ x ∈ P or y ∈ P or z ∈ P $\forall \alpha, \beta \epsilon \Gamma$.

Definition 1.1: Let M *and* Γ be two non-empty sets then a triplet $(M, \Gamma, \cdot)$ is called a ternary Left Almost – Γ semi group where $\cdot$ is ternary operation $M \times \Gamma \times M \times \Gamma \times M \to M$ such that $[(a\alpha b)\beta c]\gamma(d\delta e) = [(d\delta e)\beta c]\gamma(a\alpha b) = [(c\alpha b)\beta a]\gamma(d\delta e)$, $\forall\, a, b, c, d, e \in M, \alpha, \beta, \gamma, \delta, \in \Gamma$

Example 1.2: Let $M = \{1, -1, i, -i\}$ *and* $\Gamma = \{1\}$ then M is a ternary LA- Γ semi group under the multiplication of complex numbers

Example 1.3: Let $M = Z$, the set of all integers and $\Gamma = \mathrm{N}$, the set of all natural numbers. Then M is a ternary LA- Γ semi group under the multiplication of integers.

Definition 1.4: A non-empty subset S of a ternary LA-Γ-semigroup M is said to be a ternary LA-Γ-sub semigroup if $(S\Gamma S)\Gamma S \subseteq S$.

Definition 1.5: A non-empty subset S of a ternary LA-Γ-semigroup M is said to be a left ideal if $(M\Gamma M)\Gamma S \subseteq S$.

Example 1.6: Let $M = Z$, the set of all integers and Γ = N, the set of all natural numbers, clearly $(M, \Gamma, \cdot)$ is a ternary LA-Γ-semi group and $S = \{2x /\ x\epsilon Z\}$ is a left ideal of M.

Definition 1.7: If S is a left ideal of a ternary LA-Γ-semi group M and $\varphi \neq J, K \subseteq M$, then

$(S\!:J\!:K) = \{m \in M /\ (J\Gamma K)\Gamma m \subseteq S\}$ and it is called the extension of S by J *and* K. Let $(S\!:j\!:k)$ stands for $(S\!:\{j\}\!:\{k\})$

Definition 1.8: Let M be a ternary LA-Γ-semi group.A left ideal P is called quasi prime if $(X\Gamma Y)\Gamma Z \subseteq P$ implies $X \subseteq P$ or $Y \subseteq P$ or $Z \subseteq P$ for all ideals X, Y *and* Z in M.

2. MAIN RESULTS

Theorem 2.1: Intersection of two left ideals of a ternary LA-Γ-semi group is also a left ideal.

Proof: let M be a ternary LA-Γ-semigroup. Let S_1 *and* S_2 be two left ideals of M, then $(M\Gamma M)\Gamma S_1 \subseteq S_1$ and $(M\Gamma M)\Gamma S_2 \subseteq S_2$. Let $x\epsilon(M\Gamma M)\Gamma(S_1 \cap S_2) \Rightarrow x = (a\alpha b)\beta c$ where $a, b\epsilon M$ $c\epsilon S_1 \cap S_2$ *and* $\alpha, \beta \epsilon \Gamma$. $c\epsilon S_1 \cap S_2 \Rightarrow c\epsilon S_1$ *and* $c\epsilon S_2$. Now $a, b\epsilon M$ $\alpha, \beta\epsilon\Gamma$ *and* $c\epsilon S_1 \Rightarrow (a\alpha b)\beta c\epsilon(M\Gamma M)\Gamma S_1 \subseteq S_1 \Rightarrow (a\alpha b)\beta c\epsilon S_1 \Rightarrow x\epsilon S_1$. Also $a, b\epsilon M$ $\alpha, \beta\epsilon\Gamma$ *and* $c\epsilon S_2 \Rightarrow (a\alpha b)\beta c\epsilon(M\Gamma M)\Gamma S_2 \subseteq S_2 \Rightarrow (a\alpha b)\beta c\epsilon S_2 \Rightarrow x\epsilon S_2 \therefore x\epsilon S_1 \cap S_2$

$\therefore (M\Gamma M)\Gamma(S_1 \cap S_2) \subseteq S_1 \cap S_2$. Hence $S_1 \cap S_2$ is also a left ideal.

Lemma 2.2: Let M be a ternary LA-Γ-semi group with left identity and S be a left ideal of M then $T\Gamma(S\Gamma S)$ is a left Γ ideal of M where $\varphi \neq T \subseteq M$

Proof: Suppose that M is a ternary LA-Γ-semi group with left identity and let S be a left ideal of M. Then we have $(M\Gamma M)\Gamma S \subseteq S$. Consider, $(M\Gamma M)\Gamma(T\Gamma(S\Gamma S)) = T\Gamma((M\Gamma M)\Gamma(S\Gamma S)) = T\Gamma(S\Gamma(M\Gamma M)\Gamma S) \subseteq T\Gamma(S\Gamma S)$ $\therefore$ $T\Gamma(S\Gamma S)$ is a left ideal of M

Corollary 2.3: Let M be a ternary LA-Γ-semi group with left identity and S be a left ideal of M. Then $a\Gamma(S\Gamma S)$ is a left ideal of M where $a \in M$

Proof: As $T\Gamma(S\Gamma S)$ is a left ideal of $M \Longrightarrow$ $a\Gamma(S\Gamma S)$ is a left ideal of M for $a \in T$

$\Longrightarrow a\Gamma(S\Gamma S)$ is a left ideal of M for $a \in M$ as $T \subset M$

Proposition 2.4: Let S be a left ideal of a ternary LA-Γ-semigroup$M$ and φ≠$W, X, Y, Z \subseteq M$, then the following statements hold. $i)\ S \subseteq (S{:}\,a{:}\,b)$ Where $a, b \in M$. $ii)\ Y \subseteq W\ then\ (S{:}\,W{:}\,X) \subseteq (S{:}\,Y{:}\,X)$ $iii)\ Z \subseteq X\ then\ (S{:}\,W{:}\,X) \subseteq (S{:}\,W{:}\,Z)$ $iv)\ S \subseteq Y\ then\ (S{:}\,W{:}\,X) \subseteq (Y{:}\,W{:}\,X)$.

$v)\ W \subseteq S\ then\ (S{:}\,W{:}\,X) = M$.

Proof: (i) Let $x\epsilon S$ then $(a\alpha b)\beta x \in (M\Gamma M)\Gamma S \subseteq S$ $\Rightarrow (a\alpha b)\beta x \in S \Rightarrow x \in (S{:}\,a{:}\,b)$

$\therefore S \subseteq (S{:}\,a{:}\,b)$.

(ii) Let $x\epsilon(S{:}\,W{:}\,X) \Rightarrow (W\Gamma X)\Gamma x \subseteq S \Rightarrow (Y\Gamma X)\Gamma x \subseteq S \Rightarrow x\epsilon(S{:}\,Y{:}\,X)$

$$\therefore (S{:}\,W{:}\,X) \subseteq (S{:}\,Y{:}\,X)$$

(iii) and iv). Proofs of these are same as above.

(v). Let $W \subseteq S$, it is obvious that $(S{:}\,W{:}\,X) \subseteq M$ and Let $m\epsilon$M, Then we shall prove that $m\epsilon(S{:}\,W{:}\,X).\ i.e, (W\Gamma X)\Gamma m \subseteq S$. Let $a\epsilon(W\Gamma X)\Gamma m \Rightarrow a = (w\alpha x)\beta m$ $\Rightarrow a = (m\alpha x)\beta w$ $\Rightarrow a\epsilon(M\Gamma M)\Gamma W \Rightarrow$ $a\epsilon(M\Gamma M)\Gamma S \Rightarrow a\epsilon(M\Gamma M)\Gamma S \subseteq S \Rightarrow a\epsilon S$

$\therefore (W\Gamma X)\Gamma m \subseteq S$ $\Rightarrow m\epsilon(S{:}\,W{:}\,X)$ $\therefore M \subseteq (S{:}\,W{:}\,X)$ Hence $M = (S{:}\,W{:}\,X)$

Proposition 2.5: Let S be a left ideal of a ternary LA-Γ-semi group M with left identity and φ≠$W, X \subseteq M$. Then following holds. (i) If $X \subseteq S$ then $(S{:}\,W{:}\,X) = M$ (ii) If $W \subseteq S$ then $(S{:}\,W{:}\,X) = M$

Proof: (i) Let $X \subseteq S$, Then it is obvious that $(S{:}\,W{:}\,X) \subseteq M$. Let $m\epsilon$M. Then we shall prove that $m\epsilon(S{:}\,W{:}\,X).i.e, (W\Gamma X)\Gamma m \subseteq S$. Let $w\epsilon W, x\epsilon X, \alpha, \beta\epsilon\Gamma$. Then $a\epsilon(W\Gamma X)\Gamma m \Rightarrow a = (w\alpha x)\beta m \Rightarrow a = (w\alpha x)\beta(e\gamma m) \Rightarrow a = w\alpha(x\beta(e\gamma m)) \Rightarrow a = w\alpha(m\beta(e\gamma x)) \Rightarrow a = w\alpha(m\beta x)$ $\Rightarrow$ $a = (w\alpha m)\beta x \Rightarrow a\epsilon(M\Gamma M)\Gamma X \Rightarrow a\epsilon(M\Gamma M)\Gamma S$ $\therefore$ $M \subseteq (S{:}\,W{:}\,X)$. Hence $(S{:}\,W{:}\,X) = M$

(ii). Proof of this is same as above.

Lemma.2.6.:-If M is a ternary LA-Γ-semi group with left identity $'e'$,then

$$a\alpha(b\beta c) = b\alpha(a\beta c)\forall a, b, c\ \epsilon M\ and\ \alpha, \beta\epsilon\Gamma$$

Proof:-Let $a, b, c\ \epsilon M\ and\ \alpha, \beta\epsilon\Gamma$.For $\gamma, \delta\epsilon\Gamma$

Consider, $a\alpha(b\beta c) = ((e\gamma e)\delta a)\alpha(b\beta c) = ((b\beta c)\delta a)\alpha(e\gamma e) = ((a\beta c)\delta b)\alpha(e\gamma e)$

$$= ((e\gamma e)\delta b)\alpha(a\beta c) = b\alpha(a\beta c)$$

Lemma 2.7: Let M be a ternary LA-Γ-semi group with left identity and S be a left ideal of M then $(S{:}\,a{:}\,b)$ is a left ideal of M where $a, b\epsilon M$.

Proof: Let M be a ternary LA-Γ-semi group with left identity and Let $r, s\epsilon M$ and $x\epsilon(S{:}\,a{:}\,b)$, Then $(a\alpha b)\beta x\epsilon S$. By Lemma.2.6, For α, β, γ, $\delta\epsilon\Gamma$, We have $(a\alpha b)\beta((r\gamma s)\delta x) = (r\gamma s)\beta((a\alpha b)\delta x)\epsilon(r\gamma s)\beta S \subseteq (M\Gamma M)\Gamma S \subseteq S$. $\therefore (a\alpha b)\beta((r\gamma s)\delta x)\epsilon S \Rightarrow (r\gamma s)\delta x\epsilon(S{:}\,a{:}\,b) \Rightarrow (M\Gamma M)\Gamma(S{:}\,a{:}\,b) \subseteq (S{:}\,a{:}\,b)$. Hence $(S{:}\,a{:}\,b)$ is a left ideal of M.

Lemma 2.8: Suppose M be a ternary LA-Γ-semi group with left identity and S be a left ideal of M. Then $(M\Gamma M)\Gamma S$ is a left ideal of M.

Proof: Consider $$(M\Gamma M)\Gamma((M\Gamma M)\Gamma S) = \Big(((e\Gamma e)\Gamma M)\Gamma M\Big)\Gamma((M\Gamma M)\Gamma S)$$

$$= ((M\Gamma M)\Gamma(e\Gamma e))\Gamma((M\Gamma M)\Gamma S)$$

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$$= (((M\Gamma M)\Gamma S)\Gamma(e\Gamma e))\Gamma(M\Gamma M)$$
$$= (((e\Gamma e)\Gamma S)\ \Gamma(M\Gamma M)\Gamma(M\Gamma M)$$
$$= (M\Gamma M)\Gamma(M\Gamma M)\Gamma((e\Gamma e)\Gamma S)$$
$$= (M\Gamma((M\Gamma M)\Gamma M)\Gamma S = (M\Gamma M)\Gamma S$$

Hence $(M\Gamma M)\Gamma S$ is a left ideal of M

OR

Let S is a left ideal of $M \Rightarrow (M\Gamma M)\Gamma S \subseteq S \Rightarrow (M\Gamma M)\Gamma(M\Gamma M)\Gamma S \subseteq (M\Gamma M)\Gamma S$

$\therefore (M\Gamma M)\Gamma S$ is a left ideal of M.

Lemma 2.9: Let M be a ternary LA-Γ-semi group with left identity. If S is left ideal of M, then $(M\Gamma M)\Gamma a$ is also a left ideal of M , $\forall a \epsilon S$

Theorem 2.10: Let M be a ternary LA-Γ-semi group with left identity. Then a left ideal P of M is quasi prime ideal if and only if $(x\beta y)\beta z \in P \Rightarrow x \in P$ or $y \in P$ or $z \in P$ $\forall \alpha, \beta \epsilon \Gamma$ and $\forall x, y, z \epsilon M$

Proof: Let M be a ternary LA-Γ-semigroupwith left identityand P be a left ideal of M. Suppose that P is a quasi-prime ideal of M. Then

Consider, $((M\Gamma M)\Gamma x)\Gamma((M\Gamma M)\Gamma y)\Gamma((M\Gamma M)\Gamma z) = ((((M\Gamma M)\Gamma y)\Gamma x)\Gamma(M\Gamma M))\Gamma((M\Gamma M)\Gamma z)$

$$= (((x\Gamma y)\Gamma(M\Gamma M)\Gamma(M\Gamma M))\Gamma((M\Gamma M)\Gamma z$$
$$= (((M\Gamma M)\Gamma(M\Gamma M)\Gamma(x\Gamma y))\Gamma((M\Gamma M)\Gamma z)$$
$$= (((M\Gamma M)\Gamma z)\Gamma(x\Gamma y)\Gamma((M\Gamma M)\Gamma(M\Gamma M))$$
$$= (((x\Gamma y)\Gamma z)\Gamma(M\Gamma M))\Gamma((M\Gamma M)\Gamma(M\Gamma M))$$
$$= (((M\Gamma M)\Gamma(M\Gamma M))\Gamma(M\Gamma M))\Gamma((x\Gamma y)\Gamma z)$$
$$= ((((M\Gamma M)\Gamma M)\Gamma M)\Gamma(M\Gamma M)\Gamma((x\Gamma y)\Gamma z)$$
$$= ((M\Gamma M)\Gamma(M\Gamma M))\Gamma((x\Gamma y)\Gamma z)$$
$$= (((M\Gamma M)\Gamma M)\Gamma M)\Gamma((x\Gamma y)\Gamma z)$$
$$= (M\Gamma M)\Gamma((x\Gamma y)\Gamma z) \subseteq (M\Gamma M)\Gamma P \subseteq P$$

By lemma.2.9, $(M\Gamma M)\Gamma x$, $(M\Gamma M)\Gamma y$, $(M\Gamma M)\Gamma z$ are left ideals of M.

Since P is a quasi-prime ideal of M $((M\Gamma M)\Gamma x)\Gamma((M\Gamma M)\Gamma y)((M\Gamma M)\Gamma z) \subseteq P$

That implies $(M\Gamma M)\Gamma x \subseteq P$ or $(M\Gamma M)\Gamma y \subseteq P$ or $(M\Gamma M)\Gamma z \subseteq P$

Therefore $(e\alpha e)\beta x \in (M\Gamma M)\Gamma x \subseteq P \implies x \in P$ or $(e\alpha e)\beta y \in (M\Gamma M)\Gamma y \subseteq P \implies y \in P$ or $(e\alpha e)\beta z \in (M\Gamma M)\Gamma z \subseteq P \implies z \in P$

$\therefore$ If P is quasi prime ideal, then $(x\alpha y)\beta z \in P \implies x \in P$ or $y \in P$ or $z \in P \forall \alpha, \beta \in \Gamma$.

Conversely, Assume that $(x\alpha y)\beta z \in P \implies x \in P$ or $y \in P$ or $z \in P \forall \alpha, \beta \in \Gamma$, we shall show that P is a quasi-prime ideal of M. Suppose that $(X\Gamma Y)\Gamma Z \subseteq P$ where X, Y and Z are left ideals M. Let us assume that $X \nsubseteq P$ and $Y \nsubseteq P$.we have to show that $Z \subseteq P$. If $X \nsubseteq P$ and $Y \nsubseteq P$, then there exists $x \in X, y \in Y$ such that $x, y \notin P$. Clearly $(x\alpha y)\beta z \in (X\Gamma Y)\Gamma Z \subseteq P$ where $z \in Z$

$\therefore (x\alpha y)\beta z \in P \implies x \in P$ or $y \in P$ or $z \in P$ but $, y \notin P$, $z \in Z \implies z \in P$ and z is arbitrary $\implies Z \subseteq P$. Hence P is a quasi prime ideal of M.

Example 2.11: Let $M = Z$, the set of integers and $\Gamma = N$, the set of natural numbers. Then $(M, \Gamma, \cdot)$ is a ternary LA-Γ-semi group. Let $P = \{2x / x \in Z\}$,clearly P is quasi prime ideal of M as $(3Z\Gamma 4Z)\Gamma 5Z \subseteq P \implies 4Z \subseteq P$ where $\implies 3Z, 4Z, 5Z$ are left ideals of M

Consider $Q = \{15x / x \in Z\}$, clearly Q is left ideal but not quasi prime ideal of M as $(Z\Gamma 3Z)\Gamma 5Z \subseteq Q$ but $Z \nsubseteq Q$ or $3Z \nsubseteq Q$ or $5Z \nsubseteq Q$

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Theorem 2.12: Let M be a ternary LA-Γ-semi group with left identity. If S is a left ideal of M and P is a quasi prime ideal of M then $S \cap P$ is a quasi prime ideal of M.

Proof:-Suppose M be a ternary LA-Γ-semi group with left identity and let S is a left ideal of M and P is a quasi-prime ideal of M.Clearly $S \cap P$ is a left ideal of M.

Let $(a\alpha b)\beta c \in (S \cap P) \Longrightarrow (a\alpha b)\beta c \in P \Rightarrow a \in P \;\; or \;\; b \in P \;\; or \;\; c \in P$

$\Rightarrow a \in S \cap P \quad or \;\; b \in S \cap P \quad or \;\; c \in S \cap P \Rightarrow S \cap P$ is quasi prime ideal of M.

Theorem2.13: Let M be a ternary LA-Γ-semi group with left identity. If P is a quasi prime ideal of M then, then $(P\colon a\colon b)$ is a quasi-prime ideal of M where $a, b \in M$.

Proof:-Assume that P is a quasi-prime ideal of M.∴ $(P\colon a\colon b)$ is left ideal of M

To show that $(P\colon a\colon b)$ is quasi prime ideal.

$\therefore (x\alpha y)\beta z\epsilon(P\colon a\colon b) \Rightarrow (a\gamma b)\delta((x\alpha y)\beta z)\epsilon P$ -------------(1)

$$\Rightarrow \Big(\big((a\gamma b)\delta x\big)\alpha y\Big)\beta z\, \epsilon P$$

$\Rightarrow (a\gamma b)\delta x\epsilon P$ or $y\epsilon P$ or $z\epsilon P$

$$\Rightarrow (a\gamma b)\delta x\epsilon P$$

$$\Rightarrow x\epsilon(P\colon a\colon b)$$

Similarly, $x\delta\big((a\gamma b)\alpha y\big)\beta z\epsilon P \Rightarrow x\epsilon P \;\; or \;\; (a\gamma b)\alpha y\epsilon P \quad or \quad z\epsilon P$

Now $(a\gamma b)\alpha y\epsilon P \Rightarrow y\epsilon(P\colon a\colon b)$ and $(x\alpha y)\delta((a\gamma b)\beta z\epsilon P \Rightarrow x\epsilon P \;\; or \;\; y\epsilon P \quad or \quad (a\gamma b)\beta z\epsilon P$

$Hence \quad (a\gamma b)\beta z\epsilon P \Rightarrow z\epsilon(P\colon a\colon b)$.

Theorem 2.14: Let M be a ternary LA-Γ-semi group with left identity. A left ideal P of M is quasi prime ideal of M if and only if $M - P$ is either ternary LA-Γ-sub semi group of M or empty.

Proof: Suppose P is quasi prime ideal of M and $M - P \neq \emptyset$. Let $a, b, c\epsilon M - P$. Then $a, b, c \notin P$

By corollary we get $(a\alpha b)\beta c \notin P$ where $a, b, c\epsilon M$ and $\alpha, \beta\epsilon\Gamma$.

$\therefore (a\alpha b)\beta c\epsilon M - P \Rightarrow M - P$ is a ternary LA-Γ-sub semi group of M.

Conversely, suppose that $M - P$ is either ternary LA-Γ-sub semi group of M or empty.

If $M - P = \emptyset$. Then $M = P$. $\therefore$ P is quasi prime ideal of M.

Assume that $M - P$ is a ternary LA-Γ-sub semi group of M. And Let $(a\alpha b)\beta c\epsilon P \Rightarrow (a\alpha b)\beta c \notin M - P$. Since $M - P$ is a ternary LA-Γ-sub semi group of M we get $a \notin M - P$ or $b \notin M - P$ or $c \notin M - P$ for $\alpha, \beta\epsilon\Gamma \Rightarrow a\epsilon P$ or $b\epsilon P$ or $c\epsilon P$ for $\alpha, \beta\epsilon\Gamma$

$\therefore (a\alpha b)\beta c\epsilon P \Rightarrow a\epsilon P$ or $b\epsilon P$ or $c\epsilon P$ $\Rightarrow P$ is a quasi-prime ideal of M

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THE DEGREE SEQUENCES OF S-CORONA GRAPHS

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ABSTRACT
The topological descriptors play a prominent role in mathematical chemistry, particularly in studies of quantitative structure property and quantitative structure activity relationships. The degree sequences of a graph are obtained by the set of vertex degrees i.e., $DS(G) = \{d_1^{z_1},\ d_2^{z_2},\ \dots, d_i^{z_i}\}$. *In this paper we swot up the degree sequences of* $S-$ *vertex (edge) corona,* $S-$ *vertex (edge) neighbourhood corona of graphs and relationship.*

Keywords: $S-$ *vertex (edge) corona,* $S-$ *vertex (edge) neighbourhood corona and degree sequences.*

1. INTRODUCTION

Let $G = (V, E)$ be a nontrivial, simple, connected, undirected graph with r vertices and s edges. The vertex set is $V(G) = \{v_1,\ v_2,\ v_3 ,\dots,\ v_n\}$ and an edge set is $E(G) = \{e_1,\ e_2,\ e_3, \dots, e_m\}$. The number of edges incident on vertex v is the degree of vertex v and it is denoted by $d_G(v)$ or $deg_G(v)$ or $degree_G(v)$. The subdivision graph of G is obtained by inserting vertex to each edge. The study of Chemical graphs is called Chemical graph theory is a branch of Mathematical Chemistry **[5]**. Chemical graphs are obtained by in which atoms are represented by vertices and chemical bonds by edges of graphs. The basic idea of chemical graph theory is that physi-chemical properties of molecules can be studied by using the information encoded in their corresponding chemical graphs.

The degree sequence (DS) of G is the sequence $\{d_1,\ d_2, \dots, d_n\}$ where d_i , $1 \le i \le n$, are the degrees of the vertices v_i of a graph G in any order **[1, 7]**. The degree sequence(DS) are useful to study the $QSPR$ and $QSAR$ of chemical compounds.

Let G and H be two graphs with vertices r_1 and r_2 and edges s_1 and s_2 respectively. We recall the definition from **[2, 3, 4, 6],** the $S-$ vertex corona of graphs G and H with disjoint vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$ is obtained from the subdivision of G and $|V(G)|$ copies of H, by joining the i^{th} vertex of $V(G)$ to each vertex in the i^{th} copy of H . Then $|V(G \odot_S H)| = r_1(1 + r_2) + s_1$ and $|E(G \odot_S H)| = 2s_1 + r_1(s_2 + r_2)$.

The $S-$ edge corona of graphs G and H with disjoint vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$ is obtained from the subdivision of G and $|E(G)|$ copies of H, by joining the i^{th} vertex of $I(G)$ ($I(G)$ is the inserted vertices in G) to each vertex in the i^{th} copy of H. Then $|V(G \ominus_S H)| = r_1 + s_1(1 + r_2)$ and $|E(G \ominus_S H)| = s_1(2 + s_2 + r_2)$.

The $S-$ vertex neighbourhood corona of graphs G and H with disjoint vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$ is obtained from the subdivision of G and $|V(G)|$ copies of H, by joining the neighbours of i^{th} vertex of $V(G)$ to each vertex in the i^{th} copy of H. Then $|V(G \odot_{nS} H)| = r_1(1 + r_2) + s_1$ and $|E(G \odot_{nS} H)| = 2s_1 + r_1 s_2 + r_2 \sum_{i=1}^{r_1} d_G(v_i)$.

The $S-$ edge neighbourhood corona of graphs G and H with disjoint vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$ is obtained from the subdivision of G and $|E(G)|$ copies of H, by joining the neighbours of i^{th} vertex of $I(G)$ ($I(G)$ is the inserted vertices in G) to each vertex in the i^{th} copy of H. Then $|V(G \ominus_{nS} H)| = r_1 + s_1(1 + r_2)$ and $|E(G \ominus_{nS} H)| = s_1(2 + s_2 + 2r_2)$.

2. General Formulae for the DSs of the $S-$ vertex corona, $S-$ edge corona, $S-$ vertex neighbourhood corona and $S-$ edge neighbourhood corona.

In this section, we obtain the DS of the $S-$ vertex corona, $S-$ edge corona, $S-$ vertex neighbourhood corona and $S-$ edge neighbourhood corona of any given number of simple connected graphs. First we start with two graphs $G,\ H$ and obtain the DS of $G \odot_S H$, $G \ominus_S H$, $G \odot_{nS} H$ and $G \ominus_{nS} H$ and using mathematical induction, we obtain the general formula for $G_1 \odot_S G_2 \odot_S \dots \odot_S G_k$, $G_1 \ominus_S G_2 \ominus_S \dots \ominus_S G_k$, $G_1 \odot_{nS} G_2 \odot_{nS} \dots \odot_{nS} G_k$ and $G_1 \ominus_{nS} G_2 \ominus_{nS} \dots \ominus_{nS} G_k$ in terms of the number of vertices of G_i's.

Theorem 2.1:- Let G and H be two simple connected graphs with DSs

$DS(G) = \{\lambda_{11}^{\mu_{11}}, \lambda_{12}^{\mu_{12}}, \dots, \lambda_{1k_1}^{\mu_{1k_1}}\}$

and

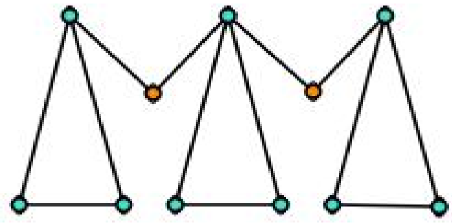
$DS(H) = \{\lambda_{21}^{\mu_{21}}, \lambda_{22}^{\mu_{22}}, \dots, \lambda_{2k_2}^{\mu_{2k_2}}\}$

respectively. Then the DS of the $S-$ vertex corona of the two graphs G and H is

$$DS(G\odot_S H) = \{(\lambda_{11}+r_2)^{\mu_{11}}, (\lambda_{12}+r_2)^{\mu_{12}}, \dots, (\lambda_{1k_1}+r_2)^{\mu_{1k_1}}, 2^{s_1}, (\lambda_{21}+1)^{r_1\mu_{21}},$$
$$(\lambda_{22}+1)^{r_1\mu_{22}}, \dots, (\lambda_{2k_2}+1)^{r_1\mu_{2k_2}}\}.$$

Note that to obtain $DS(G\odot_S H)$, we add r_2 to each λ_{1x} where $1 \le x \le k_1$, without changing the powers μ_{1x}, add number '1' to each λ_{2x} where $1 \le x \le k_2$, with changing the powers as $r_1\mu_{2x}$ and 2^{s_1}.

Example 2.1:- Let us consider $DS(P_l) = \{1^2, 2^{l-2}\}$ and $DS(P_m) = \{1^2, 2^{m-2}\}$, we will find the DS of $P_l\odot_S P_m$. Let the number of vertices of P_l be r_1 and the number of vertices of P_m be r_2.



$\boldsymbol{P_3\odot_S P_2}$

As $\lambda_{11} = 1$, $\mu_{11} = 2$, $\lambda_{12} = 2$, $\mu_{12} = 1$, $\lambda_{21} = 1$, $\mu_{21} = 2$ by the definition of $S-$ vertex corona.

We have,

$$DS(P_3\odot_S P_2) = \{1^2, 2^1\}\odot_S\{1^2\}$$
$$= \{(1+2)^2, (2+2)^1, 2^2, (1+1)^{3\times 2}\}$$

$= \{2^8, 3^2, 4^1\}$.

Theorem 2.2:- Let G and H be two simple connected graphs with DSs

$DS(G) = \{\lambda_{11}^{\mu_{11}}, \lambda_{12}^{\mu_{12}}, \dots, \lambda_{1k_1}^{\mu_{1k_1}}\}$

and

$DS(H) = \{\lambda_{21}^{\mu_{21}}, \lambda_{22}^{\mu_{22}}, \dots, \lambda_{2k_2}^{\mu_{2k_2}}\}$

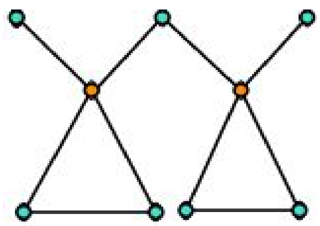
respectively. Then the DS of the $S-$ edge corona of the two graphs G and H is

$$DS(G\ominus_S H) = \{{\lambda_{11}}^{\mu_{11}}, {\lambda_{12}}^{\mu_{12}}, \dots, {\lambda_{1k_1}}^{\mu_{1k_1}}, (2+r_2)^{s_1}, (\lambda_{21}+1)^{s_1\mu_{21}},$$

$(\lambda_{22}+1)^{s_1\mu_{22}}, \dots, (\lambda_{2k_2}+1)^{s_1\mu_{2k_2}}\}$.

Note that to obtain $DS(G\ominus_S H)$, we write $DS(G)$ without changing, add number '1' to each λ_{2x} where $1 \le x \le k_2$, with changing the powers as $s_1\mu_{2x}$ and $(2+r_2)^{s_1}$.

Example 2.2:- Let us consider $DS(P_l) = \{1^2, 2^{l-2}\}$ and $DS(P_m) = \{1^2, 2^{m-2}\}$, we will find the DS of $P_l\ominus_S P_m$. Let the number of vertices of P_l be r_1 and the number of vertices of P_m be r_2.



$P_3\ominus_S P_2$

As $\lambda_{11} = 1$, $\mu_{11} = 2$, $\lambda_{12} = 2$, $\mu_{12} = 1$, $\lambda_{21} = 1$, $\mu_{21} = 2$ by the definition of $S-$ edge corona.

We have,

$$DS(P_3 \ominus_S P_2) = \{1^2, 2^1\} \ominus_S \{1^2\}$$
$$= \{1^2, 2^1, (2+2)^2, (1+1)^{2\times 2}\}$$

$= \{1^2, 2^5, 4^2\}$.

Theorem 2.3:- Let G and H be two simple connected graphs with DSs

$DS(G) = \{\lambda_{11}^{\mu_{11}}, \lambda_{12}^{\mu_{12}}, \dots, \lambda_{1k_1}^{\mu_{1k_1}}\}$

and

$DS(H) = \{\lambda_{21}^{\mu_{21}}, \lambda_{22}^{\mu_{22}}, \dots, \lambda_{2k_2}^{\mu_{2k_2}}\}$

respectively. Then the DS of the $S-$ vertex neighbourhood corona of the two graphs G and H is

$$DS(G \odot_{nS} H) = \{{\lambda_{11}}^{\mu_{11}}, {\lambda_{12}}^{\mu_{12}}, \dots, {\lambda_{1k_1}}^{\mu_{1k_1}}, (2+2r_2)^{s_1},$$
$$(\lambda_{21}+\lambda_{11})^{\mu_{11}\mu_{21}}, (\lambda_{22}+\lambda_{11})^{\mu_{11}\mu_{22}}, \dots, \left(\lambda_{2k_2}+\lambda_{11}\right)^{\mu_{11}\mu_{2k_2}},$$
$$(\lambda_{21}+\lambda_{12})^{\mu_{12}\mu_{21}}, (\lambda_{22}+\lambda_{12})^{\mu_{12}\mu_{22}}, \dots, \left(\lambda_{2k_2}+\lambda_{12}\right)^{\mu_{12}\mu_{2k_2}},$$

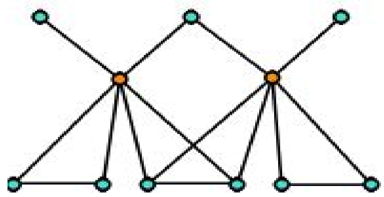
....

$$\left(\lambda_{21}+\lambda_{1k_1}\right)^{\mu_{1k_1}\mu_{21}}, \left(\lambda_{22}+\lambda_{1k_1}\right)^{\mu_{1k_1}\mu_{22}},$$

$\dots, \left(\lambda_{2k_2}+\lambda_{1k_1}\right)^{\mu_{1k_1}\mu_{2k_2}}\}$.

Note that to obtain $DS(G \odot_{nS} H)$, we write $DS(G)$ without changing, add each λ_{1x} to λ_{2y} where $1 \le x \le k_1$ *and* $1 \le y \le k_2$, with changing the powers $r_1\mu_{2y}$ and $(2+2\,r_2)^{s_1}$.

Example 2.3:- Let us consider $DS(P_l) = \{1^2, 2^{l-2}\}$ and $DS(P_m) = \{1^2, 2^{m-2}\}$, we will find the DS of $P_l \odot_{nS} P_m$. Let the number of vertices of P_l be r_1 and the number of vertices of P_m be r_2.



$P_3 \odot_{nS} P_2$

As $\lambda_{11} = 1$, $\mu_{11} = 2$, $\lambda_{12} = 2$, $\mu_{12} = 1$, $\lambda_{21} = 1$, $\mu_{21} = 2$ by the definition of $S-$ vertex neighbourhood corona.

We have,

$$DS(P_3 \odot_{nS} P_2) = \{1^2, 2^1\} \odot_{nS} \{1^2\}$$
$$= \{1^2, 2^1, (2+4)^2, (1+1)^{2\times 2}, (1+2)^{2\times 1}\}$$
$$= \{1^2, 2^5, 3^2, 6^2\}.$$

Theorem 2.4:- Let G and H be two simple connected graphs with DSs

$DS(G) = \{\lambda_{11}^{\mu_{11}}, \lambda_{12}^{\mu_{12}}, \dots, \lambda_{1k_1}^{\mu_{1k_1}}\}$

and

$DS(H) = \{\lambda_{21}^{\mu_{21}}, \lambda_{22}^{\mu_{22}}, \dots, \lambda_{2k_2}^{\mu_{2k_2}}\}$

respectively. Then the DS of the $S-$ edge neighbourhood corona of the two graphs G and H is

$$DS(G \ominus_{nS} H) = \Big\{(\lambda_{11}+\lambda_{11}r_2)^{\mu_{11}}, (\lambda_{12}+\lambda_{12}r_2)^{\mu_{12}}, \dots, \left(\lambda_{1k_1}+\lambda_{1k_1}r_2\right)^{\mu_{1k_1}}, 2^{s_1}, (\lambda_{21}+2)^{s_1\mu_{21}}, (\lambda_{22}+2)^{s_1\mu_{22}}, \dots, \left(\lambda_{2k_2}+2\right)^{s_1\mu_{2k_2}}\Big\}.$$

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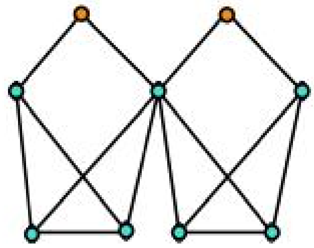
Note that to obtain $DS(G \ominus_{nS} H)$, we add $\lambda_{1x}r_2$ to each λ_{1x} where $1 \le x \le k_1$, without changing the powers μ_{1x} , add number '2' to each λ_{2x} where $1 \le x \le k_2$, with changing the powers as $s_1\mu_{2x}$ and 2^{s_1}.

Example 2.4:- Let us consider $DS(P_l) = \{1^2, 2^{l-2}\}$ and $DS(P_m) = \{1^2, 2^{m-2}\}$, we will find the DS of $P_l \ominus_{nS} P_m$. Let the number of vertices of P_l be r_1 and the number of vertices of P_m be r_2.



$P_3 \ominus_{nS} P_2$

As $\lambda_{11} = 1$, $\mu_{11} = 2$, $\lambda_{12} = 2$, $\mu_{12} = 1$, $\lambda_{21} = 1$, $\mu_{21} = 2$ by the definition of $S-$ edge neighbourhood corona.

We have,

$$DS(P_3 \ominus_{nS} P_2) = \{1^2, 2^1\} \ominus_{nS} \{1^2\}$$
$$= \{(1 + 1(2))^2, (2 + 2(2))^1, 2^2, (1 + 2)^{2\times 2}\}$$

$= \{2^2, 3^6, 6^1 \}$.

Now we take the $S-$ vertex corona, $S-$ edge corona, $S-$ vertex neighbourhood corona and $S-$ edge neighbourhood corona of l simple connected graphs $G_1, G_2, G_3, \dots, G_l$, where $l \ge 2$ is a finite integer. The DS of $G_1 \odot_S G_2 \odot_S \dots \odot_S G_l$, $G_1 \ominus_S G_2 \ominus_S \dots \ominus_S G_l$, $G_1 \odot_{nS} G_2 \odot_{nS} \dots \odot_{nS} G_l$ and $G_1 \ominus_{nS} G_2 \ominus_{nS} \dots \ominus_{nS} G_l$ is given as follows.

Theorem 2.5:- Let $G_1, G_2, G_3, \dots, G_l$ be l simple connected graphs. Let G_i have n_i vertices for $i = 1, 2, \dots, l$. Also let the DS of G_i be

$DS(G_i) = \{\lambda_{i1}^{\mu_{i1}}, \lambda_{i2}^{\mu_{i2}}, \dots, \lambda_{ik_i}^{\mu_{ik_i}}\}$.

Then the DS of the $S-$vertex corona of $G_1, G_2, G_3, \dots, G_l$ is

$$DS(G_1 \odot_S G_2 \odot_S \dots \odot_S G_l) = \{(\lambda_{11} + r_2 + r_3 + \dots + r_l)^{\mu_{11}}, \dots, (\lambda_{1k_1} + r_2 + r_3 + \dots + r_l)^{\mu_{1k_1}},$$
$$(\lambda_{21} + 1 + r_3 + r_4 + \dots + r_l)^{r_1\mu_{21}}, \dots,$$
$$(\lambda_{2k_2} + 1 + r_3 + r_4 + \dots + r_l)^{r_1\mu_{2k_2}},$$
$$(\lambda_{31} + 1 + r_4 + r_5 + \dots + r_l)^{|V(G_1 \odot_S G_2)|\mu_{31}}, \dots,$$
$$(\lambda_{3k_3} + 1 + r_4 + r_5 + \dots + r_l)^{|V(G_1 \odot_S G_2)|\mu_{3k_3}},$$

....,

$$(\lambda_{(l-1)1} + 1 + r_l)^{|V(G_1 \odot_S G_2 \odot_S \dots \odot_S G_{(l-2)})|\mu_{(l-1)1}}, \dots,$$
$$(\lambda_{(l-1)k_{(l-1)}} + 1 + r_l)^{|V(G_1 \odot_S G_2 \odot_S \dots \odot_S G_{(l-2)})|\mu_{(l-1)k_{(l-1)}}},$$
$$(\lambda_{l1} + 1)^{|V(G_1 \odot_S G_2 \odot_S \dots \odot_S G_{(l-1)})|\mu_{l1}}, \dots,$$
$$(\lambda_{lk_l} + 1)^{|V(G_1 \odot_S G_2 \odot_S \dots \odot_S G_{(l-1)})|\mu_{lk_l}},$$
$$(2 + r_3 + r_4 + \dots + r_l)^{s_1}, (2 + r_4 + r_5 + \dots + r_l)^{|E(G_1 \odot_S G_2)|},$$

$\dots, (2)^{|E(G_1 \odot_S G_2 \odot_S \dots \odot_S G_{(l-1)})|}\}$.

Theorem 2.6:- Let $G_1, G_2, G_3, \dots, G_l$ be l simple connected graphs. Let G_i have n_i vertices for $i = 1, 2, \dots, l$. Also let the DS of G_i be

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$DS(G_i) = \{\lambda_{i1}^{\mu_{i1}}, \lambda_{i2}^{\mu_{i2}}, \dots, \lambda_{ik_i}^{\mu_{ik_i}}\}$.

Then the DS of the $S-$edge corona of $G_1, G_2, G_3, \dots, G_l$ is

$$DS(G_1 \ominus_S G_2 \ominus_S \dots \ominus_S G_l) = \{\lambda_{11}^{\mu_{11}}, \lambda_{12}^{\mu_{12}}, \dots, \lambda_{1k_1}^{\mu_{1k_1}}, (\lambda_{21}+1)^{s_1\mu_{21}}, \dots, (\lambda_{2k_2}+1)^{s_1\mu_{2k_2}} (\lambda_{31}+1)^{|E(G_1\ominus_S G_2)|\mu_{31}}, \dots, (\lambda_{3k_3}+1)^{|E(G_1\ominus_S G_2)|\mu_{3k_3}} (\lambda_{(l-1)1}+1)^{|E(G_1\ominus_S G_2\ominus_S \dots \ominus_S G_{(l-2)})|\mu_{(l-1)1}}, \dots, \left(\lambda_{(l-1)k_{(l-1)}}+1\right)^{|E(G_1\ominus_S G_2\ominus_S \dots \ominus_S G_{(l-2)})|\mu_{(l-1)k_{(l-1)}}}, (\lambda_{l1}+1)^{|E(G_1\ominus_S G_2\ominus_S \dots \ominus_S G_{(l-1)})|\mu_{l1}}, \dots, (\lambda_{lk_l}+1)^{|E(G_1\ominus_S G_2\ominus_S \dots \ominus_S G_{(l-1)})|\mu_{lk_l}}, (2+r_2)^{s_1}, (2+r_3)^{|E(G_1\ominus_S G_2)|}, \dots, (2+r_l)^{|E(G_1\ominus_S G_2\ominus_S \dots \ominus_S G_{(l-1)})|}\}.$$

Theorem 2.7:- Let $G_1, G_2, G_3, \dots, G_l$ be l simple connected graphs. Let G_i have n_i vertices for $i = 1, 2, \dots, l$. Also let the DS of G_i be

$DS(G_i) = \{\lambda_{i1}^{\mu_{i1}}, \lambda_{i2}^{\mu_{i2}}, \dots, \lambda_{ik_i}^{\mu_{ik_i}}\}$.

Then the DS of the $S-$vertex neighbourhood corona of $G_1, G_2, G_3, \dots, G_l$ is

$$DS(G_1 \odot_{nS} G_2 \odot_{nS} \dots \odot_{nS} G_l) = \{\lambda_{11}^{\mu_{11}}, \lambda_{12}^{\mu_{12}}, \dots, \lambda_{1k_1}^{\mu_{1k_1}}, (\lambda_{21}+\lambda_{11})^{\mu_{21}\mu_{11}}, \dots, (\lambda_{2k_2}+\lambda_{11})^{\mu_{2k_2}\mu_{11}} (\lambda_{21}+\lambda_{1k_1})^{\mu_{21}\mu_{1k_1}}, \dots, (\lambda_{2k_2}+\lambda_{1k_1})^{\mu_{2k_2}\mu_{1k_1}} (\lambda_{l1}+\lambda_{11})^{\mu_{l1}\mu_{11}}, \dots, (\lambda_{lk_l}+\lambda_{11})^{\mu_{lk_l}\mu_{11}}, \left(\lambda_{l1}+\lambda_{1k_1}+\lambda_{2k_2}+\dots+\lambda_{(l-1)k_{(l-1)}}\right)^{\mu_{l1}\mu_{1k_1}\mu_{2k_2}\dots\mu_{(l-1)k_{(l-1)}}}, \dots, \left(\lambda_{lk_l}+\lambda_{1k_1}+\lambda_{2k_2}+\dots+\lambda_{(l-1)k_{(l-1)}}\right)^{\mu_{lk_l}\mu_{1k_1}\mu_{2k_2}\dots\mu_{(l-1)k_{(l-1)}}}, (2+2r_2)^{s_1}, (2+2r_3)^{|E(G_1\odot_{nS}G_2)|}, \dots, (2+2r_l)^{|E(G_1\odot_{nS}G_2\odot_{nS}\dots\odot_{nS}G_{(l-1)})|}\}.$$

Theorem 2.8:- Let $G_1, G_2, G_3, \dots, G_l$ be l simple connected graphs. Let G_i have n_i vertices for $i = 1, 2, \dots, l$. Also let the DS of G_i be

$DS(G_i) = \{\lambda_{i1}^{\mu_{i1}}, \lambda_{i2}^{\mu_{i2}}, \dots, \lambda_{ik_i}^{\mu_{ik_i}}\}$.

Then the DS of the $S-$edge neighbourhood corona of $G_1, G_2, G_3, \dots, G_l$ is

$$DS(G_1 \ominus_{nS} G_2 \ominus_{nS} \dots \ominus_{nS} G_l) = \{ (..((\lambda_{11}+r_2\lambda_{11}) + (\lambda_{11}+r_2\lambda_{11})r_3)r_4 + ..)^{\mu_{11}}, \dots, (..((\lambda_{1k_1}+r_2\lambda_{1k_1}) + (\lambda_{1k_1}+r_2\lambda_{1k_1})r_3)r_4 + ..)^{\mu_{1k_1}} (\lambda_{l1}+2)^{|E(G_1\ominus_{nS}G_2\ominus_{nS}\dots\ominus_{nS}G_{(l-1)})|\mu_{l1}}, \dots, (\lambda_{lk_l}+2)^{|E(G_1\ominus_{nS}G_2\ominus_{nS}\dots\ominus_{nS}G_{(l-1)})|\mu_{lk_l}}, (((2+2r_2)+(2+2r_2)r_3) + ((2+2r_2)+(2+2r_2)r_3)r_4 + \cdots)^{|E(G_1\ominus_{nS}G_2\ominus_{nS}\dots\ominus_{nS}G_{(l-1)})|}\}.$$

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3. CONCLUSION

In this article, we study the $S-$ vertex corona, $S-$ edge corona, $S-$ vertex neighbourhood corona and $S-$ edge neighbourhood corona of number of simple graphs. From these results we get information about graph that are useful to understand the problems corresponding to the graphs.

4. ACKNOWLEDGEMENT

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SOME TOPOLOGICAL INDICES COMPUTING RESULTS AND SUBDIVISION IF ARCHIMEDEAN LATTICES $L(4, 8^2)$

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ABSTRACT
Topological index is a type of molecular descriptor that is calculated based on the molecular graph of a chemical compound. In this paper we computing results and subdivision of the archimedean lattices $L(4, 8^2)$ for the topological indices Atom bond connectivity index, Randic index, Harmonic index, First zagreb index, Second zagreb index, Hyper zagreb index, Inverse sum indeg index and Forgotton index.

Keywords: Topological Index, subdivision graph, $L_{4,8^2}$ circumference.

1. INTRODUCTION AND PRELIMINARIES

A topological index of a graph G is a numeric quantity related to G which is describe molecular graph G. A molecular graph is a simple graph such that its vertices corresponds to the atoms and the edges to the bonds, as a useful tool of research chemical graph is applied to reveal the relationships between various physical characteristics and chemical structures such as biological activity, chemical reactivity. A major part of the current research in mathematical chemistry, chemical graph theory and quantitative structure-activity property relationship studies involves topological indices.

There is an undirected graph without multiple edges and loops that is considered in this paper. we define G as a graph, and define E(G) and V(G) as the edge set and vertex set of G. We also use the notation $E_{x,y}$ to express the set of edges that the degree of end vertices x and y, mathematically, $E_{x,y} = \{uv / \{x,y\} = \{d(u),d(v)\}\}$, where the notations d(u) and d(v) denote the degree of vertices u and v in the graph G. The subdivision graph S(G) is the graph obtained from G by replacing each of its edge by a path of length 2 or equivalently by inserting an additional vertex into each edge of G [9]. In the following definitions, we will remind the meanings of some topological indices that will be needed for our results in this paper.

Definition 1: Estrada et al. [3], introduced Atom-bond connectivity index as

$$ABC(G) = \sum_{uv\in E(G)} \sqrt{\frac{d(u)+d(v)-2}{d(u)d(v)}}.$$

This index has been applied up until now to study the stability of alkanes and the strain energy of cyclo alkanes.

Definition 2: The connectivity index introduced in 1975 by Milan Randic [8], who has shown this index to reflect molecular branching. Randic index was defined as follows,

$$\aleph(G) = \sum_{uv\epsilon E(G)} \frac{1}{\sqrt{d(u)d(v)}} .$$

Definition 3: The Harmonic Index $H(G)$ is defined in [13], as

$$H(G) = \sum_{uv\in E(G)} \frac{2}{d(u)+d(v)} .$$

Definition 4: Zagreb indices have been introduced more than thirty years ago by Gutman and Trinajstic [5], which are defined as

$$M_1(G) = \sum_{uv\in E(G)} d(u)^2 = \sum_{uv\in E(G)}[\, d(u) + d(v)].$$

$$M_2(G) = \sum_{uv\in E(G)} d(u)d(v).$$

Definition 5: The Hyper Zagreb index HM (G) introduced by Shirdel et. al in [10], and is defined by

$HM(G) = \sum_{uv\in E(G)}[d(u) + d(v)]^2$. For some various study of this index, we may refer the references [2, 6, 7, 11].

Definition 6: Vukicevic et. al [12], introduced the Inverse sum indeg index as

$$ISI(G) = \sum_{uv\in E(G)} \frac{d(u)d(v)}{d(u)+d(v)}.$$

Definition 7: The Forgotton topological index or F index of a graph G has more applications in the analysis of drug molecular structures which was introduced by Furtula and Gutman [4]. This index can be formulated as $F(G) = \sum_{uv\in E(G)}(d(u)^2 + d(v)^2)$.

2. MAIN RESULTS

The Archimedean latttices are uniform tilings of the plane in which all the faces are regular polygons and the symmetry group acts transitively on the vertices. It follows that all vertices are equivalent and have the same co-ordination number [1]. In this paper, we define an archimedean lattice called $(4,8^2)$. and we name this lattice $L_{4,8^2}$ circumference. The structure is as follows.

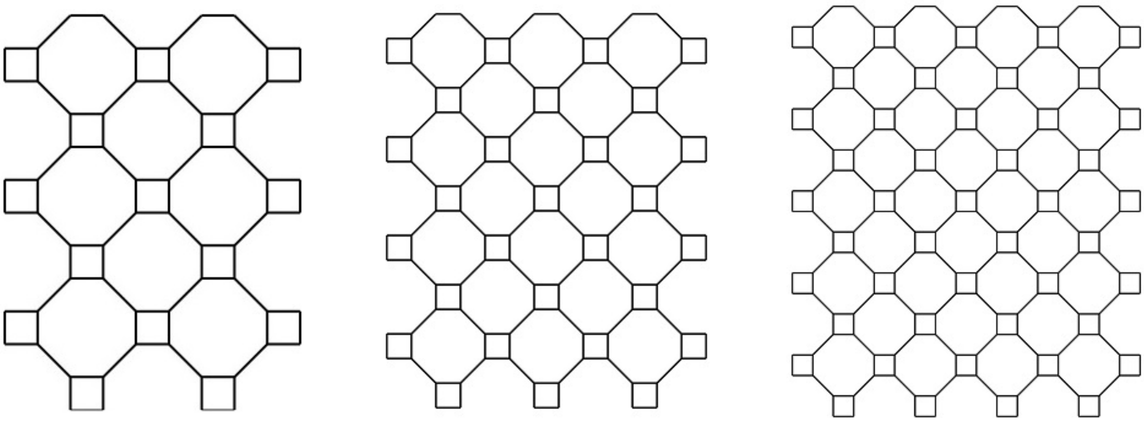

Fig-1: The Archimedean lattice $L_{4,8^2}(n)$.

The number of vertices of subdivision of $L_{4,8^2}(n)$ is $\frac{61n^2+137n+102}{2}$ and the number edges of subdivision of $L_{4,8^2}(n)$ is $29n^2 + 61n + 82$.

Table 1: The partition of the edge set of $L_{4,8^2}(n)$ in to $E_{s,t}$.

(s,t)	(2,2)	(2,3)	(3,3)
$\|E_{s,t}\|$	4n+6	8n+12	$\frac{29n^2 + 37n + 46}{2}$

Table 1: The subdivision of the partition of the edge set of $L_{4,8^2}(n)$ in to $E_{s,t}$.

(s,t)	(2,2)	(2,3)
$\|E_{s,t}\|$	16n+24	$29n^2 + 45n + 58$

In the proof of each following results, we will consider the edge partition of $L_{4,8^2}(n)$ given in Table 1 and the subdivision of the edge partition $S(L_{4,8^2}(n))$ given in Table 2, respectively.

Theorem 1: Let G be an $L_{4,8^2}(n)$ circumference. Then $ABC\left(L_{4,8^2}(n)\right) = \frac{29n^2}{3} + n\left[\frac{18\sqrt{2}+37}{3}\right] + \frac{27\sqrt{2}+46}{3}$ and $ABC\left(S(L_{4,8^2}(n))\right) = \frac{29n^2}{\sqrt{2}} + n\left[\frac{61}{\sqrt{2}}\right] + \frac{82}{\sqrt{2}}$.

Proof: By definition 1,

$$ABC\left(L_{4,8^2}(n)\right) = \sqrt{\frac{2}{4}}|E_{2,2}| + \sqrt{\frac{3}{6}}|E_{2,3}| + \sqrt{\frac{4}{9}}|E_{3,3}|$$

$= \sqrt{\frac{2}{4}}(4n + 6) + \sqrt{\frac{3}{6}}(8n + 12) + \sqrt{\frac{4}{9}}\left[\frac{29n^2+37n+46}{2}\right]$

$= \frac{29n^2}{3} + n\left[\frac{18\sqrt{2}+37}{3}\right] + \frac{27\sqrt{2}+46}{3}$

and

$$ABC\left(S(L_{4,8^2}(n))\right) = \sqrt{\frac{2}{4}}|E_{2,2}| + \sqrt{\frac{3}{6}}|E_{2,3}|$$

$$= \sqrt{\frac{2}{4}}(16n + 24) + \sqrt{\frac{3}{6}}(29n^2 + 45n + 58)$$

$$= \frac{29n^2}{\sqrt{2}} + n\left[\frac{61}{\sqrt{2}}\right] + \frac{82}{\sqrt{2}}$$

Theorem2: Let G be an $L_{4,8^2}(n)$ circumference. Then $\aleph\left(L_{4,8^2}(n)\right) = \frac{29n^2}{6} + n\left[\frac{49\sqrt{3}+24\sqrt{2}}{6\sqrt{3}}\right] + \frac{32\sqrt{6}+36}{3\sqrt{6}}$ and $\aleph\left(S\left(L_{4,8^2}(n)\right)\right) = \frac{29n^2}{\sqrt{6}} + n\left[\frac{8\sqrt{6}+45}{\sqrt{6}}\right] + \frac{12\sqrt{6}+58}{\sqrt{6}}$.

Theorem 3: Let G be an $L_{4,8^2}(n)$ circumference. Then $H\left(L_{4,8^2}(n)\right) = \frac{29n^2}{6} + n\left[\frac{341}{30}\right] + \frac{232}{15}$ and $H\left(S\left(L_{4,8^2}(n)\right)\right) = \frac{58n^2}{5} + 26n + \frac{176}{5}$.

Theorem 4: Let G be an $L_{4,8^2}(n)$ circumference. Then $M_1\left(L_{4,8^2}(n)\right) = 87n^2 + 167n + 222$ and $M_1\left(S\left(L_{4,8^2}(n)\right)\right) = 145n^2 + 289n + 386$.

Theorem 5: Let G be an $L_{4,8^2}(n)$ circumference. Then $M_2\left(L_{4,8^2}(n)\right) = \frac{261n^2}{2} + n\left[\frac{461}{2}\right] + 303$ and $M_2\left(S\left(L_{4,8^2}(n)\right)\right) = 174n^2 + 334n + 444$.

Theorem 6: Let G be an $L_{4,8^2}(n)$ circumference. Then $HM\left(L_{4,8^2}(n)\right) = \frac{2349n^2}{2} + n\left[\frac{3525}{2}\right] + 2259$ and $HM\left(S\left(L_{4,8^2}(n)\right)\right) = 725n^2 + 1381n + 1834$.

Theorem 7: Let G be an $L_{4,8^2}(n)$ circumference. Then $ISI\left(L_{4,8^2}(n)\right) = \frac{87n^2}{4} + n\left[\frac{827}{20}\right] + \frac{549}{10}$ and $ISI\left(S\left(L_{4,8^2}(n)\right)\right) = \frac{174n^2}{5} + 70n + \frac{468}{5}$.

Theorem 8: Let G be an $L_{4,8^2}(n)$ circumference. Then $F\left(L_{4,8^2}(n)\right) = 261n^2 + 469n + 618$ and $F\left(S\left(L_{4,8^2}(n)\right)\right) = 377n^2 + 713n + 946$.

The proof of the above theorems (2), (3), (4), (5), (6), (7) and (8) can be proved quite similarly.

CONCLUSION

In this study, we investigated some results and subdivision of a class of honeycomb network which is covered by C_4 and doublet of C_8 and also derived some formulas in terms of Atom-bond connectivity index, Randic index, Harmonic index, First zagreb index, Second zagreb index, Hyper zagreb index, Inverse sum indeg index, Forgotton index.

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SOME GRAPH INVARIANTS OF GRID

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ABSTRACT
In this paper, we compute First and second Zagreb indices, first and second multiplicative Zagreb indices, Augmented Zagreb index, Harmonic index of Grid.

Keywords and Phrases : First and second Zagreb indices, first and second mul- tiplicative Zagreb indices, Augmented Zagreb index, Harmonic index of Grid.

1 INTRODUCTION

Topological indices are the molecular descriptors that describes the structures of chemical compounds and it help us to predict certain physico-chemical properties like boiling point, enthalpy of vaporization, stability etc. In this paper, we determine the topological indices like First and second Zagreb indices, first and second multiplica- tive Zagreb indices, Augmented Zagreb index, Harmonic index of Grid.

All molecular graphs considered in this paper are finite, connected, loop less and without multiple edges. Let $G = (V, E)$ be a graph with n vertices and m edges. The degree of a vertex $u \in V(G)$ is denoted by d_u and is the number of vertices that are adjacent to u. The edge connecting the vertices u and v is denoted by uv. Using these terminologies, certain topological indices are defined in the following manner.

A pair of molecular descriptors (or topological index), known as the First Zagreb index $M_1(G)$ and the Second Zagreb index $M_2(G)$, first appeared in the topological formula for the total π-energy of conjugated molecules that has been derived in 1972

by I. Gutman and N.Trinajstic[7]. Soon after these indices have been used as branching indices. Later the Zagreb indices found applications in QSPR and QSAR studies. Zagreb indices are included in a number of programs used for the routine computation of topological indices POLLY, DRAGON, CERIUS, TAM, and DISSI. M1(G) and M2(G) were recognize as measures of the branching of the carbon atom molecular skeleton [8], and since then these are frequently used for structure - property modeling. De- tails on the chemical applications of the two Zagreb indices can be found in the books [10, 11]. Further studies on Zagreb indices can be found in [1, 6, 14, 15, 16].

Definition 1.1. For a simple connected graph G, the first and second zagreb indices were defined as follows

$$M_1(G) = \sum_{e=uv\in E(G)} (d_u + d_v)$$

$$M_2(G) = \sum_{e=uv\in E(G)} (d_u d_v)$$

Where d_v denotes the degree (number of first neighbors) of vertex v in G.

In 2012, M. Ghorbani and N. Azimi [4] defined the Multiple Zagreb topological indices of a graph G, based on degree of vertices of G.

Definition 1. 2. For a simple connected graph G, the first and second multiple Zagreb indices were defined as follows

$$PM_1(G) = \prod_{e=uv\in E(G)} (d_u + d_v)$$

$$PM_2(G) = \prod_{e=uv\in E(G)} (d_u d_v)$$

Properties of the first and second Multiple Zagreb indices may be found in [2, 5].

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The Augmented Zagreb index was introduced by Furtula et al [3]. This graph invariant has proven to be a valuable predictive index in the study of the heat of formation in octanes and heptanes, is a novel topological index in chemical graph theory, whose prediction power is better than atom-bond connectivity index. Some basic investigation implied that *AZI* index has better correlation properties and structural sensitivity among the very well established degree based topological indices.

Definition 1.3. Let G = (V, E) be a graph and d_u be the degree of a vertex u, then augmented Zagreb index is denoted by AZI(G) and is defined as

$$AZI(G) = \sum_{e=uv\in E(G)} \left(\frac{d_u d_v}{d_u+d_v-2}\right)^3$$

Further studies can be found in [9] and the references cited there in.

The Harmonic index was introduced by Zhong [13]. It has been found that the harmonic index correlates well with the Randic index and with the π-electronic energy of benzenoid hydrocarbons.

Definition 1.4. Let G = (V, E) be a graph and du be the degree of a vertex u then

Harmonic index is defined as

$$H(G) = \sum_{e=uv\in E(G)} \left(\frac{2}{d_u+d_v}\right)$$

Further studies on H(G) can be found in [12, 15].

1 Main results

Theorem 2.1. The First Zagreb index of grid with (m-1) rows and (n-1) columns is given by

$$M_1(G(m,n)) = \begin{cases} 16mn - 14m - 14n + 8 & \text{if } m > 2 \text{ and } n > 2 \\ 18n - 20 & \text{if } m = 2 \text{ and } n > 2 \\ 16 & \text{if } m = n = 2 \end{cases}$$

Proof. The topological structure of a grid network, denoted by $G(m, n)$, is defined

as the cartesian product $P_m X P_n$ of undirected paths P_m and P_n. The spectrum of the graph does not depend on the numbering of the vertices. However, here we adopt a particular numbering such that the edges has a pattern which is common for any dimension. We follow the sequential numbering from left to right as shown in the diagram.

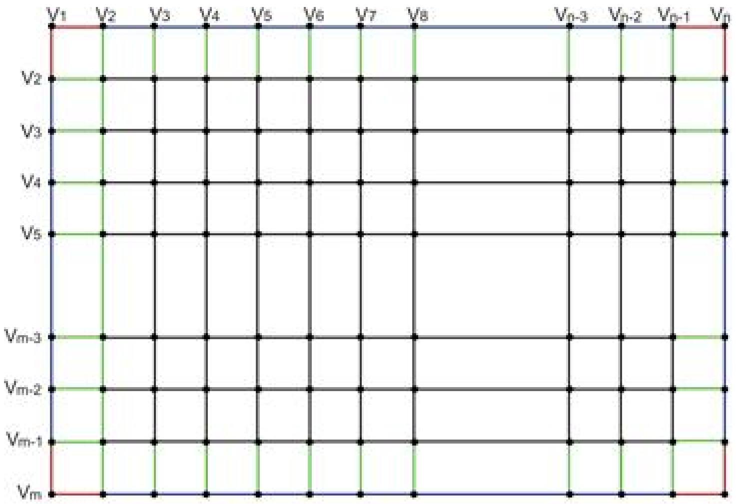

Figure 1

Consider a two-dimensional structure of Grid with $(m-1)$ rows and $(n-1)$ columns as shown in the Figure-1. Let $|e_{i,j}|$ denotes the number of edges connecting the vertices of degrees d_i and d_j.

Case 1 : If $m > 2$ and $n > 2$

Grid contains only $e_{2,3}$, $e_{3,3}$, $e_{3,4}$ and $e_{4,4}$ edges. In the above figure $e_{2,3}$, $e_{3,3}$, $e_{3,4}$ and $e_{4,4}$ edges are colored in red, blue, green and black respectively. The number of $e_{2,3}$, $e_{3,3}$, $e_{3,4}$ and $e_{4,4}$ edges in each row is mentioned in the following table.

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Row	$\lvert e_{2,3}\rvert$	$\lvert e_{3,3}\rvert$	$\lvert e_{3,4}\rvert$	$\lvert e_{4,4}\rvert$
1	4	$n-3$	n	$n-3$
2	0	2	2	$2n-5$
3	0	2	2	$2n-5$
4	0	2	2	$2n-5$
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
$m-2$	0	2	2	$2n-5$
$m-1$	4	$n-3$	$n-2$	0
Total	8	$2m+2n-12$	$2m+2n-8$	$2mn-5m-5n+12$

$\therefore |e_{2,3}| = 8$, $|e_{3,3}| = (2m+2n-12)$, $|e_{3,4}| = (2m+2n-8)$ and

$|e_{4,4}| = (2mn-5m-5n+12)$.

Consider,

$$M_1(G) = \sum_{e=uv\in E(G)} (d_u+d_v)$$

$= |e_{2,3}|\,(2+3) + |e_{3,3}|(3+3) + |e_{3,4}|\,|(3+4) + |e_{4,4}|(4+4)$

=8(5)+2(m+n-6)(3+3)+2(m+n-4)(3+4)+(2mn-5m-5n+12)(4+4).

=40+12m+12n-72+14m+14n-56+16mn-40m-40n+96

$\therefore M_1(G(m,n)) = 16mn - 14m - 14n + 8.$

Case 2 : If $m = 2$ and $n > 2$

In this case Grid contains $e_{2,2}$, $e_{2,3}$ and $e_{3,3}$ edges. The edges $e_{2,2}$, $e_{2,3}$ and $e_{3,3}$ are colored in red, blue and black respectively as shown in the Figure 2.

The number of $e_{2,2}$, $e_{2,3}$ and $e_{3,3}$ edges in each row is mentioned in the following table.

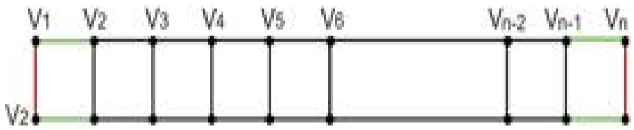

Figure 2

$\lvert e_{2,2}\rvert$	$\lvert e_{2,3}\rvert$	$\lvert e_{3,3}\rvert$
2	4	$(3n-8)$

$$M_1(G(2,n)) = \sum_{e=uv\in E(G(2,n))} (d_u+d_v)$$

$= |e_{2,2}|\,(2+2) + |e_{2,3}|(2+3) + |e_{3,3}|\,|(3+3)$

=2(4)+4(5)+(3n-8)(6)

=8+20+18n-48

$$\therefore M_1(G(2,n)) = 18n - 20$$

Similarly,

$$\therefore M_1(G(m,2)) = 18m - 20$$

Case 3: In this case the number of $e_{2,2}$ edges is as shown in Figure 3.

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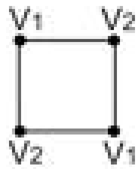

Figure 3

$\therefore M_1(G(2,2)) = \sum_{e=uv\in E(G(2,2))}(d_u+d_v) = |e_{2,2}|\ (2 + 2) = 4(2 + 2) = 16.$

Theorem 2.2. The Second Zagreb index of grid with (m−1) rows and (n−1) columns is given by

$$M_2(G(m,n)) = \begin{cases} 32mn - 38m - 38n + 36 & \text{if } m > 2 \text{ and } n > 2 \\ 27n - 40 & \text{if } m = 2 \text{ and } n > 2 \\ 16 & \text{if } m = n = 2 \end{cases}$$

Proof. **Case 1 :** If $m > 2$ and $n > 2$

Consider

$$M_2(G(m,n)) = \sum_{e=uv\in E(G)}(d_u.d_v)$$

$= |e_{2,3}|\ (2\ .\ 3) + |e_{3,3}|(3\ .\ 3) + |e_{3,4}|\ |(3\ .\ 4) + |e_{4,4}|(4\ .\ 4)$

=8(6)+2(m+n-6)(9)+2(m+n-4)(12)+(2mn-5m-5n+12)(16).

=48+18m+18n-108+24m+24n-96+32mn-80m-80n+192

$\therefore M_2(G(m,n)) = 32mn - 38m - 38n + 36.$

Case 2 : If $m = 2$ and $n > 2$

$$M_2(G(2,n)) = \sum_{e=uv\in E(G(2,n))}(d_u.d_v)$$

$= |e_{2,2}|\ (2\ .\ 2) + |e_{2,3}|(2\ .\ 3) + |e_{3,3}|\ |(3\ .\ 3)$

=2(4)+4(6)+(3n-8)(9)

=8+24+27n-72

$$\therefore M_2(G(2,n)) = 27n - 40$$

Similarly,

$$\therefore M_2(G(m,2)) = 27m - 40$$

Case 3: *If* m = 2 and n = 2

Consider

$$M_2(G(m,n)) = \sum_{e=uv\in E(G)}(d_u.d_v)$$

$= |e_{2,2}|\ (2 + 2) = 4(2 + 2) = 16.$

Theorem 2.3. The First multiple Zagreb index of Grid with (m-1) rows and (n-1) columns in each row is given by

$$PM_1(G(m,n)) = \begin{cases} 53760(m + n - 6)(m + n - 4)(2mn - 5m - 5n + 12) & \text{if } m > 2 \text{ and } n > 2 \\ 2880n - 7680 & \text{if } m = 2 \text{ and } n > 2 \\ 16 & \text{if } m = n = 2 \end{cases}$$

Proof. **Case 1 :** If $m > 2$ and $n > 2$

First multiple Zagreb index is given by

$$PM_1(G) = \prod_{e=uv\in E(G)}(d_u+d_v)$$

$= |e_{2,3}|\,(2+3)\,.\,|e_{3,3}|(3+3)\,.\,|e_{3,4}|\,|(3+4)\,.\,|e_{4,4}|(4+4)$

=8(5) . 2(m+n-6)(6) . 2(m+n-4)(7) . (2mn-5m-5n+12)(8).

=53760(m + n – 6)(m + n – 4)(2mn – 5m – 5n + 12)

$\therefore PM_1(G(m,n)) = 53760(m+n-6)(m+n-4)(2mn-5m-5n+12).$

Case 2 : If $m = 2$ and $n > 2$

Consider

$$PM_1(G(2,n)) = \prod_{e=uv\in E(G)} (d_u + d_v)$$

$= |e_{2,2}|\,(2+2)\,.\,|e_{2,3}|(2+3)\,.\,|e_{3,3}|\,|(3+3)$

=2(4). 4(5). (3n-8)(6)

=2880n-7680

$$\therefore PM_1(G(2,n)) = 2880n - 7680$$

Similarly,

$$\therefore PM_1(G(m,2)) = 2880m - 7680$$

Case 3: *If* m = 2 and n = 2

Consider

$$PM_1(G(2,2)) = \sum_{e=uv\in E(G)} (d_u + d_v)$$

$= |e_{2,2}|\,(2+2) = 4(2+2) = 16.$

Theorem 2.4. The Second multiple Zagreb index of Grid with (m-1) rows and (n-1) columns in each row is given by

$$PM_2(G(m,n)) = \begin{cases} 331776(m+n-6)(m+n-4)(2mn-5m-5n+12) & \text{if } m > 2 \text{ and } n > 2 \\ 5184n - 13824 & \text{if } m = 2 \text{ and } n > 2 \\ 16 & \text{if } m = n = 2 \end{cases}$$

Proof. **Case 1 :** If $m > 2$ and $n > 2$

Second multiple Zagreb index is given by

$$PM_2(G(m,n)) = \prod_{e=uv\in E(G(m,n))} (d_u d_v)$$

$= |e_{2,3}|\,(2\,.\,3)\,.\,|e_{3,3}|(3\,.\,3)\,.\,|e_{3,4}|\,|(3\,.\,4)\,.\,|e_{4,4}|(4\,.\,4)$

=8(6) . 2(m+n-6)(9) . 2(m+n-4)(12) . (2mn-5m-5n+12)(16).

=331776(m + n – 6)(m + n – 4)(2mn – 5m – 5n + 12)

$\therefore PM_2(G(m,n)) = 53760(m+n-6)(m+n-4)(2mn-5m-5n+12).$

Case 2 : If $m = 2$ and $n > 2$

Consider

$$PM_2(G(2,n)) = \prod_{e=uv\in E(G(2,n))} (d_u d_v)$$

$= |e_{2,2}|\,(2\,.\,2)\,.\,|e_{2,3}|(2\,.\,3)\,.\,|e_{3,3}|\,|(3\,.\,3)$

=2(4). 4(6). (3n-8)(9)

=5184n-13824

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$$\therefore PM_2(G(2,n)) = 5184\text{n} - 13824$$

Similarly,

$$\therefore PM_2(G(m,2)) = 5184\text{m} - 13824$$

Case 3: *If* m = 2 and n = 2

Consider

$$PM_2(G(2{,}2)) = \sum_{e=uv\in E(G(2,n))} (d_u d_v)$$

= $|e_{2,2}|$ (2 . 2)= 4(4)=16.

Theorem 2.5. The Augmented Zagreb index of Grid with (m-1) rows and (n-1) columns in each row is given by

$$AZI(G(m,n)) = \begin{cases} (37.925925)\text{mn} - (47.718898)\text{m} - (47.718898)\text{n} - (92.4114444) & \text{if } m > 2 \text{ } and \text{ } n > 2 \\ (34.171875)\text{n} - 13824 & \text{if m = 2 and n > 2} \\ 32 & if \text{ } m = n = 2 \end{cases}$$

Proof. **Case 1 :** If $m > 2$ and $n > 2$

The Augmented Zagreb index is given by

$$AZI(G) = \sum_{e=uv\in E(G)} \left(\frac{d_u d_v}{d_u + d_v - 2}\right)^3$$

$= |e_{2,3}|\left(\frac{2.3}{2+3-2}\right)^3 + |e_{3,3}|\left(\frac{3.3}{3+3-2}\right)^3 + |e_{3,4}|\left(\frac{3.4}{3+4-2}\right)^3 . + |e_{4,4}|\left(\frac{4.4}{4+4-2}\right)^3$

=8(8) +2(m+n-6) $\left(\frac{9}{4}\right)^3$ + 2(m+n-4) $\left(\frac{12}{5}\right)^3$ + (2mn-5m-5n+12) $\left(\frac{16}{6}\right)^3$

$\therefore AZI(G) = (37.925925)\text{mn} - (47.718898)\text{m} - (47.718898)\text{n} - (92.4114444).$

Case 2: If $m = 2$ and $n > 2$

Consider

$$AZI(G) = \sum_{e=uv\in E(G)} \left(\frac{d_u d_v}{d_u + d_v - 2}\right)^3$$

$= |e_{2,2}|\left(\frac{2.2}{2+2-2}\right)^3 + |e_{2,3}|\left(\frac{2.3}{2+3-2}\right)^3 + |e_{3,3}|\left(\frac{3.3}{3+3-2}\right)^3$

=2(8) +4(8) +(3n-8) $\left(\frac{9}{4}\right)^3$

$$\therefore AZI(G(2,n)) = (34.171875)\text{n} - 13824$$

Similarly,

$$\therefore AZI(G(m,2)) = (34.171875)\text{n} - 13824$$

Case 3: *If* m = 2 and n = 2

Consider

$$AZI(G(2,2)) = \sum_{e=uv\in E(G(2,2))} \left(\frac{d_u d_v}{d_u + d_v - 2}\right)^3$$

$= |e_{2,2}|\left(\frac{2.2}{2+2-2}\right)^3$ = 4(8) =3 2.

Theorem 2.6. The Harmonic index of Grid with (m-1) rows and (n-1) columns in each row is given by

$$H(G(m,n)) = \begin{cases} (0.5)mn - (0.0119047)m - (0.0119047)n - (0.0857142) & \text{if } m > 2 \text{ and } n > 2 \\ n - 0.06666667 & \text{if } m = 2 \text{ and } n > 2 \\ 2 & \text{if } m = n = 2 \end{cases}$$

Proof. **Case 1:** If $m > 2$ and $n > 2$

The Harmonic index is given by

$$H(G(m,n)) = \sum_{e=uv\in E(G(m,n))} \left(\frac{2}{d_u+d_v}\right)$$

$= |e_{2,3}| \left(\frac{2}{2+3}\right) + |e_{3,3}|\left(\frac{2}{3+3}\right) + |e_{3,4}| \left(\frac{2}{3+4}\right). + |e_{4,4}| \left(\frac{2}{4+4}\right)$

$=8\left(\frac{2}{5}\right) + 2(m+n-6) \left(\frac{2}{6}\right) + 2(m+n-4) \left(\frac{2}{7}\right) + (2mn-5m-5n+12) \left(\frac{2}{8}\right)$

$\therefore H(G(m,n)) = (0.5)mn - (0.0119047)m - (0.0119047)n - (0.0857142).$

Case 2: If $m = 2$ and $n > 2$

Consider

$$H(G(2,n)) = \sum_{e=uv\in E(G(m,n))} \left(\frac{2}{d_u+d_v}\right)$$

$= |e_{2,2}| \left(\frac{2}{2+2}\right) + |e_{2,3}|\left(\frac{2}{2+3}\right) + |e_{3,3}| \left(\frac{2}{3+3}\right)$

$=2\left(\frac{2}{4}\right) +4 \left(\frac{2}{5}\right) +(3n-8) \left(\frac{2}{6}\right)$

$$\therefore H(G(2,n)) = n - 0.06666667$$

Similarly,

$$H(G(m,2)) = m - 0.06666667$$

Case 3: *If* m = 2 and n = 2

Consider

$$H(G(2,2)) = \sum_{e=uv\in E(G(2,2))} \left(\frac{2}{d_u+d_v}\right)$$

$= |e_{2,2}| \left(\frac{2}{2+2}\right) = 4\left(\frac{2}{4}\right) = 2.$

2 CONCLUSION

The problem of finding First and second Zagreb indices, first and second multiplicative Zagreb indices, Augumented Zagreb index, Harmonic index of Grid is solved here analytically without using computers.

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LUCAS POLYNOMIAL COEFFICIENT BOUNDS FOR A CLASS OF BI-BAZILEVIČ FUNCTIONS

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ABSTRACT
In this paper, we defined the subclass of Bi-Bazilevič functions through the (p,q)-Lucas polynomials using the subordination principle. Further the bound for the initial coefficient and the Fekete-Szegö inequalities are determined. Some interesting relevant remarks and results are also investigated.

Keywords: Analytic functions - Bi-univalent functions – Bi-Bazilevič functions- (p,q)- Lucas polynomials – Coefficient bounds – Fekete-Szegö problem.

1. INTRODUCTION

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are normalized and analytic in the open unit disk $\Delta = \{z \backslash z \in C \text{ and } |z| < 1\}$. Let $\mathcal{S}$ be the class of all functions $f \in A$ which are univalent in Δ.

A function $f \in A$ is said to be starlike, if it satisfies the inequality

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \Delta) \tag{2}$$

We denote the class which consists of all functions $f \in A$ that are starlike by S^*.

A function $f \in A$ is said to be convex, if it satisfies the inequality

$$Re\left(1 + \frac{zf'(z)}{f'(z)}\right) > 0 \qquad (z \in \Delta) \tag{3}$$

We denote the class which consists of all functions $f \in A$ that are convex □.

For two functions f and g which are analytic in Δ, we say that the function f is subordinate to g, we write $f(z) \prec g(z)$, if there exists a Schwarz function w, that is a function w analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$ in Δ, such that $f(z) = g(w(z))$ for all $z \in \Delta$.

In particular, if the function g is univalent in Δ, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$.

The Koebe one-fourth theorem ensures that the image of Δ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $1/4$. Thus every function $f \in \mathcal{S}$ has an inverse f^{-1}, such that

$$f^{-1}(f(z)) = z \quad (z \in \Delta), \ \ and \ \ f(f^{-1}(w)) = w \quad \left(|w| < r_0(f); \ \ r_0(f) \geq \tfrac{1}{4}\right)$$

where the inverse function f^{-1} is given by [10]

$$g(w) \coloneqq f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots \tag{4}$$

A function $f \in A$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ, in the sense that f^{-1} has a univalent analytic continuation to Δ. Let Σ denote the class of bi-univalent functions in Δ given by (1). Several authors have subsequently studied similar problems in this direction. Brannan and Taha considered certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. They introduced bi-starlike functions and bi-convex functions and determined the coefficient bounds. Recently, Srivastava et al. [17] introduced the subclasses of bi-univalent functions and found the initial coefficients. Further, many authors have investigated coefficient bounds of several subclasses of functions in [1,2,3].

Let $p(x)$ and $q(x)$ be polynomials with real coefficients. The (p,q)-Lucas polynomials $L_{p,q,n}(x)$ are defined by the recurrence relation

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$$L_{p,q,n(x)} = p(x)L_{p,q,n-2}(x) + q(x)L_{p,q,n-2}(x) \qquad (n \geq 2) \qquad (5)$$

With

$$L_0(x) = 2, L_1(x) = p(x), L_2(x) = p^2(x) + 2q(x), \qquad L_3(x) = p^3(x) + 3p(x)q(x)$$

$$G_{L_{n(x)}}(z) := \sum_{n=0}^{\infty} L_n(x)z^n = \frac{2 - p(x)z}{1 - p(x)z - q(x)z^2} \qquad (6)$$

Few special cases of $L_{p,q,n}(x)$ are listed below:

1) For $p(x) = x$ and $q(x) = 1$, we get the Lucas polynomials $L_n(x)$.

2) For $p(x) = 2x$ and $q(x) = 1$, we have the Pell-Lucas polynomials $Q_n(x)$.

3) For $p(x) = 1$ and $q(x) = 2x$, we obtain the Jacobsthal-Lucas polynomials $j_n(x)$.

4) For $p(x) = 3x$ and $q(x) = -2$, we attain the Fermat Lucas polynomials $f_n(x)$.

5) For $p(x) = 2x$ and $q(x) = -1$, we get the Chebyshev polynomials $T_n(x)$.

These special polynomials and their generalizations are of great importance in a variety of branches such as physics, engineering, number theory and numerical analysis etc., These interesting polynomials have been studied in several papers [4,6,12-16,18,19]

2. THE FUNCTION CLASS $B_{\Sigma}^{\gamma,\lambda}(x)$

Definition 2.1 A function $f \in \Sigma$ of the form (1) belongs to the class $B_{\Sigma}^{\gamma,\lambda}(x)$ and $z, w \in \Delta$, if the following conditions are satisfied:

$$1 + \frac{1}{\gamma}\left(\frac{z^{1-\lambda}f'(z)}{(f(z))^{1-\lambda}} - 1\right) < G_{L_{p,q,n(x)}}(\phi(z)) - 1 \qquad ,(z \in \Delta) \qquad (7)$$

and for $g(w) = f^{-1}(w)$

$$1 + \frac{1}{\gamma}\left(\frac{w^{1-\lambda}g'(w)}{(g(w))^{1-\lambda}} - 1\right) < G_{L_{p,q,n(x)}}(\psi(w)) - 1 \qquad ,(w \in \Delta) \qquad (8)$$

3. COEFFICIENT BOUNDS AND FEKETE-SZEGö INEQUALITY

In this section, we provide the coefficient bound $|a_2|$, $|a_3|$ and Fekete-szegö inequalities for functions in the subclass $B_{\Sigma}^{\gamma,\lambda}(x)$.

Theorem 3.1: Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $B_{\Sigma}^{\gamma,\lambda}(x)$.Then $|a_2| \leq \frac{|\gamma||p(x)|\sqrt{2|p(x)|}}{\sqrt{|k|}}$ and $|a_3| \leq \frac{|\gamma||p(x)|}{2+\lambda} + \frac{\gamma^2 p^2(x)}{(1+\lambda)^2}$

and for $\nu \in R$

$$|a_3 - \nu a_2^2| \leq \begin{cases} \frac{|\gamma||p(x)|}{2+\lambda} & ; 0 \leq |h(\nu,x)| \leq \frac{|\gamma|}{2(2+\lambda)} \\ 2|p(x)||h(\nu,x)| & ; \quad |h(\nu,x)| \geq \frac{|\gamma|}{2(2+\lambda)} \end{cases} \qquad (9)$$

where $k = [\gamma(1+\lambda)(2+\lambda) - 2(1+\lambda)^2]p^2(x) - 4q(x)(1+\lambda)^2$

Proof: Let $f \in B_{\Sigma}^{\gamma,\lambda}(x)$ be given by Taylor – Maclaurin expansion (1). Then from the Definition 2.1, for some analytic functions ϕ and ψ such that

$$\phi(0) = 0, \qquad \psi(0) = 0, \qquad |\phi(z)| < 1 \quad and \; |\psi(z)| < 1 \qquad (\forall\, z, w \in \Delta)$$

we can write $1+\frac{1}{\gamma}\left(\frac{z^{1-\lambda}f'(z)}{(f(z))^{1-\lambda}}-1\right)=G_{L_{p,q,n(x)}}(\phi(z))-1$

and $1+\frac{1}{\gamma}\left(\frac{w^{1-\lambda}g'(w)}{(g(w))^{1-\lambda}}-1\right)=G_{L_{p,q,n(x)}}(\psi(w))-1$

or,

$$1+\frac{1}{\gamma}\left(\frac{z^{1-\lambda}f'(z)}{(f(z))^{1-\lambda}}-1\right)=-1+L_{p,q,0}(x)+L_{p,q,1}(x)\phi(z)+L_{p,q,2}(x)\phi^2(z)+\cdots \quad (10)$$

and

$$1+\frac{1}{\gamma}\left(\frac{w^{1-\lambda}g'(w)}{(g(w))^{1-\lambda}}-1\right)=1+L_{p,q,0}(x)+L_{p,q,1}(x)\psi(w)+L_{p,q,2}(x)\psi^2(w)+.. \quad (11)$$

or

$$1+\frac{1}{\gamma}\left(\frac{z^{1-\lambda}f'(z)}{(f(z))^{1-\lambda}}-1\right)=1+L_{p,q,1}(x)t_1z+\left[L_{p,q,1}(x)t_2+L_{p,q,2}(x)t_1^2\right]z^2+.. \quad (12)$$

and

$$1+\frac{1}{\gamma}\left(\frac{w^{1-\lambda}g'(w)}{(g(w))^{1-\lambda}}-1\right)=1+L_{p,q,1}(x)s_1w+\left[L_{p,q,1}(x)s_2+L_{p,q,2}(x)s_1^2\right]w^2+\cdots \quad (13)$$

It is well known that

$$|\phi(z)|=|t_1z+t_2z^2+t_3z^3+\cdots|<1 \quad and \quad |\psi(w)|=|s_1z+s_2z^2+s_3z^3+\cdots|<1$$

Then $|t_k|\le 1$ *and* $|s_k|\le 1 \ (k\in N)$.

LUCAS POLYNOMIAL COEFFICIENT BOUND FOR BI-BAZILEVIČ FUNCTIONS 5

when comparing the corresponding coefficients in (12) and (13), we get

$$\frac{1}{\gamma}(1+\lambda)a_2=L_{p,q,1}(x)t_1 \quad (14)$$

$$\frac{1}{\gamma}(2+\lambda)[a_3+(\lambda\text{-}1)\frac{a_2^2}{2}]=L_{p,q,1}(x)t_2+L_{p,q,2}(x)t_1^2 \quad (15)$$

$$-\frac{1}{\gamma}(1+\lambda)a_2=L_{p,q,1}(x)s_1 \quad (16)$$

and

$$\frac{1}{\gamma}(2+\lambda)\left[(3+\lambda)\frac{a_2^2}{2}-a_3\right]=L_{p,q,1}(x)s_2+L_{p,q,2}(x)s_1^2 \quad (17)$$

From (14) and (16), we can easily see that

$$t_1=-s_1 \quad (18)$$

and

$$\frac{2}{\gamma^2}(1+\lambda)^2a_2^2=\left[L_{p,q,1}(x)\right]^2(t_1^2+s_1^2)$$

$$a_2^2=\frac{\gamma^2\left[L_{p,q,1}(x)\right]^2(t_1^2+s_1^2)}{2(1+\lambda)^2} \quad (19)$$

Adding (15) and (17), we obtain

$$\frac{1}{\gamma}(1+\lambda)(2+\lambda)a_2^2=L_{p,q,1}(x)(t_2+s_2)+L_{p,q,2}(x)(t_1^2+s_1^2) \quad (20)$$

Using (18) in (20), we deduce that

$$a_2^2 = \frac{\gamma^2[L_{p,q,1}(x)]^3(t_2+s_2)}{\gamma(1+\lambda)(2+\lambda)[L_{p,q,1}(x)]^2-2L_{p,q,2}(x)(1+\lambda)^2} \qquad (21)$$

That implies

$$|a_2| \leq \frac{\sqrt{2}|\gamma||px|\sqrt{|px|}}{\sqrt{|[\gamma(1+\lambda)(2+\lambda)p-2p(1+\lambda)^2]px^2-2qa(1+\lambda)^2|}} \qquad (22)$$

Subtracting (17) from (15) and using (18), we have

$$\frac{2}{\gamma}(2+\lambda)a_3 - \frac{2}{\gamma}(2+\lambda)a_2^2 = L_{p,q,1}(x)(t_2-s_2) + L_{p,q,2}(x)(t_1^2-s_1^2)$$

$$a_3 = \frac{\gamma L_{p,q,1}(x)(t_2-s_2)}{2(2+\lambda)} + a_2^2 \qquad (23)$$

By applying (19) in (23) gives

$$a_3 = \frac{\gamma L_{p,q,1}(x)(L_2+s_2)}{2(2+\lambda)} + \frac{\gamma^2[L_{p,q,1}(x)]^2(t_1^2+s_1^2)}{2(1+\lambda)^2}$$

Using (6), we have

$$|a_3| \leq \frac{|\gamma||px|}{2+\lambda} + \frac{\gamma^2p^2x^2}{(1+\lambda)^2}$$

From (23), for ν ∊ R , we get

$$a_3 - \nu a_2^2 = \frac{\gamma L_{p,q,1}(x)(t_2-s_2)}{2(2+\lambda)} + (1-\nu)a_2^2 \qquad (24)$$

By applying (21) in (24), we obtain

$$a_3 - \nu a_2^2 = \frac{\gamma L_{p,q,1}(x)(t_2-s_2)}{2(2+\lambda)} + (1-\nu)\left(\frac{\gamma^2[L_{p,q,1}(x)]^3(s_2+t_2)}{\gamma(1+\lambda)(2+\lambda)[L_{p,q,1}(x)]^2-2L_{p,q,2}(x)(1+\lambda)^2}\right)$$

$$= L_{p,q,1}(x)\left\{\left(h(\nu,x)+\frac{\gamma}{2(2+\lambda)}\right)t_2 + \left(h(\nu,x)-\frac{\gamma}{2(2+\lambda)}\right)s_2\right\} \qquad (25)$$

where

$$h(\nu,x) = \frac{(1-\nu)\gamma^2[L_{p,q,1}(x)]^2}{\gamma(1+\lambda)(2+\lambda)[L_{p,q,1}(x)]^2-2L_{p,q,2}(x)(1+\lambda)^2}$$

Hence , by using (6), we obtain that

$$|a_3-\nu a_2^2| \leq \begin{cases} \dfrac{|\gamma||p(x)|}{2+\lambda} & ; 0 \leq |h(\nu,x)| \leq \dfrac{|\gamma|}{2(2+\lambda)} \\ 2|p(x)||h(\nu,x)| & ; |h(\nu,x)| \geq \dfrac{|\gamma|}{2(2+\lambda)} \end{cases}$$

Which completes the proof of Theorem 3.2. □

4. COROLLARIES AND CONSEQUENCES

Corollary 4.1 Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $B_{\Sigma}^{\gamma,\lambda}(x)$. Then

$$|a_2| \leq \frac{|\gamma||px|\sqrt{|px|}}{\sqrt{|[\gamma-1]p^2(x)-q(x)|}}, \quad and \quad |a_3| \leq \frac{|\gamma||px|}{2} + \gamma^2p^2(x)$$

and for ν ∈ R

$$|a_3 - \nu a_2^2| \leq \begin{cases} \dfrac{|\gamma||bx|}{2} & ; |\nu - 1| \leq \dfrac{|[\gamma b - p]bx^2 - qa|}{2|\gamma|b^2x^2} \\ \dfrac{|\gamma|^2|p(x)|^3|\nu-1|}{\sqrt{|[\gamma p(x) - 1]p(x) - 2q(x)|}} & ; |\nu - 1| \geq \dfrac{|[\gamma b - p]bx^2 - qa|}{2|\gamma|b^2x^2} \end{cases}$$

Remark 4.2 For γ=1, Corollary 4.1 coincide with results derived in [13], in Remark 1.1, Corollary 4.1 deduce to the results studied in [2,12,13].

Corollary 4.3 Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class$B_{\Sigma}^{\gamma,\lambda}(x)$. Then

$$|a_2| \leq \frac{|\gamma||px|\sqrt{|px|}}{\sqrt{|[3\gamma - 4]p^2(x) - 8q(x)|}} \qquad and \qquad |a_3| \leq \frac{|\gamma||px|}{3} + \frac{\gamma^2 p^2(x)}{4}$$

and for ν ∈ R

$$|a_3 - \nu a_2^2| \leq \begin{cases} \dfrac{|\gamma||px|}{3} & ; |\nu - 1| \leq \dfrac{|[3\gamma-4]p^2(x) - 8q(x)|}{3|\gamma|p^2(x)} \\ \dfrac{|\gamma|^2|px|^3|\nu-1|}{|[3\gamma - 4]p^2(x) - 8q(x)|} & ; |\nu - 1| \leq \dfrac{|[3\gamma - 4]p^2(x) - 8q(x)|}{3|\gamma|p^2(x)} \end{cases}$$

Remark 4.4 In view of Remark 1.1, Corollary 4.3 deduce to the results studied in [4, Corollary 2, p.88 and 13].

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INVESTIGATION OF TOPOLOGICAL INDICES OF POLYCYCLIC AROMATIC COMPOUNDS

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ABSTRACT
Triphenylenes are characterized with high resonance stability, UV-fluorescence and liquid crystallinity leading to various applications in the field of optics, electronics and Liquid Crystal Display. Polypyrenes [1] are one of the most investigated and utilized electrically conducting organic polymer with photo luminescence. Both the π-systems play challenging role in the field of organic electronics with possible chemical functionalization and derivatization leading to advance discoveries. The aromatic compounds are considered as graphs and the topological indices are determined. The aromatic compounds under the study are Triphenylene, First Generation Poly Pyrene and Second Generation Polypyrene. Harmonic index, Hyper- Zagreb index, Redefined Zagreb indices and Randić indices are determined for the said polycyclic aromatic compounds.

Keywords: Triphenylene, First Generation Polypyrene, Second Generation Polypyrene, Harmonic Index, Hyper- Zagreb index, Redefined Zagreb Indices, Randić indices.

1. INTRODUCTION AND TERMINOLOGIES

[1][12][13] suggests that electrophilic substitution of pyrene takes place preferentially at the 1-, 3-, 6-, and 8-positions, based on experimental results. Hence in this present study, an attempt is made to substitute pyrene at tetra positions mentioned above. Topological indices are determined for the tetra substituted pyrene. A study is carried out on first generation polypyrene and second generation polypyrene. Another study has been made on Triphenylene, which is an aromatic compound with 18 vertices and 21 edges. In this paper Triphenylenes are attached as a linear combination and calculated the topological indices.

In this paper, all graphs used are simple and undirected for the reason that there would not be any multiple edges or loops in the intended and resulting structure.

Let G be a simple graph, the vertex-set and edge-set of which are represented by $V(G)$ and $E(G)$ respectively. If u and v are two vertices of G then d_u, d_v denotes the degrees of the vertices u and v respectively.

Figure-1: Substitution of Pyrene [1]

The ***Harmonic index*** [5, 9, 10, 8] of a graph G is defined by

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d_u + d_v}$$

where (u,v) is an element of E(G).

The ***Hyper - Zagreb index*** [4,6,11] of a graph G is defined by

$$Hy(G) = \sum_{uv \in E(G)} (d_u + d_v)^2$$

where (u,v) is an element of E(G).

The ***Redefined First Zagreb index*** [9, 10] of a graph G is defined by

$$ReZG_1(G) = \sum_{uv \in E(G)} \frac{d_u + d_v}{d_u d_v}$$

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where (u,v) is an element of E(G).

The ***Redefined Second Zagreb index*** [9,10] of a graph *G* is defined by

$$ReZG_2(G) = \sum_{uv \in E(G)} \frac{d_u d_v}{d_u + d_v}$$

where (u,v) is an element of E(G).

The ***Redefined Third Zagreb index***[9,10] of a graph *G* is defined by

$$ReZG_3(G) = \sum_{uv \in E(G)} [d_u d_v][d_u + d_v]$$

where (u,v) is an element of E(G).

Gutman et.al.***reciprocal reduced Randić index*** [2] is defined as

$$RR(G) = \sum_{uv \in E(G)} \sqrt{d_u d_v}$$

Gutman et.al. in [2] ***reduced Second Zagreb index*** is defined as

$$RM_2(G) = \sum_{uv \in E(G)} (d_u - 1)(d_v - 1)$$

reciprocal Randić index [2] is defined as

$$RRR(G) = \sum_{uv \in E(G)} \sqrt{(d_u - 1)(d_v - 1)}$$

2. HARMONIC INDEX, HYPER-ZAGREB INDEX, REDEFINED ZAGREB INDICES OF TRIPHENYLENE

Theorem 2.1 The Harmonic Index of Triphenylene denoted by H(Tph) where n≥18, where n is the total number of vertices and m is the number of edges is

$$H(Tph)=\frac{236n+31}{30}$$

Proof: There are 16x+2 number of vertices, where x takes the values from 1,2,3… and the number of edges are 20n + 1 . A Triphenylene (Tph) will have the edges whose end vertices are (2,2), (2,3) and (3,m) . Considering the count of the edges of all the types and calculating the Harmonic index results in

$$H(Tph)=\frac{236n+31}{30}$$

Theorem 2.2 The Hyper- Zagreb index of Triphenylene denoted by HM(Tph) where n≥18, where n is the total number of vertices and m is the number of edges is

HM(Tph)=552n- 42

Proof: Same as the proof of Theorem 2.1.

Theorem 2.3 The Redefined Zagreb Indices of a Triphenylene $ReZG_1$(Tph) where n≥18, where n is the total number of vertices and m is the number of number of edges is

$ReZG_1(Tph)=16n+2$

$$ReZG_2(Tph) = \frac{128n - 2}{5}$$

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$$ReZG_3(Tph) = 736n - 88$$

Proof: Same as the proof of Theorem 2.1.

Theorem 2.4 The Randić indices of a Triphenylene (Tph) where n ≥18, where n is the total number of vertices and m is the number of edges are

$$RR(Tph) = (32 + 8\sqrt{6})n + 2(2 - \sqrt{2})$$

$$RM_2(Tph) = 52n - 7$$

$$RRR(Tph) = 4n(5 + 2\sqrt{2}) + (1 - 2\sqrt{2})$$

Proof: Same as the proof of Theorem 2.1.

3. TOPOLOGICAL INDICES OF FIRST GENERATION POLYPYRENE

Theorem 3.1 The Harmonic Index of First Generation Polypyrene denoted by H(Py(5)), where n is the total number of vertices and m is the number of edges is

$H(Py(5))= \frac{118n}{15}$

Proof. There are 16x number of vertices, where x takes the values from 1,2,3 …. and the number of edges are 20n-1. The First Generation Polypyrene Py(5) will have the edges whose end vertices are (2, 2) , (2, 3) and (3, 3) . Considering the count of the edges of all the types and calculating the Harmonic Index results in,

$H(Py(5))= \frac{118n}{15}$

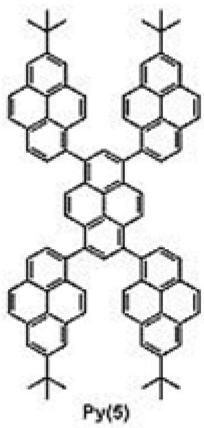

Figure-2: First Generation Polypyrene

Theorem 3.2 The Hyper- Zagreb index of First Generation Polypyrene denoted by HM(Py(5)), where n is the total number of vertices and m is the number of edges is

HM(Py(5))=552n- 76

Proof: Same as the proof of Theorem 3.1.

Theorem 3.3 The Redefined Zagreb Indices of a First Generation Polypyrene $Py(5)$, where n is the total number of vertices and m is the number of number of edges is

$ReZG_1(Py(5))=16n$

$$ReZG_2(Py(5)) = \frac{256n - 25}{10}$$

$$ReZG_3(Py(5)) = 736n - 130$$

Proof: Same as the proof of Theorem 3.1.

Theorem 3.4 The Randić indices of a First Generation Polypyrene Py(5), where n is the total number of vertices and m is the number of edges are

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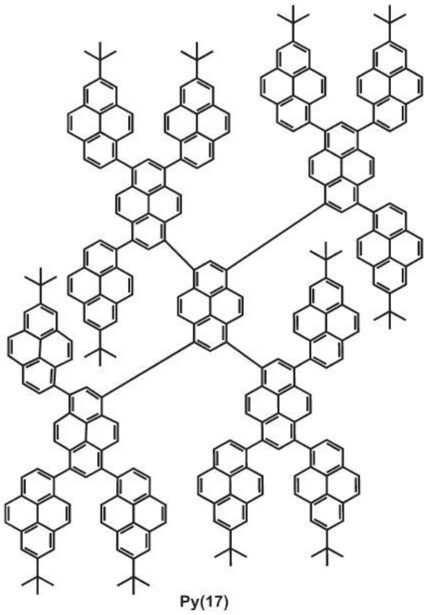
$$RR(Py(5)) = (32 + 8\sqrt{6})n - 5$$

$$RM_2(Py(5)) = 52n - 10$$

$$RRR(Py(5)) = (20 + 8\sqrt{2})n - 4$$

Proof: Same as the proof of Theorem 3.1.



Py(17)

Figure-3: Second Generation Polypyrene

4. TOPOLOGICAL INDICES OF SECOND GENERATION POLYPYRENE

Theorem 4.1 The Harmonic Index of Second Generation Polypyrene denoted by H(Py(17)), where n is the total number of vertices and m is the number of edges is

$$H(Py(17)) = \frac{60\prod_{i=1}^{n}[3^i] + 10 + 8*3^n}{2} + 2\frac{(48(3^{(n+1)} - 1) + 40)}{5} + \frac{(6\sum_{i=1}^{n}[12*3^i] + 9*\sum_{i=0}^{n}[12*3^i + 5])}{3}$$

Proof: There are 16[4 +$\sum_{i=0}^{n}[12*3^i]$+ 1] number of vertices, and the number of edges are 339+360(3^n-1) . A Second Generation Polypyrene Py(17) will have the edges whose end vertices are (2, 2) , (2,3) and (3, 3) . The number of (2, 2) edges are ($60\prod_{i=1}^{n}[3^i] + 10 + 8*3^n$) , (2,3) edges are ($48(3^{(n+1)} - 1) + 40$) and (3, 3) edges are ($6\sum_{i=1}^{n}[12*3^i] + 9*\sum_{i=0}^{n}[12*3^i + 5]$).Considering the count of the edges of all the types and calculating the Harmonic index results in

$$H(Py(17)) = \frac{60\prod_{i=1}^{n}[3^i] + 10 + 8*3^n}{2} + 2\frac{(48(3^{(n+1)} - 1) + 40)}{5} + \frac{(6\sum_{i=1}^{n}[12*3^i] + 9*\sum_{i=0}^{n}[12*3^i + 5])}{3}$$

Theorem 4.2 The Hyper- Zagreb Index of Second Generation Polypyrene denoted by HM(Py(17)), where n is the total number of vertices and m is the number of edges is

$$HM(Py(17)) = 16\left(60\prod_{i=1}^{n}[3^i] + 10 + 8*3^n\right) + 2(48(3^{(n+1)} - 1) + 40) + 36\left(6\sum_{i=1}^{n}[12*3^i] + 9*\sum_{i=0}^{n}[12*3^i + 5]\right)$$

Proof: Same as the proof of Theorem 4.1.

Theorem 4.3 The Redefined Zagreb Indices of the Second Generation Polypyrene $Py(17)$, where n is the total number of vertices and m is the number of number of edges is

$$ReZG_1\big(Py(17)\big) = \left(60\prod_{i=1}^{n}[3^i] + 10 + 8*3^n\right) + \frac{5\left(48\left(3^{(n+1)}-1\right)+40\right)}{6} + \frac{2\left(6\sum_{i=1}^{n}[12*3^i]+9*\sum_{i=0}^{n}[12*3^i+5]\right)}{3}$$

$$ReZG_2\big(Py(17)\big) = \left(60\prod_{i=1}^{n}[3^i] + 10 + 8*3^n\right) + \frac{6\left(48\left(3^{(n+1)}-1\right)+40\right)}{5} + \frac{3\left(6\sum_{i=1}^{n}[12*3^i]+9*\sum_{i=0}^{n}[12*3^i+5]\right)}{2}$$

$ReZG_3\big(Py(17)\big) = 16\left(60\prod_{i=1}^{n}[3^i] + 10 + 8*3^n\right) + 30\left(48\left(3^{(n+1)}-1\right)+40\right) + 54\big(6\sum_{i=1}^{n}[12*3^i] + 9*$
$i=0n12*3i+5$

Proof: Same as the proof of Theorem 4.1.

Theorem 4.4 The Randić indices of the Second Generation Polypyrene RR(Py(17)), RM_2(Py(17)) and RRR(Py(17))where 'n' is the total number of vertices and 'm' is the number of number of edges is

$RR\big(Py(17)\big) = 2\left(60\prod_{i=1}^{n}[3^i] + 10 + 8*3^n\right) + \sqrt{6}\left(48\left(3^{(n+1)}-1\right)+40\right) + 3\big(6\sum_{i=1}^{n}[12*3^i] + 9*$
$i=0n12*3i+5$

$RM_2\big(Py(17)\big) = \left(60\prod_{i=1}^{n}[3^i] + 10 + 8*3^n\right) + 2\left(48\left(3^{(n+1)}-1\right)+40\right) + 4\big(6\sum_{i=1}^{n}[12*3^i] + 9*$
$i=0n12*3i+5$

$RRR\big(Py(17)\big) = \left(60\prod_{i=1}^{n}[3^i] + 10 + 8*3^n\right) + \sqrt{2}\left(48\left(3^{(n+1)}-1\right)+40\right) + 2\big(6\sum_{i=1}^{n}[12*3^i] + 9*$
$i=0n12*3i+5$

Proof: Same as the proof of Theorem 4.1.

CONCLUSION

The topological indices are extensively used in mathematical chemistry in drug design and in QSPR,QSAR studies to check the toxicity of the drug with regard to the range of topological indices obtained. In this paper, the indices are determined for polycyclic aromatic compounds namely, Triphenylene, First Generation Polypyrene and Second Generation Polypyrene.

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BOUNDS FOR THE F-INDEX OF A LINK GRAPH

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ABSTRACT
Mathematical chemistry associated with graph operations are vital role in chemical and algebraic analysis. Motivated from this, here we initiated the moves class of graph operations on link graph and found result associated to F-index. Also, the explicit interpretation for F-index of in terms of graph size and maximum or minimum vertex degrees of link graphs is acquired.

Keywords and Phrases: F-index, link graph.

INTRODUCTION

Graph theory is the gateway for chemists and scientists to lightening on the topological descriptors of molecular graphs. Molecular compounds may be developed by using graph theoretic method. A topological index is a numerical quantity which is rooted mathematically in a direct and a unique manner from the structural graph of a molecule. Topological indices have been found to be useful in isomer discrimination, quantitative structure-activity relationship (QSAR) and structure-property relationship (QSPR) for predicting different properties of chemical compounds and biological activities [1].

The Zagreb indices have been introduced more than thirty years ago by Gutman and Trinajestic [5]. They are defined as:

$$M_1(G) = \sum_{uv \in E(G)} [d_u + d_v] \qquad (1)$$

$$M_2(G) = \sum_{uv \in E(G)} [d_u d_v] \qquad (2)$$

For more detail on the Zagreb indices we refer to the articles [2, 7].

Furtula and Gutman [3], introduced another topological index called forgotten index or F-index

$$F(G) = \sum_{uv \in E(G)} d_u^{\,3} . \qquad (3)$$

Let H_1 and H_2 be simple connected graphs, let $x \in V(H_1)$ and $y \in V(H_2)$. Link of H_1 and H_2 is obtained by joining x and y by an edge. By algebraic method, we have $|V(H_1)| = p_1$, $V(H_2)| = p_2$ and $|E\ (H_1)| = q_1$, $|E\ (H_2)| = q_2$. The link graph has $p_1 + p_2$ vertices and $q_1 + q_2 + 1$ edges.

Example for link graph

The subdivision graph $S(G)$ is the graph obtained from G by replacing each of its edges by a path of length two [11].

The R –graph [12] is the graph obtained from G by adding a new vertex corresponding to each edge of G and by joining each new vertex to the end vertices of the edge corresponding to it.

In [9], Liu and Lu introduced two new graph operations based on subdivision graphs namely, the subdivision vertex-neighbourhood corona and the subdivision edge-neighbourhood corona and discussed their adjacency, Laplacian and signless Laplacian spectra.

J. Lan and Zhou [8] introduced four new graph operations based on R-graphs namely the R-vertex corona, R-edge corona, R-vertex neighbourhood corona and R-edge neighbourhood corona.

Motivated from the above works, in the upcoming section derived the certain expressions of *F*-index of different types of link graphs such as S- vertex and R- vertex link, S- edge and R- edge link, S- vertex neighbourhood and R-vertex neighbourhood link, S-edge neighbourhood and *R*- edge neighbourhood link.

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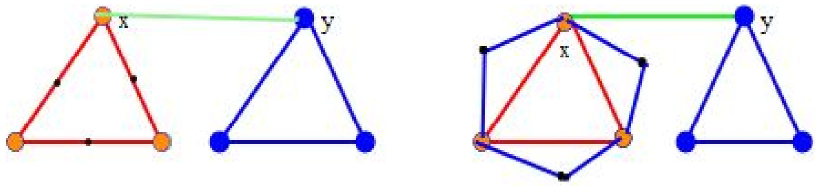
S-vertex and R-vertex link

Let H_1 and H_2 be two vertex disjoint graphs and let $x \in V(H_1)$ and $y \in V(H_2)$. The **S-verte link $\mathcal{L}_v$ (or the R-vertex link $\mathcal{L}_{vR}$)** of H_1 and H_2 is obtained from $S(H_1)$ (or from $R(H_1)$) and one copy of H_2 joining the vertices of x and y by an edge.



S-vertex link R- vertex link

Theorem 1. The bounds for the F index of the $F(\mathcal{L}_v)$ and $F(\mathcal{L}_{vR})$ is given by

$$\delta_{H_1}{}^2 + \delta_{H_1} + \delta_{H_2}{}^2 + \delta_{H_2} \le \frac{F(\mathcal{L}_v)-\alpha_1}{3} \le \Delta_{H_1}^2 + \Delta_{H_1} + \Delta_{H_2}{}^2 + \Delta_{H_2}$$

$$\delta_{H_1}{}^2 + \delta_{H_1} + 4\delta_{H_2}{}^2 + 2\delta_{H_2} \le \frac{F(\mathcal{L}_v)-\alpha_2}{3} \le \Delta_{H_1}{}^2 + \Delta_{H_1} + 4\Delta_{H_2}{}^2 + 2\Delta_{H_2}$$

Where $\alpha_1 = F(H_1) + F(H_2) + 8q_1 + 2$

$$\alpha_2 = 8F(H_1) + F(H_2) + 8q_1 + 2$$

Proof: By using equation 3, then

$$F(\mathcal{L}_v) = \sum_{i=1}^{p_1-1}(d_{H_1}(x_i))^3 + \sum_{i=1}^{p_2-1}(d_{H_2}(y_i))^3 + \sum_{i=1}^{q_1}(2)^3 + (d_{H_1}(x) + 1)^3 + (d_{H_2}(y) + 1)^3$$

$$=F(H_1) - (d_{H_1}(x))^3 + F(H_2) - (d_{H_2}(y))^3 + 8q_1 + (d_{H_1}(x))^3 + 1$$

$$+3(d_{H_1}(x)(d_{H_1}(x) + 1) + (d_{H_2}(y))^3 + 1 + 3(d_{H_2}(y)(d_{H_2}(y) + 1)$$

$$=F(H_1) + F(H_2) + 3\left((d_{H_1}(x))^2 + d_{H_1}(x) + (d_{H_2}(y))^2 + (d_{H_2}(y)\right) + \quad + 8q_1 + 2.$$

Note that for all vertices $x \in H_1$, $\delta_{H_1} \le d_{H_1}(x)$ *and* $d_{H_1}(x) \le \Delta_{H_1}$ with equalities hold if and only if H_1 is a regular graph. Therefore,

$$F(\mathcal{L}_v) \le F(H_1) + F(H_2) + 3(\Delta_{H_1}{}^2 + \Delta_{H_1} + \Delta_{H_2}{}^2 + \Delta_{H_2}) + 8q_1 + 2. \quad (4)$$

Similarly, the following inequalities computed:

$$F(\mathcal{L}_v) \ge F(H_1) + F(H_2) + 3(\delta_{H_1}{}^2 + \delta_{H_1} + \delta_{H_2}{}^2 + \delta_{H_2}) + 8q_1 + 2. \quad (5)$$

Combining equation (4) and (5), then

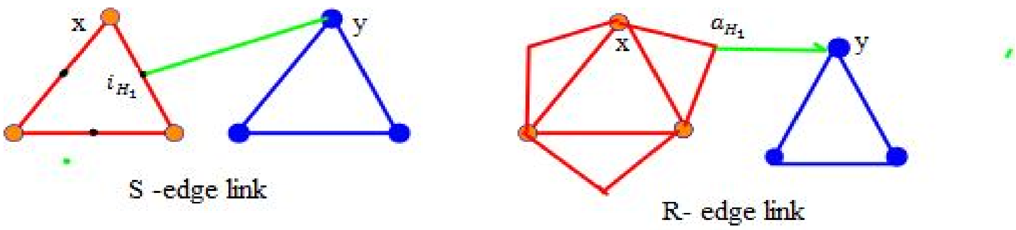
$$\delta_{H_1}{}^2 + \delta_{H_1} + \delta_{H_2}{}^2 + \delta_{H_2} \le \frac{F(\mathcal{L}_v)-\alpha_1}{3} \le \Delta_{H_1}^2 + \Delta_{H_1} + \Delta_{H_2}{}^2 + \Delta_{H_2}$$

With similar idea, the following inequity obtained.

$$\delta_{H_1}{}^2 + \delta_{H_1} + 4\delta_{H_2}{}^2 + 2\delta_{H_2} \le \frac{F(\mathcal{L}_v)-\alpha_2}{3} \le \Delta_{H_1}{}^2 + \Delta_{H_1} + 4\Delta_{H_2}{}^2 + 2\Delta_{H_2}$$

S-edge and R-edge link

Let H_1 and H_2 be two vertex disjoint graphs and let $'i_{H_1}'$ be the inserted vertex of $S(H_1)$ (or $'a_{H_1}'$ be the adding vertex of $R(H_1)$) and $y \in V(H_2)$. The **S-edge link $\mathcal{L}_v$ (or the R-edge link $\mathcal{L}_{vR}$)** of H_1 and H_2 is obtained from $S(H_1)$ (or from $R(H_1)$) and one copy of H_2 joining the vertices of i_{H_1} and y by an edge.



Theorem 2. The bounds for the F index of the $F(\mathcal{L}_e)$ and $F(\mathcal{L}_{eR})$ is given by

$$\delta_{H_2}(\delta_{H_2}+1) \le \frac{F(\mathcal{L}_e)-\beta_1}{3} \le \Delta_{H_2}(\Delta_{H_2}+1)$$

$$\delta_{H_2}(\delta_{H_2}+1) \le \frac{F(\mathcal{L}_{eR})-\beta_2}{3} \le \Delta_{H_2}(\Delta_{H_2}+1)$$

Where, $\beta_1 = F(H_1) + F(H_2) + 8q_1 + 20$

$$\beta_2 = 8F(H_1) + F(H_2) + 8q_1 + 20$$

Proof: By definition of F- index, then

$$F(\mathcal{L}_e) = \sum_{i=1}^{p_1}(d_{H_1}(x_i))^3 + \sum_{i=1}^{p_2-1}(d_{H_2}(y_i))^3 + \sum_{i=1}^{q_1-1}(2)^3 + (3)^3 + (d_{H_2}(y)+1)^3$$

$$= F(H_1) + F(H_2) - (d_{H_2}(y))^3 + (d_{H_2}(y))^3 + 8(q_1-1) + 1$$

$$+ 27 + 3d_{H_2}(y)(d_{H_2}(y)+1)$$

$$= F(H_1) + F(H_2) + 8q_1 + 20 + 3(d_{H_2}(y)(d_{H_2}(y)+1)$$

$$\le F(H_1) + F(H_2) + 3\Delta_{H_2}(\Delta_{H_2}+1) + 8q_1 + 20$$

$$\delta_{H_2}(\delta_{H_2}+1) \le \frac{F(\mathcal{L}_e)-\beta_1}{3} \le \Delta_{H_2}(\Delta_{H_2}+1).$$

Where, $\beta_1 = F(H_1) + F(H_2) + 3\delta_{H_2}(\delta_{H_2}+1) + 8q_1 + 20.$

With similar idea, the following inequality established:

$$\delta_{H_2}(\delta_{H_2}+1) \le \frac{F(\mathcal{L}_e)-\beta_2}{3} \le \Delta_{H_2}(\Delta_{H_2}+1).$$

S-vertex neighbourhood and R- vertex neighbourhood link

Let H_1 and H_2 be two vertex disjoint graphs and let $x \in V(H_1)$ and $y \in V(H_2)$. The **S-vertex neighbourhood link $\mathcal{L}_{nv}$ (or the R-vertex neighbourhood link $\mathcal{L}_{nvR}$)** of H_1 and H_2 is obtained from identifying the neighbourhood vertices of $'x'$ of $S(H_1)$ (or $R(H_1)$)and join H_2 to corresponding neighbourhood separated vertices by an edge.

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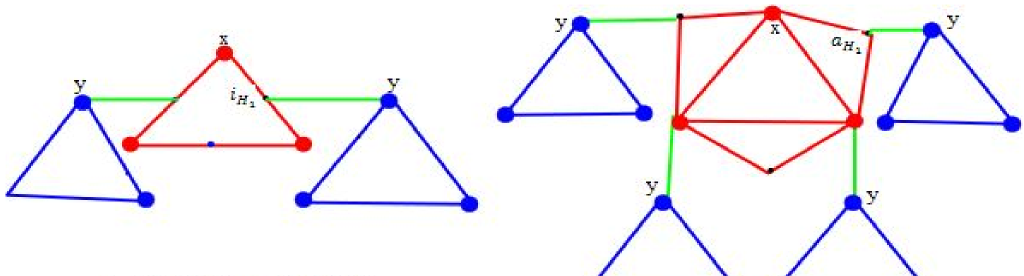


S - vertex neighbourhood link   R - vertex neighbourhood link

Theorem 3: The bounds for the F index of the $F(\mathcal{L}_{nv})$ and $F(\mathcal{L}_{nvR})$ is given by

$$F(\mathcal{L}_{nv}) \leq F(H_1) + \Delta_{H_1}F(H_2) + 3\Delta_{H_2}(\Delta_{H_2} + 1) + 8q_1 + 20.$$

$$F(\mathcal{L}_{nv}) \geq F(H_1) + \delta_{H_1}F(H_2) + 3\delta_{H_2}(\delta_{H_2} + 1) + 8q_1 + 20$$

and

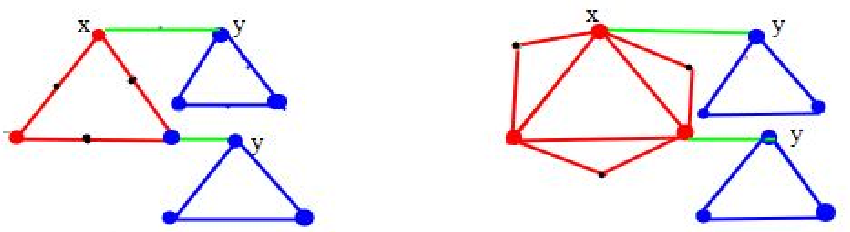
$$F(\mathcal{L}_{nvR}) \leq 8F(H_1) + 2\Delta_{H_1}F(H_2) + 6\Delta_{H_2}(\Delta_{H_1}(2\Delta_{H_1} + 1) + \Delta_{H_2}(\Delta_{H_2} + 1)) + 8q_1 + 22\Delta_{H_1}$$

$$F(\mathcal{L}_{nvR}) \geq 8F(H_1) + 2\delta_{H_1}F(H_2) + 6\delta_{H_2}(\delta_{H_1}(2\delta_{H_1} + 1) + \delta_{H_2}(\delta_{H_2} + 1)) + 8q_1 + 22\delta_{H_1}.$$

Proof: Proof: The proof technique of the Theorem is same as that of the above Theorem.

$\boldsymbol{S}$-edge neighbourhood and $\boldsymbol{R}$- edge neighbourhood link

Let H_1 and H_2 be two vertex disjoint graphs and let $'i_{H_1}'$ be the inserted vertex of $S(H_1)$ (or $'a_{H_1}'$ be the adding vertex of $R(H_1)$) and $y \in V(H_2)$. The **$\boldsymbol{S}$-edge neighbourhood link $\boldsymbol{\mathcal{L}_{ne}}$(or the $\boldsymbol{R}$-edge neighbourhood link $\boldsymbol{\mathcal{L}_{neR}}$)** of H_1 and H_2 is obtained from identifying the neighbourhood vertices of $'i_{H_1}'$ (correspondingly $'a_{H_1}'$) and join H_2 to corresponding neighbourhood separated vertices by an edge.



S - edge neighbourhood link   R - edge neighbourhood link

Theorem 4: The bounds for the F index of the $F(\mathcal{L}_{ne})$ and $F(\mathcal{L}_{neR})$ is given by

$$F(\mathcal{L}_{ne}) \leq F(H_1) + 2F(H_2) + 6({\Delta_{H_1}}^2 + \Delta_{H_1} + {\Delta_{H_2}}^2 + \Delta_{H_2}) + 8q_1 + 4$$

$$F(\mathcal{L}_{ne}) \geq F(H_1) + 2F(H_2) + 6({\delta_{H_1}}^2 + \delta_{H_1} + {\delta_{H_2}}^2 + \delta_{H_2}) + 8q_1 + 4$$

and

$$F(\mathcal{L}_{neR}) \leq 8F(H_1) + 2F(H_2) + 12\Delta_{H_1}(2\Delta_{H_1} + 1) + 6\Delta_{H_2}(\Delta_{H_2} + 1) + 8q_1 + 4$$

$$F(\mathcal{L}_{neR}) \geq 8F(H_1) + 2F(H_2) + 12\delta_{H_1}(2\delta_{H_1} + 1) + 6\delta_{H_2}(\delta_{H_2} + 1) + 8q_1 + 4.$$

Proof: The proof technique of the Theorem is exactly same as that of the previous Theorem.

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REVERSE DEGREE-BASED TOPOLOGICAL INDICES OF SOME ARCHIMEDEAN LATTICES

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ABSTRACT
Graph theory is one of the modern mathematics used in many branches such as chemistry, physics and biology etc. One of the most widely used applications of molecular topology is in terms of topological indices. Topological indices are one of the tool used by chemists in a graph theory. Here, we study the Archimedean lattice using reverse Zagreb index, reverse hyper-Zagreb index, reverse Zagreb polynomials and reverse hyper-Zagreb polynomials.

Keywords: Archimedean lattices, reverse Zagreb index, reverse hyper-Zagreb index, reverse Zagreb polynomials and reverse hyper-Zagreb polynomials.

1. INTRODUCTION

A topological index is a graph invarient number which represents physical, chemical or biological parameters of molecules in QSAR and QSPR from a graph representing a molecule. In mathematical chemistry, a great number of topological indices are introduced [2, 3, 10, 11]. In chemical graph theory, a topological representation of a molecule is called a molecular graph. A molecular graph is a simple graph having no loops and multiple edges in that atoms and chemical bonds are represented by vertices and edges respectively [4]. A graph $G(V,E)$ with vertex set $V(G)$ and edge set $E(G)$ is connected if there exist a connection between any pair of vertices in G. The degree of a vertex $d_v(G)$, is the number of vertices which are connected to that fixed vertex by the edges [6]. Also, $E_{a,b}$ indicates the set of edges that the degrees of end vertices a and b, i.e., $E_{a,b} = \{uv \mid \{a,b\} = \{d_u(G), d_v(G)\}\}$. Kulli [7] introduced the concept of reverse vertex degree c_v, as $c_v = \Delta(G) - d_v(G) + 1$, where $\Delta(G)$ is the maximum degree of a vertex among the vertices of a graph G and $E'_{a,b}$ indicates the set of edges that the reverse vertex degrees of end vertices a and b, i.e., $E'_{a,b} = \{uv \mid \{a,b\} = \{c_u, c_v\}\}$.

Archimedean lattices [1, 9] are uniform tilings of the plane in which all the faces are regular polygons and the symmetry group acts transitivily on the vertices. It follows that all vertices are equivalent and have the same coordination number z. All vertices are shared by the same set of polygons and thus we can associate to each Archimedean lattice a set of integers $(p_1, p_{2,...})$ indicating, in cyclic order, the polygons meeting at a given vertex. When a polygon appears more than one time consecutively, for example as in $(..., p, p, ...)$, we abbreviate the notation writing $(..., p^2, ...)$. In this way, the uniform tilings of the plane, made with squares, triangles and hexagons, are indicated respectively by (4^4), (3^6) and (6^3). There exists exactly 11 Archimedean lattices. Being planar graphs, the Archimedean lattices have duals, 3 of which are Archimedean, the other 8 are called Laves lattices. The eleven uniform tilings of the plane made with more than one regular polygon are shown in following Figure1.

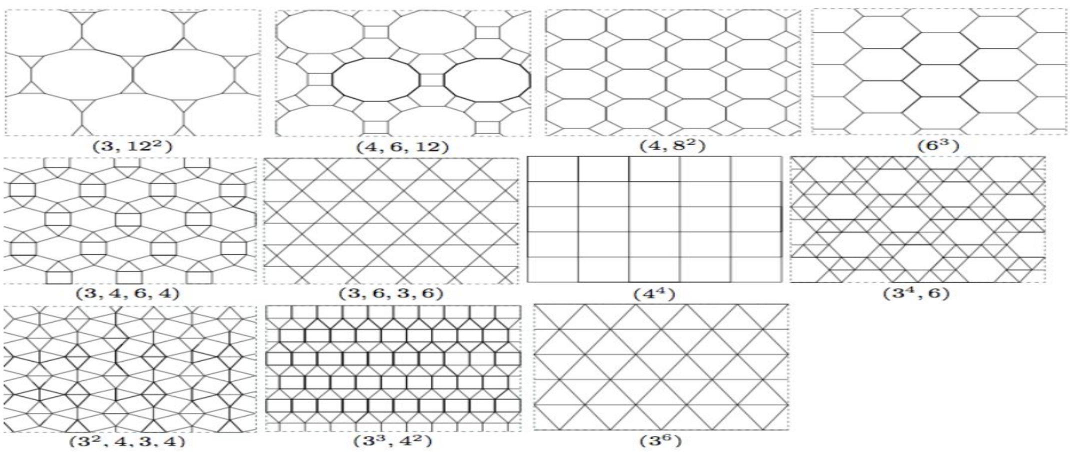

Figure-1

The paper organised as basic introduction in this section. Section 2, provide definitions are recall from [7] that are required for development of the paper. Final section, results related to different structures of Archimedean lattices are studied utilizing the Topological indices. We are inspired by the results given in [8, 13].

2. PRELIMINARIES

In [7], Kulli introduce the following definitions:

Definition 2.1: The first and second reverse Zagreb indices of a graph G are defined as:

$$M_1(G)=\sum_{uv\in E(G)}(c_u+c_v) \text{ and } M_2(G)=\sum_{uv\in E(G)}(c_u.c_v)$$

Definition 2.2: The first and second reverse hyper-Zagreb indices of a graph G are defined as:

$$HCM_1(G)=\sum_{uv\in E(G)}(c_u+c_v)^2 \text{ and } HCM_2(G)=\sum_{uv\in E(G)}(c_u.c_v)^2$$

Definition 2.3: For a simply connected graph G, the first and second reverse Zagreb polynomials are defined as:

$$CM_1(G,x)=\sum_{uv\in E(G)}x^{(c_u+c_v)} \text{ and } CM_2(G,x)=\sum_{uv\in E(G)}x^{(c_u.c_v)}$$

Definition 2.4: For a simply connected graph G, the first and second reverse hyper-Zagreb polynomials are defined as:

$$HCM_1(G,x)=\sum_{uv\in E(G)}x^{(c_u+c_v)^2} \text{ and } HCM_2(G,x)=\sum_{uv\in E(G)}x^{(c_u.c_v)^2}$$

3. MAIN RESULTS

In this section, our aim to study different types of Archimedean lattices are L(4, 6, 12), L(3, 6, 3, 6) and L(3, 12, 12).

I. Archimedean Lattice, L (4, 6, 12)

The notation (4, 6, 12) indicates that around a given vertex, going in the clockwise direction, one encounters first a square, second encounters a hexagons and third encounters a dodecagon. The Archimedean lattice (4, 6, 12) is also known as extended ruby Lattice. Archimedean lattice $L_{(4,6,12)}(n)$ as shown in the following Figure 2 [8].

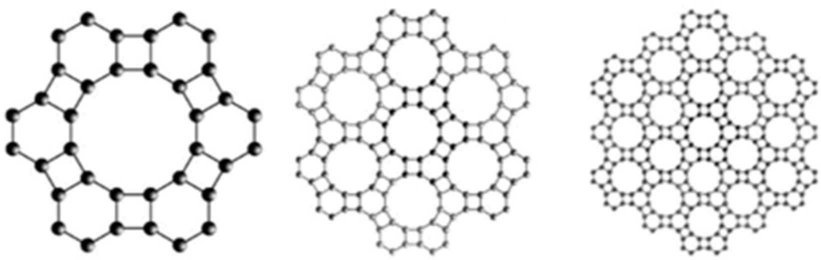

Figure-2: Archimedean Lattice $L_{(4,6,12)}(n)$

Consider the edge partition of $L_{(4,6,12)}(n)$ is as follows:

Table-1: Edge partition of $L_{(4,6,12)}(n)$ into $E_{a,b}$

(a,b)	(3, 3)	(2, 3)	(2, 2)
$\|E_{a,b}\|$	$54n^2-24n$	$12n$	$6n$

The maximum vertex degree $\Delta(G)$ in $L_{(4,6,12)}(n)$ is 3 i.e. $\Delta(G)=3$, so Table 2 gives the reverse edge partition of $L_{(4,6,12)}(n)$.

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Table-2: Reverse edge partition of $L_{(4,6,12)}(n)$ into $E'_{a,b}$

(a,b)	(1, 1)	(2, 1)	(2, 2)
$\lvert E'_{a,b}\rvert$	$54n^2-24n$	$12n$	$6n$

Theorem 1. Let G be an $L_{(4,6,12)}(n)$ circumference. Then

(i) $M_1(G)=108n^2+12n.$

(ii) $M_2(G)=54n^2+24n.$

Proof:- Recall the definition 2.1. and using Table 2, then

(i) $M_1(G)=(2)\lvert E'_{1,1}\rvert+(3)\lvert E'_{2,1}\rvert+(4)\lvert E'_{2,2}\rvert$

$=(2)(54n^2-24n)+(3)(12n)+(4)(6n)$

$=108n^2+12n$.

(ii) $M_2(G)=(1)\lvert E'_{1,1}\rvert+(2)\lvert E'_{2,1}\rvert+(4)\lvert E'_{2,2}\rvert$

$=(1)(54n^2-24n)+(2)(12n)+(4)(6n)$

$=54n^2+24n$.

Hence, $M_1(G)=108n^2+12n$ and $M_2(G)=54n^2+24n.$

Observation: (i) $M_1(G)=2M_2(G)-36n.$

(ii) $M_2(G)=\frac{1}{2}[M_1(G)+36n].$

Theorem 2. Let G be an $L_{(4,6,12)}(n)$ circumference. Then

(i) $HCM_1(G)=216n^2+108n.$

(ii) $HCM_2(G)=54n^2+120n.$

Proof :- Recall the definition 2.2. and using Table 2, then

(i) $HCM_1(G)=(4)\lvert E'_{1,1}\rvert+(9)\lvert E'_{2,1}\rvert+(16)\lvert E'_{2,2}\rvert$

$=(4)(54n^2-24n)+(9)(12n)+(16)(6n)$

$=216n^2+108n.$

(ii) $HCM_2(G)=(1)\lvert E'_{1,1}\rvert+(4)\lvert E'_{2,1}\rvert+(16)\lvert E'_{2,2}\rvert$

$=(1)(54n^2-24n)+(4)(12n)+(16)(6n)$

$=54n^2+120n.$

Hence, $HCM_1(G)=216n^2+108n$ and $HCM_2(G)=54n^2+120n.$

Observation: (i) $HCM_1(G)=4HCM_2(G)-372n.$

(ii) $HCM_2(G)=\frac{1}{4}[HCM_1(G)+372n].$

Theorem 3. Let G be an $L_{(4,6,12)}(n)$ circumference. Then

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(i) $CM_1(G,x) = (54n^2 - 24n)x^2 + 12nx^3 + 6nx^4$.

(ii) $CM_2(G,x) = (54n^2 - 24n)x + 12nx^2 + 6nx^4$.

Proof:- Recall the definition 2.3. and using Table 2, then

(i) $CM_1(G,x) = x^2 \,|E'_{1,1}| + x^3 \,|E'_{2,1}| + x^4 \,|E'_{2,2}|$

$= (54n^2 - 24n)x^2 + 12nx^3 + 6nx^4$.

(ii) $CM_2(G,x) = x^1 \,|E'_{1,1}| + x^2 \,|E'_{2,1}| + x^4 \,|E'_{2,2}|$

$= (54n^2 - 24n)x + 12nx^2 + 6nx^4$.

Hence, $CM_1(G,x) = (54n^2 - 24n)x^2 + 12nx^3 + 6nx^4$ and

$CM_2(G,x) = (54n^2 - 24n)x + 12nx^2 + 6nx^4$.

Theorem 4: Let G be an $L_{(4,6,12)}(n)$ circumference. Then

(i) $HCM_1(G,x) = (54n^2 - 24n)x^4 + 12nx^9 + 6nx^{16}$.

(ii) $HCM_2(G,x) = (54n^2 - 24n)x + 12nx^4 + 6nx^{16}$.

Proof:- Recall the definition 2.4. and using Table 2, then

(i) $HCM_1(G,x) = x^4 \,|E'_{1,1}| + x^9 \,|E'_{2,1}| + x^{16} \,|E'_{2,2}|$

$= (54n^2 - 24n)x^4 + 12nx^9 + 6nx^{16}$.

(ii) $HCM_2(G,x) = x^1 \,|E'_{1,1}| + x^4 \,|E'_{2,1}| + x^{16} \,|E'_{2,2}|$

$= (54n^2 - 24n)x + 12nx^4 + 6nx^{16}$.

Hence, $HCM_1(G,x) = (54n^2 - 24n)x^4 + 12nx^9 + 6nx^{16}$ and

$HCM_2(G,x) = (54n^2 - 24n)x + 12nx^4 + 6nx^{16}$.

The values of first and second reverse Zagreb indices and first and second reverse hyper-Zagreb indices of $L_{(4,6,12)}(n)$ for specific values of n are given in the following Table 3.

Table-3

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$
$M_1(G)$	120	456	1008	1776	2760	3960	5376	7008	8856
$M_2(G)$	78	264	558	960	1470	2088	2814	3648	4590
$HCM_1(G)$	324	1080	2268	3888	5940	8424	11340	14688	18468
$HCM_2(G)$	174	456	846	1344	1950	2664	3486	4416	5454

II. Archimedean Lattice, L (3, 6, 3, 6)

The (3, 6, 3, 6) lattice have name from Japanese, it is known as the Kagome' lattice(Kagome' means "woven bamboo pattern"). The notation (3, 6, 3, 6) indicates that around a given vertex, going in the clockwise direction, one encounters first a two triangle and second encounters a two hexagons. Archimedean lattice $L_{(3,6,3,6)}(n)$ as shown in following Figure 3.

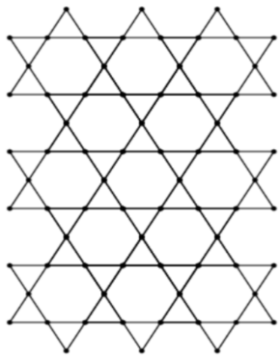

Figure-3: Archimedean lattice $L_{(3,6,3,6)}(n)$

Consider the edge partition of $L_{(3,6,3,6)}(n)$ is as follows:

Table-4: Edge partition of $L_{(3,6,3,6)}(n)$ into $E_{a,b}$

(a,b)	(4, 4)	(2, 4)
$\lvert E_{a,b}\rvert$	$18n^2-12n$	$12n$

The maximum vertex degree $\Delta(G)$ in $L_{(3,6,3,6)}(n)$ is 4 i.e. $\Delta(G)=$ 4, so Table 5 gives the reverse edge partition of $L_{(3,6,3,6)}(n)$.

Table-5: Reverse edge partition of $L_{(3,6,3,6)}(n)$ into $E'_{a,b}$

(a,b)	(1, 1)	(3, 1)
$\lvert E'_{a,b}\rvert$	$18n^2-12n$	$12n$

Theorem 5. Let G be an $L_{(3,6,3,6)}(n)$ circumference. Then

(i) $M_1(G)=36n^2+24n.$

(ii) $M_2(G)=18n^2+24n.$

Proof:- Recall the definition 2.1. and using Table 5, then

(i) $M_1(G)=(2)\lvert E'_{1,1}\rvert+(4)\lvert E'_{3,1}\rvert$

$=(2)(18n^2-12n)+(4)(12n)$

$=36n^2+24n.$.

(ii) $M_2(G)=(1)\lvert E'_{1,1}\rvert+(3)\lvert E'_{3,1}\rvert$

$=(1)(18n^2-12n)+(3)(12n)$

$=18n^2+24n$.

Hence, $M_1(G)=36n^2+24n$ and $M_2(G)=18n^2+24n.$

Observation: (i) $M_1(G)=2M_2(G)-24n.$

(ii) $M_2(G)=\frac{1}{2}[M_1(G)+24n].$

Theorem 6. Let G be an $L_{(3,6,3,6)}(n)$ circumference. Then

(i) $HCM_1(G)=72n^2+144n.$

(ii) $HCM_2(G)=18n^2+96n.$

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Proof :- Recall the definition 2.2. and using Table 5, then

(i) $HCM_1(G) = (4)|E'_{1,1}| + (16)|E'_{3,1}|$

$= (4)(18n^2 - 12n) + (16)(12n)$

$= 72n^2 + 144n.$

(ii) $HCM_2(G) = (1)|E'_{1,1}| + (9)|E'_{3,1}|$

$= (1)(18n^2 - 12n) + (9)(12n)$

$= 18n^2 + 96n.$

Hence, $HCM_1(G) = 72n^2 + 144n$ and $HCM_2(G) = 18n^2 + 96n.$

Observation: (i) $HCM_1(G) = 4HCM_2(G) - 240n.$

(ii) $HCM_2(G) = \frac{1}{4}[HCM_1(G) + 240n].$

Theorem 7. Let G be an $L_{(3,6,3,6)}(n)$ circumference. Then

(i) $CM_1(G,x) = (18n^2 - 12n)x^2 + 12nx^4.$

(ii) $CM_2(G,x) = (18n^2 - 12n)x + 12nx^3.$

Proof:- Recall the definition 2.3. and using Table 5, then

(i) $CM_1(G,x) = x^2|E'_{1,1}| + x^4|E'_{3,1}|$

$= (18n^2 - 12n)x^2 + 12nx^4.$

(ii) $CM_2(G,x) = x^1|E'_{1,1}| + x^3|E'_{3,1}|$

$= (18n^2 - 12n)x + 12nx^3.$

Hence, $CM_1(G,x) = (18n^2 - 12n)x^2 + 12nx^4$ and $CM_2(G,x) = (18n^2 - 12n)x + 12nx^3.$

Observation: (i) $CM_1(G,x) = x[CM_2(G,x)]$.

(ii) $CM_2(G,x) = \frac{1}{x}[CM_1(G,x)]$.

Theorem 8: Let G be an $L_{(3,6,3,6)}(n)$ circumference. Then

(i) $HCM_1(G,x) = (18n^2 - 12n)x^4 + 12nx^{16}.$

(ii) $HCM_2(G,x) = (18n^2 - 12n)x + 12nx^9.$

Proof:- Recall the definition 2.4. and using Table 5, then

(i) $HCM_1(G,x) = x^4|E'_{1,1}| + x^{16}|E'_{3,1}|$

$= (18n^2 - 12n)x^4 + 12nx^{16}.$

(ii) $HCM_2(G,x) = x^1|E'_{1,1}| + x^9|E'_{3,1}|$

$= (18n^2 - 12n)x + 12nx^9.$

Hence, $HCM_1(G,x)=(18n^2-12n)x^4+12nx^6$ and $HCM_2(G,x)=(18n^2-12n)x+12nx^9$.

The values of first and second reverse Zagreb indices and first and second reverse hyper-Zagreb indices of $L_{(3,6,3,6)}(n)$ for specific values of n are given in the following Table 6.

Table-6

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$
$M_1(G)$	60	192	396	672	1020	1440	1932	2496	3132
$M_2(G)$	42	120	234	384	570	792	1050	1344	1674
$HCM_1(G)$	216	576	1080	1728	2520	3456	4536	5760	7128
$HCM_2(G)$	114	264	450	672	930	1224	1554	1920	2322

III. Archimedean Lattice, L (3, 12, 12)

This lattice is also known as the extended Kagome' lattice. The notation (3, 12, 12) indicates that around a given vertex, going in the clockwise direction, one encounters first a triangle and second encounters a two dodecagon. Archimedean lattice $L_{(3,12,12)}(n)$ as shown in following Figure 4.

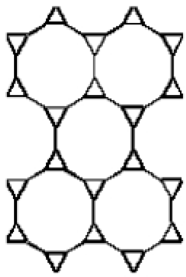

Figure-4: Archimedean lattice $L_{(3,12,12)}(n)$

Consider the edge partition of $L_{(3,12,12)}(n)$ is as follows:

Table-7: Edge partition of $L_{(3,12,12)}(n)$ into $E_{a,b}$

(a,b)	(3, 3)	(2, 3)
$\|E_{a,b}\|$	$27n^2-15n$	$12n$

The maximum vertex degree $\Delta(G)$ in $L_{(3,12,12)}(n)$ is 3 i.e. $\Delta(G)=3$, so Table 8 gives the reverse edge partition of $L_{(3,12,12)}(n)$.

Table-8: Reverse edge partition of $L_{(3,12,12)}(n)$ into $E'_{a,b}$

(a,b)	(1, 1)	(2, 1)
$\|E'_{a,b}\|$	$27n^2-15n$	$12n$

Theorem 9. Let G be an $L_{(3,12,12)}(n)$ circumference. Then

(i) $M_1(G)=54n^2+6n.$

(ii) $M_2(G)=27n^2+9n.$

Proof:- Recall the definition 2.1. and using Table 8, then

(i) $M_1(G)=(2)|E'_{1,1}|+(3)|E'_{2,1}|$

$=(2)(27n^2-15n)+(3)(12n)$

$= 54n^2 + 6n.$

(ii) $M_2(G) = (1)|E'_{1,1}| + (2)|E'_{2,1}|$

$= (1)(27n^2 - 15n) + (2)(12n)$

$= 27n^2 + 9n..$

Hence, $M_1(G) = 54n^2 + 6n$ and $M_2(G) = 27n^2 + 9n.$

Observation: (i) $M_1(G) = 2M_2(G) - 12n.$

(ii) $M_2(G) = \frac{1}{2}[M_1(G) + 12n].$

Theorem 10. Let G be an $L_{(3,12,12)}(n)$ circumference. Then

(i) $HCM_1(G) = 108n^2 + 48n.$

(ii) $HCM_2(G) = 27n^2 + 33n.$

Proof:- Recall the definition 2.2. and using Table 8, then

(i) $HCM_1(G) = (4)|E'_{1,1}| + (9)|E'_{2,1}|$

$= (4)(27n^2 - 15n) + (9)(12n)$

$= 108n^2 + 48n.$

(ii) $HCM_2(G) = (1)|E'_{1,1}| + (4)|E'_{2,1}|$

$= (1)(27n^2 - 15n) + (4)(12n)$

$= 27n^2 + 33n.$

Hence, $HCM_1(G) = 108n^2 + 48n$ and $HCM_2(G) = 27n^2 + 33n.$

Observation: (i) $HCM_1(G) = 4HCM_2(G) - 84n.$

(ii) $HCM_2(G) = \frac{1}{4}[HCM_1(G) + 84n].$

Theorem 11. Let G be an $L_{(3,12,12)}(n)$ circumference. Then

(i) $CM_1(G,x) = (27n^2 - 15n)x^2 + 12nx^3.$

(ii) $CM_2(G,x) = (27n^2 - 15n)x + 12nx^2.$

Proof:- Recall the definition 2.3. and using Table 8, then

(i) $CM_1(G,x) = x^2|E'_{1,1}| + x^3|E'_{2,1}|$

$= (27n^2 - 15n)x^2 + 12nx^3.$

(ii) $CM_2(G,x) = x^1|E'_{1,1}| + x^2|E'_{2,1}|$

$= (27n^2 - 15n)x + 12nx^2.$

Hence, $CM_1(G,x) = (27n^2 - 15n)x^2 + 12nx^3$ and $CM_2(G,x) = (27n^2 - 15n)x + 12nx^2.$

Observation: (i) $CM_1(G,x) = x[CM_2(G,x)]$.

(ii) $CM_2(G,x) = \frac{1}{x}[CM_1(G,x)]$.

Theorem 12. Let G be an $L_{(3,12,12)}(n)$ circumference. Then

(i) $HCM_1(G,x) = (27n^2 - 15n)x^4 + 12nx^9$.

(ii) $HCM_2(G,x) = (27n^2 - 15n)x + 12nx^4$.

Proof:- Recall the definition 2.4. and using Table 8, then

(i) $HCM_1(G,x) = x^4 \,|\, E'_{1,1} \,| + x^9 \,|\, E'_{2,1} \,|$

$= (27n^2 - 15n)x^4 + 12nx^9$.

(ii) $HCM_2(G,x) = x^1 \,|\, E'_{1,1} \,| + x^4 \,|\, E'_{2,1} \,|$

$= (27n^2 - 15n)x + 12nx^4$.

Hence, $HCM_1(G,x) = (27n^2 - 15n)x^4 + 12nx^9$ and $HCM_2(G,x) = (27n^2 - 15n)x + 12nx^4$.

The values of first and second reverse Zagreb indices and first and second reverse hyper-Zagreb indices of $L_{(3,12,12)}(n)$ for specific values of n are given in the following Table 9.

Table-9

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$
$M_1(G)$	60	228	504	888	1380	1980	2688	3504	4428
$M_2(G)$	36	126	270	468	720	1026	1386	1800	2268
$HCM_1(G)$	156	528	1116	1920	2940	4176	5628	7296	9180
$HCM_2(G)$	60	174	342	564	840	1170	1554	1992	2484

4. GRAPHICAL COMPARISON AND CONCLUDING REMARKS

In this paper, we compute first and second reverse Zagreb indices and first and second reverse hyper-Zagreb indices of some Archimedean lattices. Figure 5 shows that Archimedean lattice L (3, 12, 12) get highest value of the first reverse Zagreb index and Archimedean lattice L (4, 6 12) get least value of first reverse Zagreb index. Figure 6 shows that Archimedean lattice L (3, 12, 12) get highest value of second reverse Zagreb index and Archimedean lattice L (4, 6, 12) get least value of second reverse Zagreb index. Figure 7 shows that Archimedean lattice L (3, 12, 12) get highest value of second reverse hyper Zagreb index and Archimedean lattice L (4, 6, 12) get least value of second reverse hyper Zagreb index. Figure 8 shows that Archimedean lattice L (3, 12, 12) get highest value of second reverse hyper Zagreb index and Archimedean lattice L (4, 6, 12) get least value of second reverse hyper Zagreb index.

In the following graphs, different lines are indicates that

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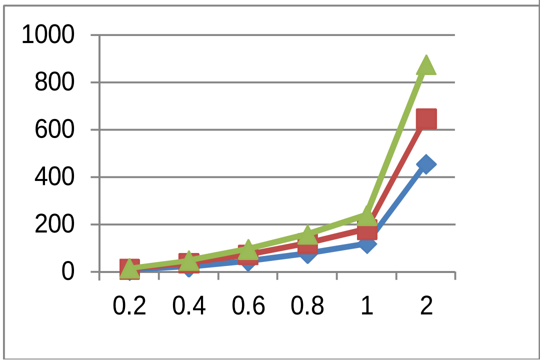

Figure-5: First reverse Zagreb index

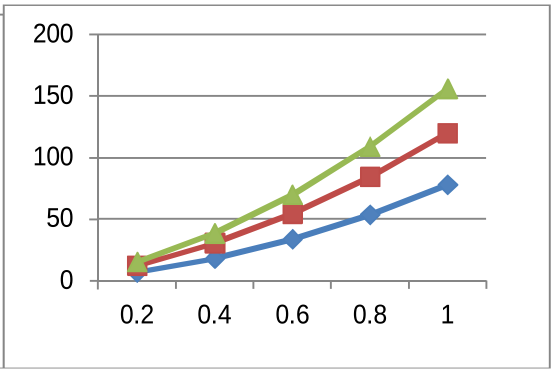

Figure-6: Second reverse Zagreb index

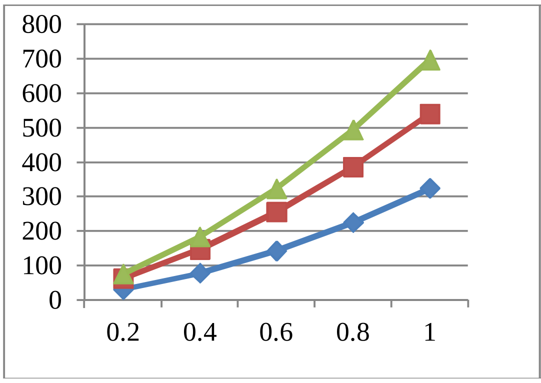

Figure-7: First reverse Zagreb index

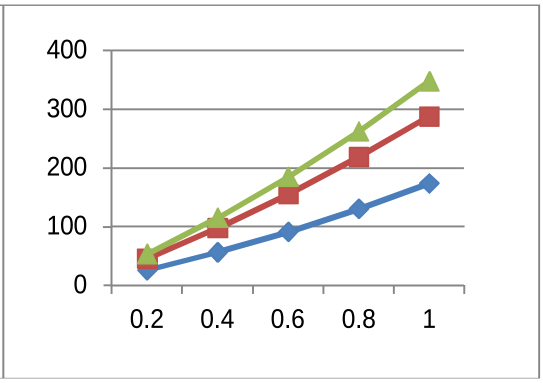

Figure-8: Second reverse hyper-Zagreb Index

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SOME RESULTS ON THE VALUE DISTRIBUTION OF DIFFERENTIAL POLYNOMIAL

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ABSTRACT
In this paper, we study the value distribution of differential polynomials and obtain some results. We consider $F = P_1[f]Q_1[f] + P_2[f]Q_2[f]$ *and study the influence of* $Q_1[f]$ *on value distribution of* F *which is an improvement of results due to S. S. B Math and K. S. L. N. Prasad.*

1 INTRODUCTION AND DEFINITIONS

Throughout this paper, a "meromorphic" function means, the function is meromorphic in the whole complex plane. We assume the reader is familiar with the standard notations of Nevanlinna theory which can be referred from [4] and [7]. Nevanlinna characteristic function of a meromorphic function f is denoted by $T(r,f)$ where $r \to \infty$ outside a set E which has finite linear measure. $S(r,f)$ is the quantity satisfying $S(r,f) = o\{T(r,f)\}$. The order of f is defined as $\rho = \limsup_{r\to\infty} \frac{log\ T(r,f)}{log\ r}$. Let f, g be two meromorphic functions, and $a \in \mathbb{C} \cup \{\infty\}$. We say that f and g share a CM(IM) if $f - a$ and $g - a$ have the same zeros counting multiplicity(ignoring multiplicity).

Definition 1.1 *Let m be a positive integer. $N(r,a;f\backslash\leq m)$ $(N(r,a;f\backslash\geq m))$ denotes the counting function of those a-points of f whose multiplicities are not greater(lesser) than m, where each a-point is counted according to its multiplicity. In a similar manner, $\overline{N}(r,a;f\backslash\leq m), \overline{N}(r,a;f\backslash\geq m)$ denotes the corresponding reduced counting function, where the multiplicity is ignored.*

Definition 1.2 *Let f, g be two meromorphic functions and m be a positive integer. $N(r,a;f\backslash g = b, > m)$ denotes the counting function of those a-points of f, counted with proper multiplicities, which are the b-points of g with multiplicities greater than m and $N_m(r,a;f)$ denotes the counting function of a-points of f, where the a-point of multiplicity μ is counted μ times if $\mu \leq m$ and m times if $\mu > m$.*

Throughout this paper, we denote $N(r,f) = N(r,\infty;f)$, $\overline{N}(r,f) = \overline{N}(r,\infty;f)$, $\overline{N}(r,a;f\backslash\leq\infty) \equiv \overline{N}(r,a;f)$ and $N(r,a;f\backslash\leq\infty) \equiv N(r,a;f)$.

Definition 1.3 *Let f be a transcedental meromorphic function of finite order ρ and $\alpha \in \mathbb{C} \cup \{\infty\}$. The relative Nevanlinna deficiency of $'\alpha'$ with respect to $Q[f]$ given by*

$$\delta_r^Q(\alpha;f) = 1 - \limsup_{r\to\infty} \frac{N(r,\alpha;Q[f])}{T(r,f)}.$$

The proximate Nevanlinna deficiency of $'\alpha'$ with respect to $Q[f]$ given by

$$\delta_\rho(\alpha;f) = 1 - \limsup_{r\to\infty} \frac{N(r,\alpha;Q[f])}{r^\rho}.$$

For distinct α-points of f,

$$\Theta_\rho(\alpha;f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,\alpha;Q[f])}{r^\rho}.$$

Clearly $\delta_r^Q(\alpha;f) \geq \delta_\rho(\alpha;f)$.

Definition 1.4 *A monomial in f, is an expression of the form $M[f] = f^{n_0}(f')^{n_1} \ldots (f^{(k)})^{n_k}$ where $n_0, n_1, \ldots, n_k$ are non negative integers. $\gamma_M = n_0 + n_1 + \cdots + n_k$ and $\Gamma_m = n_0 + 2n_1 + \cdots + (k+1)n_k$ represents the degree and weight of the monomial respectively. Let $M_1[f], M_2[f], \ldots, M_n[f]$ be the monomials in f, then $Q[f] = a_1M_1[f] + a_2M_2[f] + \cdots + a_nM_n[f]$ where $a_i \neq 0, (i = 1,2,3,\ldots,n)$ is called a differential polynomial of degree $\gamma_Q = Max\{\gamma_{M_j}: 1 \leq j \leq n\}$ and weight $\Gamma_Q = Max\{\Gamma_{M_j}: 1 \leq j \leq n\}$.*

We denote $\underline{\gamma_Q} = min_{1\leq j\leq n}\gamma_{M_j}$ as the lower degree of $Q[f]$ and if $\underline{\gamma_Q} = \gamma_Q$, then $Q[f]$ is called a homogeneous differential polynomial.

W. K. Hayman [4] in his well known problem book "Problems in Function Theory" has raised some interesting open problems related to the value distribution of differential polynomials. In 1991, H. X. Yi [5] proved the following result.

Theorem 1.1 *Let f be a transcendental meromorphic function in the plane and $Q_1[f] \neq 0, Q_2[f] \neq 0$ be differential polynomial in f. Let $P_1[f] = a_n f^n + a_{n-1} f^{n-1} + \cdots + a_0,\ a_n(z) \neq 0$. If $F = P_1[f]Q_1[f] + Q_2[f]$, then*

$$(n - \gamma_{Q_2})T(r,f) \leq \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{P_1[F]}\right) + (\Gamma_{Q_2} - \gamma_{Q_2})\overline{N}(r,f) + S(r,f).$$

In 2009 S. S. B math and K. S. L. N. Prasad [2] considered a different combination of differential polynomials and proved the following theorem.

Theorem 1.2 *Let f be a transcendental meromorphic function in the plane and $Q_1[f] \neq 0, Q_2[f] \neq 0$ be differential polynomial in f. Let $P_1[f] = a_n f^n + a_{n-1} f^{n-1} + \cdots + a_0,\ a_n(z) \neq 0$ and $P_2[f] = a_m f^m + a_{m-1} f^{m-1} + \cdots + a_0,\ a_m(z) \neq 0$ where $n > m$. If $F = P_1[f]Q_1[f] + P_1[f]Q_2[f]$, then*

$$\begin{aligned}(n - \gamma_{Q_2})T(r,f) &\leq \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{P_1[F]}\right) + 3\overline{N}\left(r,\frac{1}{P_2[F]}\right)\\ &+(\Gamma_{Q_2} - \gamma_{Q_2} + 1)\overline{N}(r,f) + S(r,f).\end{aligned}$$

In Theorem 1.2, we see that the influence of $Q_1[f]$ on the value distribution of F is ignored. In this article, we show that Theorem 1.2 can further be improved if the influence of $Q_1[f]$ is taken into consideration. Throughout this paper, we ignore zeros and poles of any small function of f because the corresponding counting function is absorbed in $S(r,f)$. We also establish the deficiencies of differential polynomials and prove the following theorem.

Theorem 1.3 *Let f be a transcendental meromorphic function in the open complex plane. Let $Q_1[f] \neq 0$, $Q_2[f] \neq 0$, be two differential polynomials generated by f such that k and $\underline{\gamma}_{Q_1}$ be the order and lower degree of $Q_1[f]$ respectively. For $P_1[f] = a_n f^n + a_{n-1} f^{n-1} + \cdots + a_0, a_n(z) \neq 0$ and $P_2[f] = a_m f^m + a_{m-1} f^{m-1} + \cdots + a_0, a_m(z) \neq 0$ where $n > m$, If $F = P_1[f]Q_1[f] + P_2[f]Q_2[f]$ then*

$$\begin{aligned}&\Theta_r^F(0;f) + \Theta_r^{P_1}(0;f) + 3\Theta_r^{P_2}(0;f) - (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\Theta_r^F(\infty;f)\\ &+\underline{\gamma}_{Q_1}[\delta_r^F(0;f) - \delta_{k+1}^F(0;f)] + n \leq \Gamma_{Q_2} + 6.\end{aligned} \tag{1.1}$$

$$\begin{aligned}&\Theta_\rho^F(0;f) + \Theta_\rho^{P_1}(0;f) + 3\Theta_\rho^{P_2}(0;f) - (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\Theta_\rho^F(\infty;f)\\ &+\underline{\gamma}_{Q_1}[\delta_\rho^F(0;f) - \delta_{(k+1)\rho}^F(0;f)] + n \leq \Gamma_{Q_2} + 6.\end{aligned} \tag{1.2}$$

2 LEMMAS

Lemma 2.1 *[3] Let f be a nonconstant meromorphic function and $Q^*[f], Q[f]$ denote differential polynomials generated by f with arbitrary meromorphic coefficients $q_1^*, q_2^*, \dots, q_s^*$ and $q_1, q_2, \dots, q_t$, respectively. Further let $P[f] = \sum_{j=0}^{n} a_j f^j, (a_n \neq 0)$ and $\gamma_Q \leq n$. If $P[f]Q^*[f] = Q[f]$, then $m(r, Q^*[f]) \leq \sum_{j=1}^{s} m(r, q_j^*) + \sum_{j=1}^{t} m(r, q_j) + S(r,f)$.*

Lemma 2.2 *[5] Let $Q[f] = \sum_{j=1}^{l} b_j M_j[f]$ be a differential polynomial generated by f of order k and lower degree $\underline{\gamma}_Q$. If z_0 is a zero of f with multiplicity $\mu(> k)$ and z_0 is not a pole of any of the coefficients $b_j (j = 1,2,\dots,l)$, then z_0 is a zero of $Q[f]$ with multiplicity at least $(\mu - k)\underline{\gamma}_Q$.*

Lemma 2.3 *[1, 6] Suppose $P[f]$ is a differential polynomial in f, with γ_P and Γ_P being the degree and weight of $P[f]$ respectively, then*

$$m(r, P[f]) = n\ m(r,f) + S(r,f),$$

$$N(r, P[f]) \leq \gamma_P N(r,f) + (\Gamma_P - \gamma_P)\overline{N}(r,f) + S(r,f),$$

$$T(r, P[f]) = n\ T(r,f) + S(r,f).$$

Lemma 2.4 *[3] If $Q[f]$ is a differential polynomial in f of degree γ_Q with arbitrary meromorphic coefficients q_j, then*

$$m(r, Q[f]) \leq \gamma_Q\, m(r,f) + \sum_{j=1}^{n} m(r, q_j) + S(r,f).$$

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3 PROOF OF THEOREM

Proof: If $n < \gamma_{Q_2}$, the theorem is obvious. Suppose that $n \geq \gamma_{Q_2}$, we have,

$$F = P_1[f]Q_1[f] + P_2[f]Q_2[f]. \quad (3.1)$$

Now,

$$F' = \frac{F'}{F}P_1[f]Q_1[f] + \frac{F'}{F}P_2[f]Q_2[f]. \quad (3.2)$$

Also,

$$F' = P_1[f](Q_1[f])' + (P_1[f])'Q_1[f] + P_2[f](Q_2[f])' + (P_2[f])'Q_2[f]. \quad (3.3)$$

Combining equations (3.2) and (3.3), we write

$$\frac{F'}{F}P_1[f]Q_1[f] + \frac{F'}{F}P_2[f]Q_2[f] = (P_1[f])'Q_1[f] + P_1[f](Q_1[f])' + P_2[f](Q_2[f])' + (P_2[f])'Q_2[f]. \quad (3.4)$$

The above equation can be rewritten as,

$$P_1[f]Q^*[f] = Q[f], \quad (3.5)$$

where

$$Q^*[f] = \frac{1}{P_2[f]}\left[\frac{F'}{F}Q_1[f] - (Q_1[f])' - \frac{(P_1[f])'Q_1[f]}{P_1[f]} - \frac{(P_2[f])'Q_2[f]}{P_1[f]}\right],$$
$$Q[f] = -\frac{F'}{F}Q_2[f] + (Q_2[f])'. \quad (3.6)$$

Without loss of generality, let us assume that $Q^*[f] \neq 0$. By Lemma 2.1, we have $m(r, Q^*[f]) = S(r, f)$. Again from (3.1), we have

$$P_1[f] = \frac{Q[f]}{Q^*[f]}. \quad (3.7)$$

Therefore,

$$m(r, P_1[f]) \leq m(r, Q[f]) + m(r, 0; Q^*[f]). \quad (3.8)$$

Using Lemma 2.3 and (3.6), we have

$$m(r, Q[f]) \leq \gamma_{Q_2} m(r, f) + S(r, f). \quad (3.9)$$

From the first fundamental theorem of Nevanlinna, we have

$$m(r, 0; Q^*[f]) = N(r, Q^*[f]) - N(r, 0; Q^*[f]). \quad (3.10)$$

By lemma 2.3, we get

$$(n - \gamma_{Q_2})m(r, f) = N(r, \infty; Q^*[f]) - N(r, 0; Q^*[f]) + S(r, f). \quad (3.11)$$

From (3.6), we see that possible poles of $Q^*[f]$ occur at the poles of f and zeros of F, $P_1[f]$ and $P_2[f]$. Also we note that the the zeros of F, $P_1[f]$ and $P_2[f]$ are at most simple poles of $Q^*[f]$. Let z_0 be a pole of f with multiplicity μ, then z_0 is a pole of $Q[f]$ with multiplicity not exceeding $(\mu - 1)\gamma_{Q_2} + \Gamma_{Q_2} + 1 = \mu\gamma_{Q_2} + \Gamma_{Q_2} - \gamma_{Q_2} + 1$ and z_0 is a pole of $P[f]$ with multiplicity $n\mu$. Hence from (3.5) it follows that z_0 is a pole of $Q^*[f]$ with multiplicity not exceeding $\mu\gamma_{Q_2} + \Gamma_{Q_2} - \gamma_{Q_2} + 1 - n\mu = \Gamma_{Q_2} - \gamma_{Q_2} + 1 - (n - \gamma_{Q_2})\mu$. Therefore

$$\begin{aligned} N(r, Q^*[f]) &\leq \overline{N}(r, F) + \overline{N}(r, 0; P_1[f]) + 3\overline{N}(r, 0; P_2[f]) \\ &\quad + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\overline{N}(r, f) - (n - \gamma_{Q_2})N(r, f) + S(r, f). \end{aligned} \quad (3.12)$$

We note that the order of the differential polynomial $(Q_1[f])'$ is $k + 1$. Let z_0 be a zero of f with multiplicity $\mu > k + 1$. For $\gamma_{Q_1} \geq 1$, by lemma 2.2, we see that z_0 is a zero of $Q_1[f]$ with multiplicity at least $(\mu - 1)\gamma_{Q_1}$. Also z_0 may be a pole of $(\frac{F'}{F} - \frac{P_1'[f]}{P_1[f]})$ with multiplicity not exceeding 1. So z_0 is a zero of $(\frac{F'}{F} - \frac{P_1'[f]}{P_1[f]})Q_1[f]$ with multiplicity at least $(\mu - k)\gamma_{Q_1} - 1$. Since the lower degree of $(Q_1[f])'$ is $\underline{\gamma}_{Q_1}$, it follows from lemma 2.2 that z_0 is a zero of $(Q_1[f])'$ with multiplicity at least $(\mu - k - 1\underline{\gamma}_{Q_1})$. Hence

$$N(r, 0; Q^*[f]) \geq \ N(r, 0; Q^*[f] \ f = 0, > k + 1)$$

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$$\geq \underline{\gamma}_{Q_1} N(r,0;f \;\; > k+1) - \gamma_{Q_1}(k+1)\overline{N}(r,0;f > k+1) + S(r,f)$$

$$= \underline{\gamma}_{Q_1} N(r,0;f) - \underline{\gamma}_{Q_1} N(r,0;f \;\; \leq k+1) + (k+1)\overline{N}(r,0;f > k+1)$$

$$+S(r,f)$$

$$N(r,0;Q^*[f]) \geq \underline{\gamma}_{Q_1} N(r,0;f) - N_{k+1}(r,0;f) + S(r,f). \quad (3.13)$$

If $\underline{\gamma}_{Q_1} = 0$, inequality (3.13) obviously holds. Now from (3.11), (3.12) and (3.13) we get,

$$(n-\gamma_{Q_2})T(r,f) \leq \overline{N}(r,0;F) + \overline{N}(r,0;P_1[f]) + 3\overline{N}(r,0;P_2[f]) \\ +(\Gamma_{Q_2} - \gamma_{Q_2} + 1)\overline{N}(r,f) - \underline{\gamma}_{Q_1} N(r,0;f) - N_{k+1}(r,0;f) + S(r,f). \quad (3.14)$$

Next we suppose that $Q^*[f] \equiv 0$. From (3.5) it follows that $Q[f] \equiv 0$, so using (3.1) we get $P[f]Q_1[f] = c\ Q_2[f]$, where c is a non-zero constant. Then in a similar line of calculations for inequalities (3.11),(3.12) and (3.13) we get,

$$(n-\gamma_{Q_2})m(r,f) \leq N(r,Q_1[f]) - N(r,0;Q_1[f]) + S(r,f),$$

$$N(r,Q_1[f]) \leq (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\overline{N}(r,f) - (n-\gamma_{Q_2})N(r,f) + S(r,f),$$

$$N(r,0;Q_1[f]) \geq \underline{\gamma}_{Q_1} N(r,0;f) - N_{k+1}(r,0;f) + S(r,f).$$

Therefore we get,

$$(n-\gamma_{Q_2})T(r,f) \leq \overline{N}(r,0;F) + \overline{N}(r,0;P_1[f]) + 3\overline{N}\big(r,0;P_2[f] + (\Gamma_{Q_2} - \gamma_{Q_2} \\ +1)\overline{N}(r,f) - \underline{\gamma}_{Q_1} N(r,0;f) - N_{k+1}(r,0;f) + S(r,f). \quad (3.15)$$

From the above equation we get,

$$(n-\gamma_{Q_2}) \quad \leq \limsup_{r\to\infty} \frac{\overline{N}(r,0;F)}{T(r,f)} + \limsup_{r\to\infty} \frac{\overline{N}(r,0;P_1[f])}{T(r,f)} + 3\limsup_{r\to\infty} \frac{\overline{N}(r,0;P_2[f])}{T(r,f)} \\ +(\Gamma_{Q_2} - \gamma_{Q_2} + 1)\limsup_{r\to\infty} \frac{\overline{N}(r,f)}{T(r,f)} - \underline{\gamma}_{Q_1}\{\frac{N(r,0;f)}{T(r,f)} - \frac{N_{k+1}(r,0;f)}{T(r,f)}\} \quad (3.16)$$

i.e.,

$$(n-\gamma_{Q_2}) \quad \leq \{1-\Theta_r^F(0;f)\} + \{1-\Theta_r^{P_1}(0;f)\} + 3\{1-\Theta_r^{P_2}(0;f)\} \\ +(\Gamma_{Q_2} - \gamma_{Q_2} + 1)\{1-\Theta_r^F(\infty;f)\} - \underline{\gamma}_{Q_1}[\{1-\delta_r^F(0;f)\} \\ -\{1-\delta_{k+1}^F(0;f)\}] \quad (3.17)$$

i.e.,

$$\Theta_r^F(0;f) \quad +\Theta_r^{P_1}(0;f) + 3\Theta_r^{P_2}(0;f) - (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\Theta_r^F(\infty;f) \\ -\underline{\gamma}_{Q_1}[\delta_r^F(0;f) - \delta_{k+1}^F(0;f)] + n \leq \Gamma_{Q_2} + 6, \quad (3.18)$$

which proves (1.1).

We know that $\limsup_{r\to\infty} \frac{T(r,f)}{r^\rho} = 1$ hence (3.15) becomes

$$(n-\gamma_{Q_2}) \quad \leq \limsup_{r\to\infty} \frac{\overline{N}(r,0;F)}{r^\rho} + \limsup_{r\to\infty} \frac{\overline{N}(r,0;P_1[f])}{r^\rho} + 3\limsup_{r\to\infty} \frac{\overline{N}(r,0;P_2[f])}{r^\rho} \\ +(\Gamma_{Q_2} - \gamma_{Q_2} + 1)\limsup_{r\to\infty} \frac{\overline{N}(r,f)}{r^\rho} - \underline{\gamma}_{Q_1}\{\frac{N(r,0;f)}{r^\rho} - \frac{N_{k+1}(r,0;f)}{r^\rho}\} \quad (3.19)$$

i.e.,

$$(n-\gamma_{Q_2}) \quad \leq \{1-\Theta_\rho^F(0;f)\} + \{1-\Theta_\rho^{P_1}(0;f)\} + 3\{1-\Theta_\rho^{P_2}(0;f)\} \\ +(\Gamma_{Q_2} - \gamma_{Q_2} + 1)\{1-\Theta_\rho^F(\infty;f)\} - \underline{\gamma}_{Q_1}[\{1-\delta_\rho^F(0;f)\} \\ -\{1-\delta_{k+1}\rho^F(0;f)\}] \quad (3.20)$$

i.e.,

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$$\Theta_\rho^F(0;f) \quad +\Theta_\rho^{P_1}(0;f) + 3\Theta_\rho^{P_2}(0;f) - (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\Theta_\rho^F(\infty;f) \\ -\underline{\gamma}_{Q_1}[\delta_\rho^F(0;f) - \delta_{(k+1)\rho}^F(0;f)] + n \leq \Gamma_{Q_2} + 6, \tag{3.21}$$

which proves (1.2).

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EFFECT OF MAGNETIC FIELD ON PERISTALTIC TRANSPORT OF NANOFLUID CONTAINING GYROTACTIC MICROORGANISMS

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ABSTRACT
This paper examines the effect of the magnetic field on the peristaltic transport of nanofluid in a channel containing gyrotactic microorganism under long wavelength and low Reynolds number assumptions. The advantages of adding motile micro-organisms to nanofluids suspension enhanced the heat transfer, mass transfer and improve the nanofluid stability. The resulting nonlinear system of coupled differential equations has been described for velocity, temperature, concentration and density of motile microorganism distributions are solved with the help of Homotopy Analysis Method. The effects of various physical parameters on the flow characteristics are shown and discussed with the help of graphs.

Keywords: Peristaltic Transport, Nanofluid, Magnetic field, gyrotactic microorganisms.

5. INTRODUCTION

Nowadays, Peristaltic transport is one of the most important pumping mechanisms. Peristaltic transport problems have been special attention due to its wide range of applications in numerous fields of chemical industries, biomedical, engineering, nuclear reactors and physiology. In particular, peristaltic pumping mechanisms involved in many biological systems like swallowing of food through the esophagus, movement of chime in the gastrointestinal tract and urine transport from the kidney to the bladder through the ureter. Peristalsis was first initiated by Latham [1] in 1966. Further, this mechanism has become an important concept of research to the above-mentioned applications. This work was further extended by Shapiro et al. [2] and Jaffrin et al. [3]. Several investigators have been studied the peristaltic transport of fluids is mentioned in the reference list [4-5].

The word nano comes from the Greek word "nanos" meaning "dwarf". Nanoparticles dimensions of size ranging from 1 to 100 nm [6]. Nanoparticles are widely used in many fields like biomedicine, photochemical, chemistry and so on. Drug delivery has been proved to be important process for targeting brain tumor and prevention of cardiovascular diseases. The recent development of nanofluids has received very much awareness due to its applications in biological, industries and biotechnology. Researchers and scientists investigated the connection of peristaltic transport and nanofluid problems with different geometries and also various models. Some of these investigations have been cited in the reference list [7-8]. Peristaltic transport in the presence of a magnetic field has also various applications in physics, engineering, and chemistry. Examples include blood pumps, MHD generators, MHD pumps, magnetic drug targeting for cancer diseases. Some more studies on the magnetic field on peristaltic transport can be found from references [9-10]. Bioconvection has a large number of applications in the fields of biomedical and biotechnology. The Bioconvection is defined as flow induced by collective swimming of motile microorganisms which are little denser than water is studied by John [11]. Bioconvection instability is developed from an initially uniform suspension without an unstable density disturbance was given by Pedley et al. [12]. Many researchers working on the bioconvection with different geometries given in the references [13-15].

Review of literature and above analysis in mind, the current research paper is based on influence of magnetic field on peristaltic transport of nanofluids containing gyrotactic microorganism through channel. The present study has wide range of applications in biomedical science and engineering. Since the microorganisms are favorable in decomposition of organic material, producing oxygen and maintaining human health. The dilution of microorganisms in the nanofluids modifies the thermal conductivity. In the present paper, the solution for velocity, pressure gradient, temperature, concentration and motile microorganism density along with boundary conditions are obtained by using the Homotopy Analysis Method. The effects of physical parameters are analyzed through graphs.

6. MODELING

Consider a peristaltic transport of nanofluid in a two dimensional channel. Here we consider Cartesian coordinate system ($\tilde{X}, \tilde{Y}$). The physical model of the channel wall surface can be written as

$$h'\left(\tilde{X},\tilde{t}\right) = b(\tilde{X}) + d\sin\left(\frac{2\pi}{\lambda}\left(\tilde{X} - c\tilde{t}\right)\right), \tag{1}$$

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here $b(\tilde{X}) = a_0 + k\tilde{X}$ is the half width of the channel. Let $\tilde{U}$ and $\tilde{V}$ are velocity components respectively, the velocity field V can be written as

$$V = \left(\tilde{U}, \tilde{V}, 0\right). \tag{2}$$

The governing equations of mass, momentum, energy, nanoparticle mass transfer and density of motile microorganism for the nanofluid can be formulated as follows

$$\frac{\partial \tilde{U}}{\partial \tilde{X}} + \frac{\partial \tilde{U}}{\partial \tilde{Y}} = 0. \tag{3}$$

$$\begin{aligned}\rho_f\left(\frac{\partial \tilde{U}}{\partial \tilde{t}} + \tilde{U}\frac{\partial \tilde{U}}{\partial \tilde{X}} + \tilde{V}\frac{\partial \tilde{U}}{\partial \tilde{Y}}\right) = &-\frac{\partial \tilde{p}}{\partial \tilde{X}} + \mu\left(\frac{\partial^2 \tilde{U}}{\partial \tilde{X}^2} + \frac{\partial^2 \tilde{U}}{\partial \tilde{Y}^2}\right) - \sigma^* B_0^2 \tilde{U} + \left(1-\phi_1\right)\rho_f g\beta\left(\tilde{T} - \tilde{T}_0\right) \\ &-\left(\rho_p - \rho_f\right)g\left(\tilde{C} - \tilde{C}_0\right) - \left(\rho_m - \rho_f\right)\gamma g\left(\tilde{n} - \tilde{n}_0\right),\end{aligned} \tag{4}$$

$$\rho_f\left(\frac{\partial \tilde{V}}{\partial \tilde{t}} + \tilde{U}\frac{\partial \tilde{V}}{\partial \tilde{X}} + \tilde{V}\frac{\partial \tilde{V}}{\partial \tilde{Y}}\right) = -\frac{\partial \tilde{p}}{\partial \tilde{Y}} + \mu\left(\frac{\partial^2 \tilde{V}}{\partial \tilde{X}^2} + \frac{\partial^2 \tilde{V}}{\partial \tilde{Y}^2}\right) - \sigma^* B_0^2 \tilde{V}. \tag{5}$$

$$\begin{aligned}(\rho c)_f\left(\frac{\partial \tilde{T}}{\partial \tilde{t}} + \tilde{U}\frac{\partial \tilde{T}}{\partial \tilde{X}} + \tilde{V}\frac{\partial \tilde{T}}{\partial \tilde{Y}}\right) = &\, k^*\left(\frac{\partial^2 \tilde{T}}{\partial \tilde{X}^2} + \frac{\partial^2 \tilde{T}}{\partial \tilde{Y}^2}\right) + (\rho c)_p D_B\left(\frac{\partial \tilde{C}}{\partial \tilde{X}} + \frac{\partial \tilde{C}}{\partial \tilde{Y}}\right)\left(\frac{\partial \tilde{T}}{\partial \tilde{X}} + \frac{\partial \tilde{T}}{\partial \tilde{Y}}\right) \\ &+ (\rho c)_p \frac{D_T}{T_m}\left(\frac{\partial^2 \tilde{T}}{\partial \tilde{X}^2} + \frac{\partial^2 \tilde{T}}{\partial \tilde{Y}^2}\right)^2.\end{aligned} \tag{6}$$

$$\frac{\partial \tilde{C}}{\partial \tilde{t}} + \tilde{U}\frac{\partial \tilde{C}}{\partial \tilde{X}} + \tilde{V}\frac{\partial \tilde{C}}{\partial \tilde{Y}} = D_B\left(\frac{\partial^2 \tilde{C}}{\partial \tilde{X}^2} + \frac{\partial^2 \tilde{C}}{\partial \tilde{Y}^2}\right) + \frac{D_T}{T_m}\left(\frac{\partial^2 \tilde{T}}{\partial \tilde{X}^2} + \frac{\partial^2 \tilde{T}}{\partial \tilde{Y}^2}\right). \tag{7}$$

$$\frac{\partial \tilde{n}}{\partial \tilde{t}} + \tilde{U}\frac{\partial \tilde{n}}{\partial \tilde{X}} + \tilde{V}\frac{\partial \tilde{n}}{\partial \tilde{Y}} = D_n\frac{\partial^2 \tilde{n}}{\partial \tilde{Y}^2} - \frac{bW_C}{\left(\tilde{C}_1 - \tilde{C}_0\right)}\frac{\partial}{\partial \tilde{Y}}\left(\tilde{n}\frac{\partial \tilde{C}}{\partial \tilde{Y}}\right), \tag{8}$$

where ρ_p is nanoparticle mass density, ρ_f is the density of the fluid, $\left(\rho c\right)_f$ and $\left(\rho c\right)_p$ are the heat capacity of the fluid and effective heat capacity of the nanoparticle material, k^* is thermal conductivity of the fluid, g is the acceleration due to gravity, β is volume expansion coefficient, $\tilde{C}$ is the nanoparticle concentration, $\tilde{T}$ is the temperature of the fluid. Further ρ_f is the effective density, D_B and D_T are the Brownian diffusion and thermophoresis diffusion coefficients, T_m is the mean fluid temperature, b and W_C are the chemotaxis and assumed constants of the microorganism, ϕ_1 is the nanoparticles solid volume fraction. The relationship between the laboratory frame and wave frame are introduced through

$$\begin{aligned}&\tilde{x} = \tilde{X} - c\tilde{t}, &&\tilde{y} = \tilde{Y}, \\ &\tilde{u}(\tilde{x}, \tilde{y}) = \tilde{U} - c, &&\tilde{v}(\tilde{x}, \tilde{y}) = \tilde{V},\end{aligned} \tag{9}$$

where $(\tilde{u}, \tilde{v})$ are velocity components, $(\tilde{x}, \tilde{y})$ are coordinates in an wave frame.

Corresponding boundary conditions are

$$\left.\begin{aligned}&\tilde{u} = 0, \quad \tilde{T} = \tilde{T}_0, \quad \tilde{C} = \tilde{C}_0, \quad \tilde{n} = \tilde{n}_0 \quad \text{at} \quad \tilde{y} = 0, \\ &\tilde{u} = -c, \quad \tilde{T} = \tilde{T}_1, \quad \tilde{C} = \tilde{C}_1, \quad \tilde{n} = \tilde{n}_1 \quad \text{at} \quad \tilde{y} = h' = b(\tilde{x}) + d\sin\frac{2\pi}{\lambda}(\tilde{x}).\end{aligned}\right\} \tag{10}$$

Introducing the following non dimensional quantities

$$\left.\begin{aligned}
&x=\frac{\tilde{x}}{\lambda},\ y=\frac{\tilde{y}}{b},\ t=\frac{c\tilde{t}}{\lambda},\ M=\sqrt{\frac{\sigma^*}{\mu}}B_0 a,\ p=\frac{b^2\tilde{p}}{c\lambda\mu},\ \sigma=\frac{\tilde{n}_0}{\tilde{n}_1-\tilde{n}_0},\ v=\frac{\tilde{v}}{c},\ \delta=\frac{b}{\lambda},\ u=\frac{\tilde{u}}{c},\\
&\mathrm{Re}=\frac{\rho_f cb}{\mu},\ \beta^*=\frac{k^*}{(\rho c)_f},\ Rb=\frac{\left(\rho_m-\rho_f\right)\gamma\left(\tilde{n}_1-\tilde{n}_0\right)}{(1-\phi_1)\beta\left(\tilde{T}_1-\tilde{T}_0\right)\rho_f},\ Gr=\frac{\left(1-\phi_1\right)\rho_f g\beta b^2\left(\tilde{T}_1-\tilde{T}_0\right)}{c\mu},\\
&Nr=\frac{\left(\rho_p-\rho_f\right)\left(\tilde{C}_1-\tilde{C}_0\right)}{(1-\phi_1)\beta\left(\tilde{T}_1-\tilde{T}_0\right)\rho_f},\ \mathrm{Pr}=\frac{\nu}{\beta^*},\ Nb=\frac{(\rho c)_p D_B\left(\tilde{C}_1-\tilde{C}_0\right)}{(\rho c)_f\nu},\ Nt=\frac{(\rho c)_p D_T\left(\tilde{T}_1-\tilde{T}_0\right)}{(\rho c)_f T_m\nu},\\
&\Omega=\frac{\tilde{C}-\tilde{C}_0}{\tilde{C}_1-\tilde{C}_0},\ \theta=\frac{\tilde{T}-\tilde{T}_0}{\tilde{T}_1-\tilde{T}_0},\ \chi=\frac{\tilde{n}-\tilde{n}_0}{\tilde{n}_1-\tilde{n}_0},\ h=\frac{h'}{a_0}=1+\frac{\lambda kx}{a_0}+\alpha\sin 2\pi x,\ \alpha=\frac{d}{a_0},\ Pe=\frac{bW_C}{D_n}.
\end{aligned}\right\} \tag{11}$$

where Pr is the Prandtl number, Gr is the Grashof number of the local temperature, Nr is the buoyancy ratio respectively, Pe and Rb are the Bioconvection Peclet number and Bioconvection Rayleigh number, Nb and Nt are the Brownian motion and thermophoresis parameters, α is the amplitude ratio.

Using the above non-dimensional variables and the basic equations (1)-(11) can be reduce to

$$\frac{\partial p}{\partial x}=\frac{\partial^2 u}{\partial y^2}-M^2u+Gr\left(\theta-Nr\Omega-Rb\chi\right), \tag{12}$$

$$\frac{\partial p}{\partial y}=0, \tag{13}$$

$$\frac{\partial^2\theta}{\partial y^2}+\mathrm{Pr}\,Nb\frac{\partial\theta}{\partial y}\frac{\partial\Omega}{\partial y}+\mathrm{Pr}\,Nt\left(\frac{\partial\theta}{\partial y}\right)^2=0, \tag{14}$$

$$\frac{\partial^2\Omega}{\partial y^2}+\frac{Nt}{Nb}\frac{\partial^2\theta}{\partial y^2}=0, \tag{15}$$

$$\frac{\partial^2\chi}{\partial y^2}-Pe\frac{\partial\Omega}{\partial y}\frac{\partial\chi}{\partial y}-Pe\chi\frac{\partial^2\Omega}{\partial y^2}-Pe\sigma\frac{\partial^2\Omega}{\partial y^2}=0. \tag{16}$$

The corresponding dimensionless boundary conditions in the wave frame of the problem are defined as

$$\left.\begin{aligned}
&u=0,\quad \theta=0,\quad \Omega=0,\quad \chi=0\quad \text{at}\quad y=0,\\
&u=-1,\quad \theta=1,\quad \Omega=1,\quad \chi=1\quad \text{at}\quad y=h=1+\frac{\lambda kx}{a_0}+\alpha\sin 2\pi x,
\end{aligned}\right\} \tag{17}$$

7. RESULTS AND DISCUSSION

From the above mentioned description of the considered problem the system of non-linear coupled partial differential equations is obtained. Such system is difficult to solve explicitly to get exact solutions. However the advancement of techniques during last few decades has provided more efficient ways to solve the complex non-linear models. Thus, the problem in hand is approximated semi-analytically using Homotopy Analysis Method with Mathematica software and graphical results are drawn in Origin software. Therefore this section comprises the development of velocity, temperature, nanoparticle concentration and motile microorganism density profiles corresponding to variation of Hartmann number is M, Grashof number of the local temperature is Gr, the buoyancy ratio is Nr, Bioconvection Peclet number and Bioconvection Rayleigh number are Pe and Rb, the Brownian motion and thermophoresis parameters are Nb and Nt.

3.1. Velocity profile

The effects of M, Gr, Nr and Rb on velocity profile $u(y)$ are presented through the figures 1 to 4. It is observed from Fig. 1 that velocity in all the regions of the peristaltic pumping decreases with an increase in the values of Hartmann number M. Fig. 2 demonstrates the behavior of velocity profile for various values of nanoparticle Grashof number Gr. It is depicted from Fig. 2 that the velocity profile increases with an increasing

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value of Gr. The variation of velocity for different values of buoyancy ratio Nr is shown in Fig. 3. It is observed from Fig. 3 that velocity in all the regions of the peristaltic pumping decreases with increasing values of buoyancy ratio Nr. Fig. 4 demonstrates the behavior of velocity profile for various values of Bioconvection Rayleigh number Rb. It is depicted from Fig. 4 that the velocity profile decreases with increasing values of Rb.

3.2. Temperature profile

Figures 5 to 6 are prepared to examine the temperature via Nb and Nt. Brownian motion parameter Nb has an increasing effect on temperature (see fig. 5). Substantial increase in temperature is seen by enhancing thermophoresis parameter Nt (see fig. 6).

3.3. Nanoparticle concentration profile

Figures 7 to 8 shows the nanoparticle concentration profile are presented under the effects of Nb and Nt. Brownian motion parameter Nb have a decreasing effect on nanoparticle concentration (see fig. 7). Substantial decreases in nanoparticle concentration is seen by enhancing thermophoresis parameter Nt (see fig. 8)

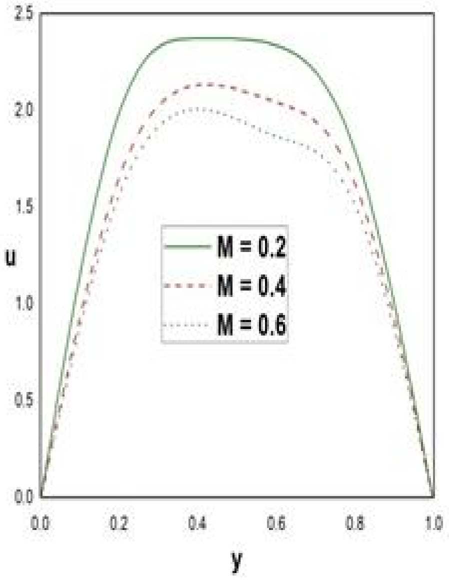

Fig-1: Influence of M on velocity

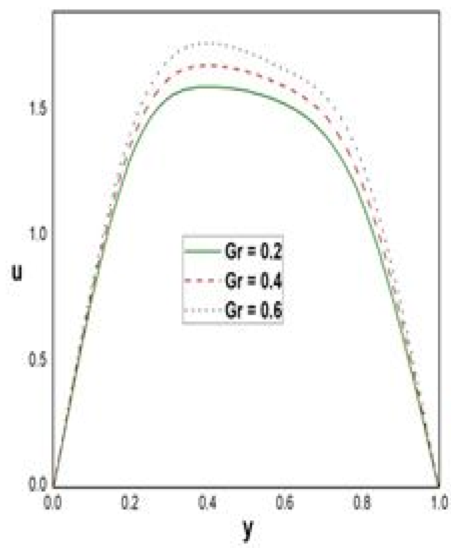

Fig-2: Influence of Gr on velocity

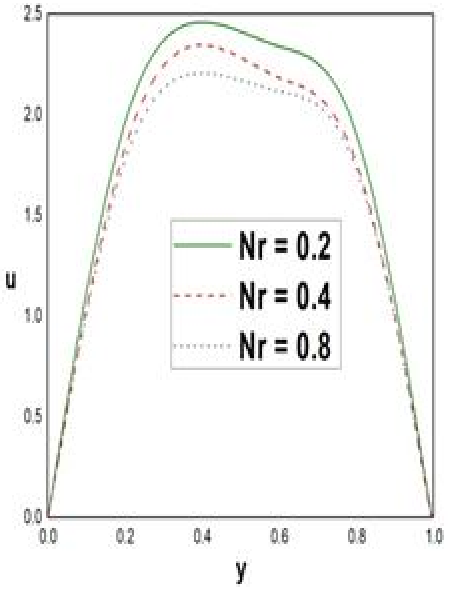

Fig-3: Influence of Nr on velocity

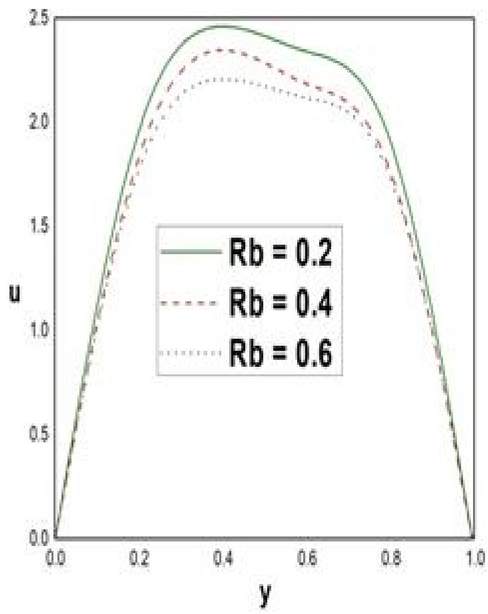

Fig-4: Influence of Rb on velocity

7.4. Motile microorganism density profile

The behavior of motile microorganism density profile for different values of Pe and σ are shown in figures 9 to 10. It is observed from figures 9 to 10 that motile microorganism density decreases in the values of Pe and decreases with an increase in the values of σ.

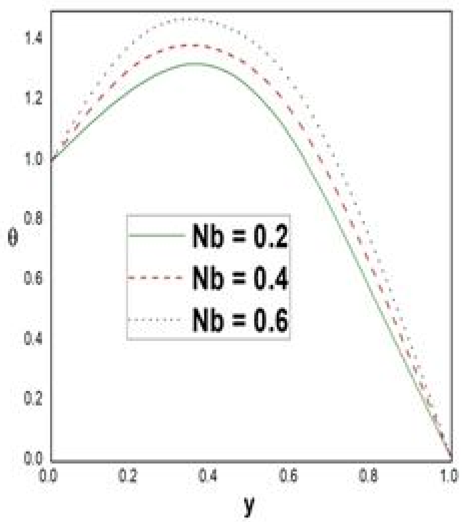

Fig-5: Influence of Nb on temperature

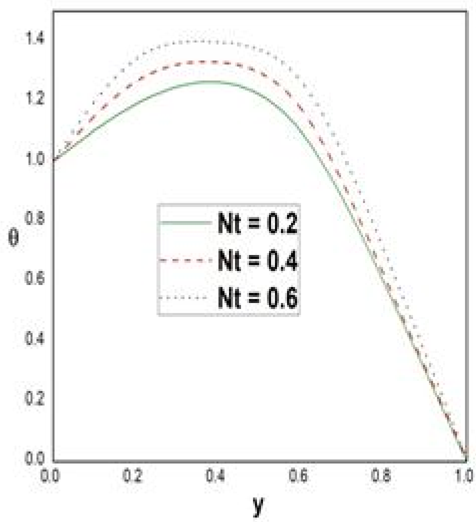

Fig-6: Influence of Nt on temperature

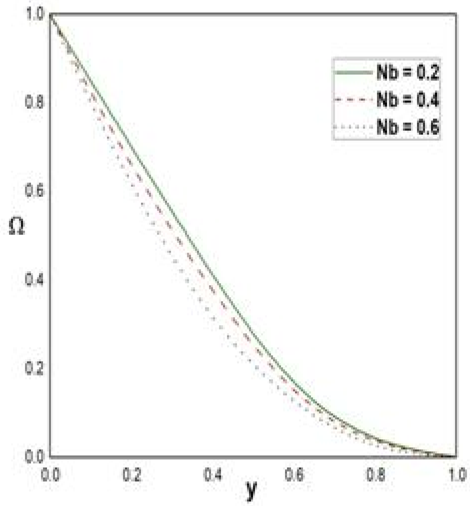

Fig-7

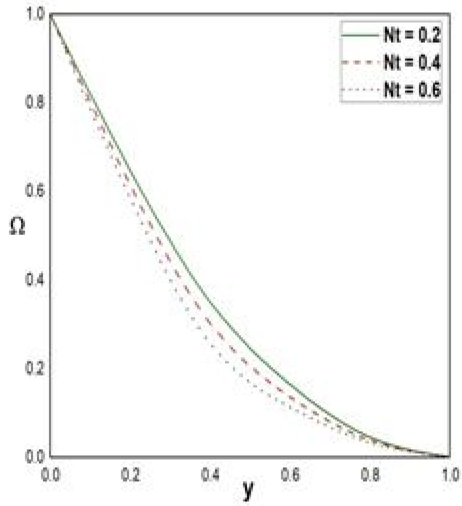

Fig-8

Fig-7: Influence of Nb on nanoparticle concentration.

Fig-8: Influence of Nt on nanoparticle concentration.

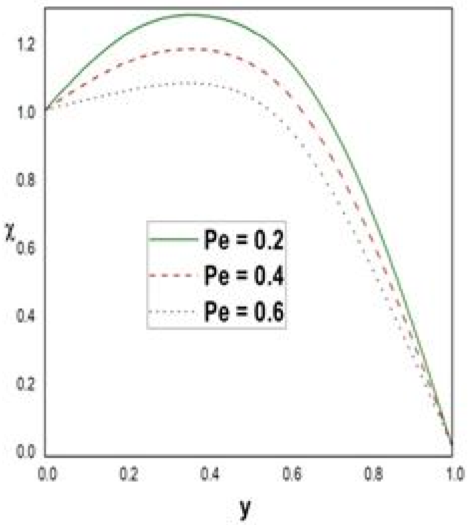

Fig-9

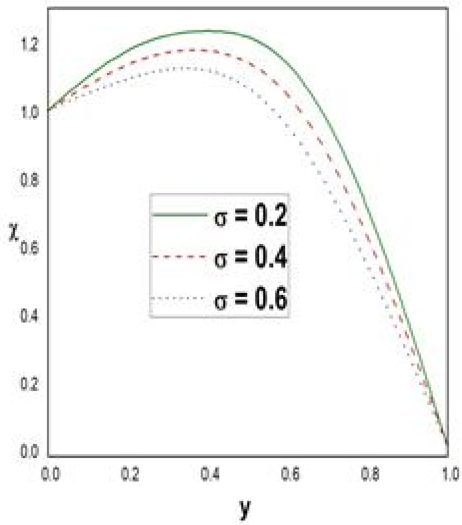

Fig-10

Fig-9: Influence of Pe on density of motile microorganism.

Fig-10: Influence of σ on density of motile microorganism.

8. CONCLUSION

In this study we discuss the effect of magnetic field on peristaltic transport of a nanofluid containing gyrotactic microorganism is discussed. The main findings of the presented study are listed below.

- It is observed that the velocity field decreases whole domain of channel when we increase the thermal Grashof number.
- It is observed that temperature is increasing with increase in thermophoresis parameter.
- It is observed that nanoparticle concentration will be decreasing as Brownian motion parameter increases whole domain of channel.
- It is observed that motile microorganism density decreases, when increasing the both Bioconvection Peclet number and Bioconvection Rayleigh number

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SYNERGIC EFFECT OF ORGANIZATIONAL CULTURE AND KNOWLEDGE SHARING BEHAVIOR ON INNOVATION CAPABILITY OF INDIAN SMEs.

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ABSTRACT

Small and Medium Enterprises portray the strength of an economy in maintaining an appreciable success rate and in generating employment opportunities .Contribution of SMEs to the Indian economy in terms of employment generation, fostering proper economic growth and enhancing the export intensity of the country has been completely phenomenal. There has been an increasing attention supposing to the power of innovativeness in SMEs.

This paper investigates how creating a knowledge sharing behavior with an organization Culture focuses on collaboration, trust and learning gave rise to incrased level of innovation capability. Data were collected from thirty four small and medium enterprises from the eastern part of India. Self designed questionnaire established to collect responses related to the study constructs . All the three scales related to organizational culture, knowledge sharing behavior and innovation capability were enumerated in five point Likert scale. Structural equation modeling (SEM) technique was applied using SPSS 20 software to examine the proposed relationships. After reliability and validity was examined, Confirmatory factor analysis (CFA) was done using maximum likelihood techniques (MLE) on all the extracted factors to assess the constructs.

The findings provide empirical evidence for the hypothesis that suggest a positive link between enterprises' cultural dimensions and innovation capability of SMEs.

Keywords: Organizational Culture, Knowledge Sharing Behavior, Innovation Capability, Small Medium Enterprises

INTRODUCTION

Small and Medium Enterprises characterize the strength of an economy in maintaining an appreciable success rate and in generating employment opportunities. Contribution of SMEs to the Indian economy in terms of employment generation, fostering proper economic growth and enhancing export intensity of the country has been completely phenomenal. There has been an increasing attention supposing to the power of innovativeness in SMEs.

Innovation is a practice enterprises manage challenges in developing new products and services, as well as to improve processes and management to offer added value to their customers. Innovation has been generally accepted as a driving force in creating economic value since Schumpeter marked it in 1934. To earn competitive advantage innovation capability deemed to be one of the practically important factor. Therefore constructs familiar with enhancing the innovation capability of organizations have approved popularity in organization literature.

This paper focuses on how being efficient at innovation is about creating knowledge sharing behavior with a Culture focusing on collaboration, trust and learning. The basis of creating employees attitudes showing in their behaviors in distinctive situations are the assumptions, values and norms. This implies that organizational culture of any enterprise is incidental as shared by employee's cultural values, beliefs and having an urge on the possibility of knowledge sharing and this impact can be in turn strengthening or hindering. Therefore it is imperative to look the organizational culture, knowledge sharing behavior shaping and improving innovation capability.

RESEARCH PROBLEM AND STUDY OBJECTIVE

India is raising in the global innovation index (81 to 60 out of 127 countries), which in a way reflects much more unrepresented innovation capacity of Indian firms and Indian small and medium-sized enterprises (SMEs) in particular having more potential in raising GII . One of the major challenges an organization faces is innovativeness in business. This study investigates the role of organizational culture, knowledge sharing behavior and innovation capability with reference to Indian SMEs.

The objective of this study is to investigate the relationship among organizational culture, knowledge sharing behavior and innovation capability. Although previous studies have established a link between culture and sharing behavior (Sanjaghi et al. 2013; Kathiravelu et al,2014; Yang and Chen, 2007),

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Razak N.A., Aziz R.A., Rahman Z.A., Ali S.2018 study highlights a comprehensive literature review on how practices of the knowledge sharing embedded the activities. The discussion lead by understanding nature of knowledge sharing, process and continued with the practices of knowledge sharing. Finally, study contributes to the strategy on how to improve the practices of knowledge sharing and perform well in business.

There is scarcity of evidence relates to elements of organizational culture e.g. collaboration, trust and learning affect the knowledge sharing behavior of employees. Still, there is a lack of studies relating sharing behavior of employee which may likely to have a positive impact on developing innovation capability in the context of small and medium enterprises.

LITERATURE REVIEW

Organizational Culture

Organizational culture is an imperative build that influences both individual and organizational related process and results. In writing there is no accord on definition or what constitute organizational culture (Hatch and Zilber; 2012). They additionally set that cultures can't be precisely or totally depicted by any means. Abu-Jarad et al, (2010) opined that it is characterized from alternate point of view. Organizational culture research and its effect on other organizational factors ended up uncontrolled a mid 1980s. As indicated by Lund (2003).

1980s saw a flood in examine on effect of organizational culture on representatives and associations execution. The definitions take distinctive shapes relying upon the idea they mirror, the creators' point of view methodologies and accentuations. Research and down to earth involvement of the 1980s uncovered two distinct ways to deal with characterizing organizational culture. Culture is dealt with as an inner subsystem of the association, enabling people to adjust to the earth while in this approach, the organization has a culture, it is illustrative and self explanatory. In the ensuing methodology, the organization is dealt with as a culture as seen by their workers.

INNOVATION CAPABILITIES

Innovation is the ability of creation and execution of new ideas by people who over time fit into place in transactions with others within an institutional setting (Van De Ven, 2006). Innovation is encouraged through appropriate cultural norms and support systems. Ahmed (1998) claims that innovation is the engine of change and culture is a primary determinant of innovation. Innovation capability influence organizational performance in several ways. Capabilities that firm possess in general are crucial in obtaining and sustaining competitive advantage (Akman and Yilmaz, 2011)In particular, innovation capability is associated with several strategic advantages.

For instance, Shan and Zhang (2009) noted that sustained competitive advantage can be achieved by raising independent innovation capability continually in the firms. Innovation capability seems to be associated with the organizational potential to convert new ideas into commercial and community value. Innovation capability is related to a variety of factors and thus is affected by different internal and external factors (Bullinger, et al; 2009).

While innovation is a complex concept, research identifies five key areas that influence the ability of organization to innovate. These influences relate to leadership; opportunistic behavior; culture and change; learning; and networking and relationship building. This study suggests that organizational culture as an important organizational factor affecting innovation capability of the firms

Knowledge sharing behavior

Knowledge sharing is delineated as an arrangement of practices about learning trade which involve the on-screen characters, learning content, organizational setting, suitable media, and societal environment(M. Shin,2004 ;V. Pale skinned person et al.1999; C.K. Lee and Suliman A,2002). Hendriks (1999) recommended a conceptualized display which comprises of two principle exercises for effective information sharing: transmission and absorption.Lee,2001 proposed two kinds of learning: unequivocal learning, which can be obviously articulated in composed records (e.g. business reports), and verifiable information, which is inserted into an individual’s involvement (e.g. know-how Lee and Suliman ,2001 proposed a learning sharing framework which is influenced by five variables – the on-screen characters who partake in the movement, the characteristics of the common information, the organizational concerns, the channel which is utilized to communicate with others, and the natural atmosphere. Ruggles ,1998 found that the important impediments for learning sharing incorporate culture (54%), organizational structure (28%), information communication innovation (22%), motivating force framework (19%), and staff turnover (8%), by investigating 431 US and European associations. There is an expanding accentuation on the importance of knowledge sharing for organizational execution and adequacy in both the private and open parts. Learning sharing exercises make open doors for private associations to maximize their capacity to meet clients' changing needs and to produce answers for increase focused advantage (Argote, Beckman and Epple 1990; Baum and Ingram 1998; Beckman 1997).

Knowledge sharing requires the dispersal of individual representatives' business related encounters and joint effort between and among people, subsystems, and associations; coordinated effort with different organizations and partners is likewise required for enhanced learning sharing (Dyer 1997; Inkpen and Beamish 1997). As De Long and Fahey (2000) specified, organizational culture impacts workers' observations and practices indispensable to information creation, sharing, and utilize. Organizations require looking into making and also keeping up a culture whereby laborers are willing and ready to share learning (Nya Ling, 2011). . Culture may go about as a spark for knowledge sharing. Gottschalk &Karlsen (2004) maintain that culture assumes a critical part in the knowledge sharing procedure.

Association culture is said to be an important factor to make, offer, and utilize learning in that it establishes standards in regards to Knowledge Sharing (De Long and Fahey 2000)and makes a situation in which people are motivated to impart their insight to others (Cabrera and Cabrera 2002). OC will influence an association's learning and capabilities, and will control it to change and improve (Lynn 1999OC is considered to be a key component of overseeing organizational change and renewal (Pettigrew 1990).Hu et al. (2009) find that if firms expect to accomplish high-benefit advancement performance, they first need to create Knowledge Sharing behaviors in addition to a superior team culture. Zheng et al. (2010) recommend that KM completely mediates the effect of OC on organizational viability. Based on the investigation of Cao and Long (2009), the outcomes bolster that OC has a positive roundabout effect on development by affecting KS.

Albeit Indian SMEs have expanding commitment in monetary development, it keeps on being dismissed in examine settings particularly in the field of organizational culture, learning sharing and advancement capacity. Subsequently, in the ebb and flow inquire about examinations the linkage between association culture ,knowledge offering conduct and development ability to confirm from Indian SMEs.

Research structure and hypotheses

This exploration looks at a firms innovation capacity and the variables that influence it’s enhancement. The examination hypotheses test the connections among knowledge sharing, organization culture and innovation capability. Lee demonstrated that learning sharing is a noteworthy pointer of regardless of whether the outsourcing movement succeeds. Organizational capacity to learn or procure the required information from different associations is a key wellspring of effective knowledge sharing, and that association quality is a huge mediating factor between knowledge sharing and development ability.

There are different ramifications that consider trust and culture as critical variables for learning sharing. Ribière (2001) featured that in an organizational culture, the measurement of a culture i.e. 'trust' and 'solidarity' are the primary pre-conditions to cultivate knowledge sharing. Augment Wulff and Ginman (2004) inspect how organizational culture assumes a focal part in how promptly workers share information. Organizational culture likewise influences organizational development capacity essentially. Vincent et al., (2009) pushed that part of condition; organizational capacities, organizational socioeconomics and organizational structure factors influence advancement in firms. Specifically, organizational capacities and structure represent the best level of remarkable difference on development. Ahmed (1998) recommended that culture is an essential determinant of development and ownership of positive social attributes furnishes the association with important fixings to advance. A few qualities of culture can serve to upgrade or restrain the inclination to enhance in firms (Ahmed, 1998 and McLean, 2012). McLean (2012) talked about that organizational culture related attributes and organizational atmosphere measurements are identified with the backings of or hindrances to imagination and advancement. While, organizational consolation, supervisory support, work gather support, flexibility/self-governance, and assets bolster the capacity to improve, the control diminishes imaginative and creative capacity of the associations. The way unique measurement of culture and related attributes influence advancement capacity and development in the organizations appear to be uncertain. A conclusion from these examinations is that proposing certain organizational social measurements and qualities as general solution for development cannot mirror the truth experienced with associations. Or maybe, every one of the qualities identified with various measurements of organizational culture with shifting degrees may influence advancement capacity of the organizations.

Consequently, hypotheses got from these hypothetical and experimental investigations would be as per the following:

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H1: Organizational culture positively relates to knowledge sharing behavior

H2: Knowledge sharing behavior positively relates to innovation capability

H3: Organizational culture positively relates to innovation capability

H4: organization culture mediates the relationship between knowledge sharing and innovation capability of the organization.

METHODOLOGY

Present study is an attempt to reveal the aspects of organizational culture in Indian SMEs, in which, the investigator seeks to gather primary data concerning knowledge sharing behavior and organization culture for enhancing innovation capabilities of SMEs located in Cuttack and Bhubaneswar. Hence, it takes into account, the analytical tools with recognizable dimensions of organizational culture such as collaboration, trust, learning etc. to reveal relationship among these constructs and innovation capabilities.

Measurement

A five-point Likert scale was used to measure the constructs related to the study. The study was conducted on a convenience sample of 214 employees in 34 small and medium sized organizations in odisha. In this research framework, knowledge sharing and organization culture were independent variables and innovation capability as dependent variable.knowledge sharing is described as the employees' interaction with their individual knowledge, and with that of other employees in the company. OC deals with collaboration,trust and learning aspect and innovation capability in terms of product ,process and management innovation.Since the current research objective and aim differ from the original studies, some of the questions from the original measurements were modified to more effectively measure innovation capability.

Reliability and validity

Data were first analyzed to ensure instrument quality by convergent and discriminant validity. Applying SPSS, the principal components analysis (PCA) was conducted to measure the underlying dimension associated with 18 items. The constructs validity was measured using Bartlett's test of Sphericity and Kaiser–Mayer–Olkin (KMO) measure of the sampling adequacy of individual variables. KMO overall should be 0.6 or over to perform factor analysis (Ozdamar, 2002). According to the results of Bartlett's test of Sphericity and KMO revealed that both are significant and suitable for the factor analysis. The cumulative variance explained is 69.39% which go above the acceptable limit of 60% (Ozdamar, 2002). The factor loading of all items of each scale exceeds 0.5 (Hair, Black, Babin, Anderson, & Tatham, 1998). All the factors loaded above 0.7 thus these values constitute of evidence of convergent validity. This data analysis demonstrates that measurements possessed an acceptable convergent validity. The composite reliability of the measurements must reach 0.6 or above . The results indicated that all the latent variables reached the standard or above.

Table-1: KMO Bartlett's test.

Kaiser–Mayer–Olkin measures of sampling adequacy	0.796
Bartlett's Test of Sphericity	
Approx. chi-square	4825.726
Df	213
Sig	0.000

Source: Prepared by the authors

Table-2: Factors loading and reliability analysis.

Scale	Items	Factor loading	Cronbach's α	AVE	CR
Knowledge sharing (KS)	Knowledge receiving	0.843	0.897	0.810	0.860
	Knowledge giving	0.816			
Organization culture (oc)	Trust	0.788	0.789	0.579	0.854
	Collaboration	0.776			
	Learning	0.734			
Innovation capability (IC)	Management innovation	0.823	0.842	0.565	0.849
	Product innovation	0.731			
	Process innovation	0.725			

Source: Prepared by the authors

Then reliability coefficient was also tested by using Cronbach alpha (α) in order to measure the reliability for the set of two or more constructs. According to the Cronbach alpha test, the total scales of reliability varies from 0.77 to 0.842, which exceed the threshold point of 0.7 introduced by Nunnally (1978). The value of 0.7 or greater is indicated that good scale of reliability (O'Leary-Kelly & Vokurka, 1998). Moreover, convergent and discriminant validities were measured using the average variance extracted. According to the Bagozzi, Baumgartner, and Youjae (1988) the basis test's criterion on each value of average variance extracted should be 0.5. All of the average variance extracted for measurements range 0.56–0.810 exceeds the threshold of 0.5 (Bagozzi & Yi, 1988) which indicate that study had adequate levels of convergent and discriminant validity

Hypothesis testing

The SEM model was employed to examine the relationship between constructs developed by study. Hence SEM analysis was performed by AMOS 24 version and analyses simultaneously goodness-of-fit indices.

Table-3: Model Fit.

Goodness of fit indices	Constructs
χ^2/degree of freedom	1.146
CFI (comparative fit index)	0.975
TLT (Tusker–Lewis fit index)	0.943
IFI (incremental fit index)	0.984
RMSEA (root mean square error)	0.23
GFI (goodness fit index)	0.920
SRMR (root mean square residual)	0.48

Source: Prepared by the authors

Using the SEM investigated that impact of organization culture ,knowledge sharing on innovation capability.

Table 4. KS Predicting IC and the Mediating Role of OC

Step	Path	*B*	SEB	***b***	CR	Probability	Inference
1	OC→KS	0.875	0.15	0.52	5.92	0.001	Positive relationship
2	KS→IC	0.803	0.11	0.63	5.05	0.000	Positive relationship
3	OC→IC	0.869	0.15	0.59	5.43	0.001	Positive relationship
4	OC→KS	0.45	0.13	0.25	2.34	0.011	Mediation effect supported
	KS→IC	0.58	0.11	0.64	5.11	0.002	
	OC→IC	0.42	0.16	0.43	3.59	0.001	

Source: Prepared by the authors

Notes: *B* is un-standardized beta; SE is the standard error of beta; ***b*** is standardized beta; CR is critical ratio; OC is organizational cultures; IC is innovation capabiliy and KS is knowledge sharing.

KS positively predicted IC and OC, and OC in turn positively predicted IC. These results for the first three steps of mediator analysis are reported in Table 2. In the full model (Figure 1), when K and OC were explanatory variables, the strength of KS predicting IC decreased compared to the first step. Therefore, it is concluded that OC partially mediated the relations of KS with IC.. The findings derived from the structural regression model had acceptable fit indices.

FINDINGS

This study examined the effect of KS on IC with the mediating effect of OC in Indian SME$_s$. Results suggest that the stronger the KS, the higher the IC. Strong knowledge sharing process paves the way for better culture creation in organization and in turn organization culture enhances IC , implying that KS impacts IC through OC.

Organizational culture leads to increased employee participation and consensus on strategic matters, which provide aligned perspectives between organizational and individual objectives (Peters & Waterman, 1982). Wider agreement on values, practices, mission and goals are crucial for a culture to lead to effectiveness and innovation. When the culture is widely shared and practiced, it provides a collaborative and free-standing environment making individual and organizational abilities useful. In a culture with both intensity and consensus, organizational members come to know and share a common set of beliefs and expectations that are consistently valued and followed across the institution. This consistency in values and their reinforcement contributes to IC.

Free flow of information on organizational values, beliefs, practices and goals enhances an environment of free

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communication in the organization. Interactions, discussions and arguments to reach consensus on issues tend to minimize differences and promote Knowledge sharing in the organization. The involvement of members in organizational customs and practices contributes to better professional relationship among all, irrespective of departments and levels, contributing to a healthier communication in the organization.

Better communication establishes a clear set of rules and regulations for all stakeholders of the organization. It leads to clear expectations that convey what is expected from them, what they can expect from the organization and how their performance will impact the organization. Free and open feedback enables them to realize their potential, to remove their shortcomings, and it provides them with guidelines to follow and often motivates for better performance. Open channels of communication often lead to new ideas, unconventional thinking and innovation. Employees understand what is needed for their organization's success, find opportunities and make improvements.. Effective OC stimulates teamwork and cooperation among organizational members. It provides the requisite knnowledge, structure and positive work environment need to deal with a wide range of issues, resulting in higher innovativation capability.

The findings of the research supported to claim that organization culture , knowledge sharing in SMEs have a positive and strong impact on innovation capability. The aspects of organization culture have similar effects on forming innovation capability. A SEM model divulges the innovation capability is directly and positively affectsthe organizational, production, process and marketing innovation efforts. The entire paths were significant at $p < 0.000$.. Thus hypothesis H1, H2, H3, are supported. These findings are very important because innovation capability is one of the most influential factors for developing the innovation activities within the firm. Knowledge sharing, motivation, and creative thinking would lead to defining clear and effective innovation strategy. An organization with a culture that nurtures innovation and organization supported by right people, the process will provide the path to create a broad diverse set of ideas, especially converting them into the profitable business concepts.

IMPLICATIONS

This study sheds light on the significance of organizational culture consisting collaboration, trust and learning and how culture helps to enhance knowledge sharing behaviour in organizational which will ultimately lead toward enhancing innovation capabilities. Organizations may envisage the potential opportunities and threats affecting people's orientation to learn that would likely to affect the organizational knowledge performance in long term. Further, the study results are helpful to research practitioners and academicians in identifying the antecedents of innovation capabilities specifically with reference to small and medium enterprises.

CONCLUSION

Execution of innovation in Indian SMEs by and large spotlight on assets, procedures and estimation of progress, i.e. the effectively quantifiable components. They frequently give substantially less regard for individuals arranged determinants of the way of life, learning sharing conduct of development, which are more hard to quantify, for example, qualities, practices and hierarchical atmosphere. Despite the fact that everything that alludes to qualities and practices of individuals and atmosphere in the work environment is more tricky and hard to controlhave the best capacity to shape the advancement arranged culture and make supportable upper hand.

organization culture might be a component positive to the improvement of innovation capability and knowledge sharing behavior. It is critical to fittingly shape the professional advancement in organizational culture from the perspective of intensity of every businesssince innovation capbility is essential component that decides the aggressive position in the market.

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ROLE OF INDIAN PRIVATE UNIVERSITIES IN MODERN HIGHER EDUCATION ECOSYSTEM

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ABSTRACT
University education has changed like never before by the integration of digital technology in all spheres of life. The technological disruption has also changed the way education was transferred and viewed. The rate of change is unprecedented and to keep pace with faster changes process of doing business has to match up. Private Universities are autonomous and flexible which make them agile enough to quickly adapt to new requirements in doing business, adopting technology for its advantages and realigning curriculums as per requirements of market and industry. How private universities are performing and meeting expectations of all stakeholders is necessary to examine to furthering their effectiveness with proper support from policies and stakeholder support.

Keywords: Private Universities, technological intervention, empowerment of learner, quality of faculty, relevance

INTRODUCTION

The trend indicates that Private Universities will play a crucial role in the increase in overall enrolment. Private universities hence have a major role in changing the higher education ecosystem.

Federation of Indian Chambers of Commerce and Industry (FICCI) has divided the education and time matrix into four categories namely Education 1.0 referring to the ancient Indian education system where India was a hub of educational institutes that had many universities with foreign students. Education 2.0 came in the British Era with colonial agenda and extreme rigidity in systems evolving due to the Industrial revolution to perform repetitive work. 3.0 is the recent time when reforms were slowly introduced and minor flexibilities were achieved. Exponential technologies such as artificial intelligence, automation, robotics, and the internet have bought us to Education 4.0 where student-centric learning, relevance, freedom from space constraint, new methods of evaluation and total flexibility in curriculums and durations are becoming the new convention and new normal. Private universities have a potential to respond to the new demands in the sector due to its characteristics of Complete Autonomy, Self-Financing from its own resource generation capacity and its compulsion to be governed by market needs.

Traditionally the Indian higher education sector has been controlled by the government and was not changed significantly the way it functioned. Only recently government is indicating reforms in higher education in keeping with the requirements of time and resource constraints but major decisions are yet to come. In the face of the altered role of higher education a major structural alternative was introduced in 2003 in India. Private universities were facilitated by the government to share its investment liabilities and leverage achievement of objectives like improve accessibility, operational freedom, resilience, relevance and sustained quality to be relevant for open global market; through State Private University Act by individuals or private trusts. Private universities are not –affiliating, self-sustaining and region bound in nature and are regulated by the University Grants Commission (Establishment of and Maintenance of Standards in Private Universities) Regulations, 2003. The setting up of Manipal Institute in 1993 in Bangalore marked the establishment of the first private university in the country in its nascent form.

Twelfth Plan (2013 onwards) suggested the following challenges and proposed several initiatives around six focus areas –

- Expansion - to augment capacity in existing institutions,
- Equity- creating targeted schemes for backward and minority communities,
- Excellence- building excellence through research and innovation, faculty development, and internationalization,
- Governance- enhancing institutional autonomy and transparency,
- Funding– increasing public and private funding and linking them to outcomes.
- Implementation and monitoring– improving coordination across ministries and agencies.

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After establishing NITI Ayog and abolishing Five Year Plan method of planning the new government has said in All India Survey of Higher Education (AISHE) Report 2018 that the vision for the higher education is based on three important aspects namely Quality, Autonomy, Research, and Innovation. This seems to indicate a clear shift of emphasis on modernization and relevance.

CURRENT SCENARIO

1. Impact of Privatization

There are more than 78% of colleges running in Private sector; aided and unaided taken together, and it caters to 67% of the total enrolment.

The number of universities and similar institutions listed on All India Survey for Higher Education (AISHE) portal has increased from 621 in 2010-11 to 799 in 2015-16, 864 in 2016-17. According to the 2017-18 AISHE report, the total number of universities has increased to 903. This increase is

Similarly, in the year 2013-14, 75 percent colleges were privately managed (60 percent private unaided and 15 percent private aided), increased to 78 percent (64.7 percent private unaided and 13.3 percent private aided) by the 2017-18 as per AISHE.

On the contrary, the proportion of government colleges decreased from 25 percent to 22 percent between 2013-14 and 2017-18. Hence, there has also been an increase in the enrolment of students in private colleges. Between 2013-14 and 2017-18 enrolment has increased from 65 percent to 67.3 percent.

There has been a decrease in the enrolment of students in government colleges, from 35 percent in 2013-14 to 32.7 percent in 2017-18. These trends show that there is an environment of acceptability for Private Higher Education.

But as per the education minister Shri Prakash Javadekar as quoted in AISHE 2018 Report that there is an increase in overall enrolment from 27.5 million in 2010-11 to 35.7 million in 2016-17 and improvement in Gross Enrolment Ratio (GER) which is a ratio of enrolment in higher education to population in the eligible age group (18-23 years) , from 19.4% in 2010-11 to 25.2% in 2016-17 which is a significant achievement. He expressed the hope that the GER ratio of 30% would be achieved by the year 2022.

Globally also, over the last 15 years, private higher education has been the fastest-growing sector of higher education. Government promotion of private providers in higher education and the growth of private higher education is much more significant in Asia than in other regions of the world. According to UNESCO the percentage of students enrolled in private higher education institution in Asia is Korea 81%, Japan 79%, Singapore 64 %, China 63%, Philippines 63%.

2. Relevance

On the other side According to All India Council for Technical Education (AICTE), out of the eight lakh graduate engineers from technical institutions in the country in 2017 more than 60 percent remain unemployed. The reason it gives is lack of exposure, suitability of curriculum, creditable entry process.

According to The Economics Time e-edition 25th November 2017, Out of nearly 5,000 management institutes across the country, about 200,000 students passed out in 2016-17. Just 47% of MBAs were placed, 4% less than the previous year, and at a five-year low. But the biggest reason behind the decline in job offers to MBAs is the outdated curriculum.

In Times Higher Education (THE) World University Rankings 2018, IISc is the top university in India. It is placed in the 251-300 groupings of the best universities in the world. It fell from the 201-250 grouping of the previous edition of the ranking. (After first 200 ranks the UK-based University ranking agency THE places universities in groups instead of assigning them individual ranks). According to the ranking survey, IISc slid largely due to drops in its research influence score and research income. While the Indian Institute of Technology (IIT) Bombay continues to be in the 351-400 band, IIT Delhi and IIT Kanpur have dropped by one grouping from 401-500 to 501-600.

According to a report in The Hindu, December 17, 2017, Delhi edition, 55% of Indian parents, were eyeing varsities overseas for both for undergraduate and postgraduate courses for their wards which is much higher than the global average of 41%. The reasons cited for sending their wards abroad, (globally) the parents surveyed done by The Hindu indicate that learning foreign languages, gaining international work experience and exposure to new ideas are the main motivations. Indian parents there is a widespread belief that the quality of foreign educational institutions, their faculty, and research opportunities, are vastly superior to what is on offer at home. Data from global agencies tell us that this trend of young people heading abroad to pursue a

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college education is the strongest in Asia. But lately, India has emerged as the torch-bearer of this trend. The total number of Indian students pursuing college education abroad has vaulted from 62,350 to 2.55 lakh between 2000 and 2016, as data from UNESCO-UIS showed. That is a moderate growth of 6% annually. But student migration from India has gathered steam in the last three years, even as that from other origin countries slowed. India received 47,576 foreign students as per AISHE Report 2016-17.

3. Research

Research is an area of importance and focuses globally. At this juncture, it is pertinent to develop research enablers and facilities in our higher education system. The current requirement is to create an encouraging environment for academic research and patenting in all spheres of higher learning. The government is committed to creating the right policy framework. Through NAAC (National Assessment and Accreditation Council), NIRF (National Institute Ranking Framework) and government-sponsored grants research is being promoted in higher education institutions. NAAC and NIRF have assigned higher weightage to Research and patents.

4. The Technological Intervention in Higher Education

Digital technology has intervened into all walks of life and all spheres of work and changed the dynamism of the educational ecosystem. The disruptive nature of the technology and its power to build efficiencies and conveniences has enabled rapid acceptability. Higher education is greatly adapting the technology for the befit of learner's ease and requirements of lifelong learning needs. Virtual classrooms, Mobile app-based learning, Massive Open Online Courses (MOOCs) are now way of life along with conventional teaching. New technology has also given rise to the need for updating curriculums at a pace of industry needs.

Learning Resources can be shared and accessed by learners beyond geographic locations. Students are empowered to choose their own pace and courses despite the locational or physical disabilities. Online examinations, instant evaluation, total objective transparency has revolutionized evaluation system for mass multi-locational examinations. Teaching methods and teaching outcomes can be shared for constant improvements.

Futuristic and visionary leaders and teachers are already engaged in redesigning their approach to be more responsive to technological interventions to accelerate expansion, accessibility, affordability, and manageability benefitting teachers, learners, and institutions.

METHODOLOGY

The data is collected from secondary sources. The websites of Private universities from the top 50 list of NIRF was visited. Another 15 randomly selected university sites were also visited. The websites contain Vision, Mission, activities and achievements of the universities. Data is also taken from AISHE Reports and MHRD Reports. The data is then compiled under headings.

The private universities have a definite role to play successfully because of the following –

THE ROLE OF PRIVATE UNIVERSITIES

1. Access

Access is a complicated issue in modern times and it is not only the availability of Higher Education close to home or colleges and eligible population ratio or availability of seats vs total applications. Now access is about the availability of a suitable or desired course in desired time duration, flexible timing and place. Lifelong learning, mid-career learning, and part-time learning are gaining importance and non-conventional learners are showing an upward trend in proportion to the conventional or traditional learner. Non-conventional learners are the learners who have delayed enrolment after undergraduate studies for gaining financial strength to go independent of education loan or financial aid or do not have conventional entry degrees or certificates, or seeking career advancement or switch fields. They need their own pace of learning and clear outcomes of learning.

Access also includes good quality shared learning resources, access to the student community and share views and access to faculty unbound by the limitation of space and time.

Accessible university management to answer their queries is also important for the positive perception of professionalism in managing the institution. Investment in the right technology enables a university for providing high-quality access and transparency to the learners.

2. Increased Employability

Learners looking for employment has to face the rigidity and of curriculums and structure of time-bound evaluation. Private universities by virtue of responsive decision-making process have the potential to be very flexible in upgrading and innovation curriculums, build flexibility in entry and exit and forge Industry academia linkages. Employability deficit rises due to an unresponsive attitude toward industry requirements and introducing short courses to augment similar but outdated skills. Practical short term skilling courses can deal with all types of un-employability. Private institutions have massive potential and a conscious willingness can enable self-sustainable professional courses.

Now globally ecosystem for start-ups is being boosted by government and employment seekers are turning into self-employed and employment generators. India took a leap in this regard. India has the 2nd largest start up ecosystem in the world; expected to witness Year on Year growth of 10-12% Among approximately 20,000 start-ups in India; around 4,750 of these are technology led start-ups. 1,400 new tech start-ups were born in 2016 alone; implying there are 3-4 tech start-ups born every day. The main drivers are incubators and co-working space. Private universities are joining hand and leading the boost by providing incubation and innovation centres. Many private universities are providing incubation facilities like BITS Pilani and Sri Sri University.

3. Empowerment of Learner

Empowerment of learner and learner autonomy is the key to enter into education 4.0. Learner empowerment entails the generous amount of flexibility in curriculum design as per needs and desire of learner, flexible entry and exit options, self-paced learning out of a classroom structure. Learner resources are also available within and from outsourced web facilities. Students can also be linked and form communities.

Private universities like Sikkim Manipal University and Symbiosis University are offering MOOCs. Many universities have adopted mobile learning and discussion groups, mobile notice boards and mobile attendance bringing ease and empowerment to palmtop.

4. Quality of Faculty

Quality of faculty is the most important factor for the success of embedding education 4.0 in institutions. The current system is moving heutagogy i.e. more personalized, peer to peer and continuous. Learning is likely to break free from our old mind sets in the coming years with repacked structures of delivery and evaluation. This is a bold and futuristic step aligned with global concepts and also responds to the needs of industrial ecosystems in a proactive manner. Teaching becomes an amalgamated skill where mentoring, enabling, global-connectivity, technology-fuelled teaching environment are also joined. Faculty continuously need to upgrade and have to opt for continuous learning themselves to establishes a blueprint for learning. Increased self-effort and pressure from student and institutions are mounting. Good faculty will form the main differential. Private universities are hiring top talents with attractive compensation and a promising career path to faculty.

5. Transparency

Digital processes and e-governance bring in required level of transparency for the stakeholder. Private University systems as regulated by UGC, State Government, and regulatory and accrediting bodies have clear guidelines on transparency in the admission process, evaluation process, recruitment Process, etc. The private universities are following the guidelines and also declaring statistics demanded by AISHE regularly. Some Private universities also have made their websites interactive and have clear directions for raising queries. Private universities are more transparent due to demands from the Government and market.

6. Scope for constructively engaging Stakeholders

A stakeholder is someone who has an interest in the university or who could be affected by either positively or negatively by the activities of the university, or who affect the university either positively or negatively. For long term success addressing stakeholder's needs is important. As in the literature, one can largely list stakeholder of a University as Students, Their Parents, Industry, Society, and Government. Balancing needs, wants and desires of stakeholders can lead to a positive image statement for the University but beyond that, Stakeholders are directly linked and their welfare or perception is affected directly be efficiency and commitment of university.

Entrepreneurial nature of Private University can increase efficiency, reduce student and employee turnover, boost returns through engaging stakeholder. Stakeholder engagement requires regular communication through various activities such as consultatio

n, collaboration, and seminars, etc. Private universities have a large scope for extending stakeholder benefits. Private universities have successfully made industry linkages and many universities are doing CSR activities around them in villages and communities.

7. Research

Private universities are publishing more papers in order to get a better global rating for research work in order to attract foreign students. A report in Times of India e-paper published on 7th June 2014 says that Private universities beat IITs in the number of research papers published. Private universities are also showing a keen interest in Government schemes for the promotion of research. Private universities have the potential to fill an academic research gap. National Institute of Ranking Framework and NAAC has higher ratings for research work by faculty and student. Private Universities are interested in improving their ranking for greater acceptance among all the stakeholders are focusing on academic research and their quality. The first position for filing patents goes to Amity University in 2014.

8. Leveraging the Options for Self Financing

Private universities are self-financed and self-sustained. This burden presents to the shareholders an opportunity for designed courses with flexibility in timing, place, and duration with premium pricing, penetration pricing, economy pricing or price skimming strategies. Private university course fee tends to be higher but a properly designed portfolio can take care of bottom lines. Also their innovation centres and labs present opportunities for Licensing, Patenting, Consultancy and resource sharing without red tape hassles add to opportunities for revenue generation. The entrepreneurial universities will have to act as academic capitalist and view itself as capitalist in public sector (Sheila Slaughter, Larry L. Leslie, 2001).

9. Market Sensitivity and Flexibility

The flexibility and promptness of private universities allow them to react to any market situations in time. This is attributed to the lean and faster decision-making matrix. Private universities have come forward to capitalize on market situations. Foreign universities are looking for access to India's booming higher-education market and looking for tie-ups. Many private universities have twinning programs with them. Amity University has successfully opened operations in America, Britain, China, Singapore, and the United Arab Emirates; They plan to open its latest foreign outpost, in Romania. Next, on its list are Australia, Germany, Brazil, and Japan, among others soon.

Bennett University, BML Munjal University, OP Jindal University, and Flame University have tied foreign universities around the world to extend the advantage to the students in the job market. Bennett's tie-up with Babson Global and Georgia Tech has made their academic learning more rigorous with a tailored, curated syllabus in various professional and technical courses.

10. Support to Government

Private Universities have introduced a paradigm shift in promising and significant contributions in terms of quality and quantity. The continuous rise in numbers of Private universities and the subsequent rise in enrolments they are providing visible support to government's role in expansion. Private universities will hand hold Government in having an ambitious target to increase enrolments up to 30% in 2020.

There is a clear shift in government policy and support for start-ups and entrepreneurial endeavours. Many private universities have started programs in Entrepreneurship and start-ups to support job creation rather than overloading the job market. They are supporting the start-up ventures through incubation centres and tying up for financial support with donors and banks. Lovely Professional University has successful Green Start-ups in its Agriculture Department approved by Indian Council of Agriculture Research (ICAR). This is one of many successful start-up bedrocks supported by Private University.

Indian Government during Five Year plans could not spend more than 2% of GDP at an average. Maximum spend was a little more than 3 percent which happened only twice. Government expenditure on welfare takes priority on the likes of food subsidies, agriculture subsidies, and medical services, etc. Private equity is bringing in support of the government's budget for education.

11. Social Responsibility

Private universities are also actively taking their social functions earnestly. Many universities like Azim Premji University have clearly stated their social orientation in their Vision and Mission statements. They are supporting the government vision of social equality, justice, and upliftment through education and social research data.

12. The paucity of Quality of Faculty

Shortage of faculty is a pressing problem in Indian Universities and higher education sector as such. While the new central universities in Haryana, Gujarat, Odisha, Bihar, Rajasthan, Tamil Nadu, and Jammu & Kashmir are operating with around 52% of the sanctioned faculty strength, Allahabad University and Delhi University, have vacancies of 64.44% and 47.7%, respectively. Overall, the total vacancies in new central universities are nearly 48%; the older universities are better off with 33% vacancies. These Universities are managing through a large number of poorly paid Ad hoc faculty.

Few hardworking and bright people take to academia as most Indian universities are not inspiring them to make a career choice with them resulting in a dwindled supply of quality faculty. Secondly, the decision-making process and its rigidity pose problems in recruitment. Thirdly good faculty has a large financial burden to the universities.

Private universities are recruiting from industry and encouraging their own students to take up teaching in the university. MOOC and online courses reduce faculty requirements. Profit sharing is another way private universities are leveraging the liability of payments to faculty. But private universities are able to fill the seats majorly by the merit of their faculty hence quality enhancement of faculty is necessary for long term survival for them. They are ready to pay as per the market to academic leaders or technically well-equipped professionals.

Though there is a lot to be done to improve faculty quality and availability of good faculty, private universities are necessitating the raising the bar higher.

CONCLUSION

Private Universities have a promising role to play in nation building. The Private University Act 2003 gives them resilience in decision making which is a strong tool to make the organization sustainable for the long term. There are challenges also and policymakers must address them. A lot still has to be done in terms of quality as in NIRF list of top ten universities only one private university is listed. Below that till 50% score are another four private universities. Most of the private universities have scored less than 50% irrespective of their ranks. This speaks volume in terms of the scope of quality improvement. Yet the journey of 16 years is commendable and cannot in any way be looked as lesser in significance. Private universities are there to stay and grow and they also need hand holding from all stakeholders and policymakers in order to be meaningful support to the Government and nation.

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AN OVERVIEW OF SYNTACTIC ERRORS COMMITTED BY SECOND LANGUAGE LEARNERS

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ABSTRACT
The study is a brief discussion about the cause, scope and nature of the syntactic deviations made by L2 learners getting their education at junior level in the schools affiliated to UP Board of Education and CBSE in Uttar Pradesh. Since the area of syntactic deviations is very broad, the researcher, in the present study, has specified the deviations for the analysis in the certain areas such as tenses, modals, auxiliaries, S-V agreement and verb forms. The study identifies, describes and analyzes the syntactic deviations found from the written scripts of L2 learners. The result of the study reflects that the students belonging to UP Board of Education struggle more in using appropriate syntactic structures to a given context than that of CBSE and deviates from the norms of the appropriate sentence constructions.

Keywords: Second language (L2), Central Board of Secondary Education (CBSE), English Language Teaching (ELT), First Language or mother tongue (L1), Transitional competence (TC), Syntactic Errors, Second Language Acquisition (SLA) etc.

INTRODUCTION
In SLA (Second Language Acquisition), the term ‘syntactic error’ has been given various names such as approximative system (Nemser, 1971), idiosyncratic dialect (Corder, 1971), Interlanguage (Selinker, 1972 in Freeman & Long, 1991: 60). Syntactic errors reflect the linguistic system constructed by L2 learners for themselves, which are different from the linguistic system of learners’ mother tongue (MT) as well as the target language (L2). According to William Nemser, (1974: 55) an approximative system is ‘the deviant linguistic system actually employed by the learner attempting to utilize the target language. Such approximative systems vary in character in accordance with proficiency level; variation is also introduced by learning experience (including exposure to a target language script system), communication function, personal learning characteristics etc.’

For many decades, syntactic errors or deviations were supposed to be checked and corrected at very beginning in L2 class but with the publication of Pit Corder’s article titled “The Significance of Learners’ Errors” (1967), educationists’ view towards errors or syntactic deviations got changed. In this article, Corder presents a different point of view that errors are not something to be eradicated; these are productive outcome of language learning and they can tell something about the learning process and it is ‘errors’ which provide important evidence to those who attempt to describe his knowledge of the language at any point in its development (Corder, 1974: 23). The teachers, therefore, can use the learner’s errors or syntactic deviations as tool to diagnose their students through interpreting and analyzing the errors as they speak a lot about the learner’s competence in L2, which may be helpful for the teachers in modifying their lesson according to their learners. James defines EA as ‘the study of linguistic ignorance, the investigation of what people do not know and how they attempt to deal with their ignorance’ (James, 2001: 62). Therefore, EA provides insights into the strategies employed in second language acquisition and the process of language learning.

THE RESPONDENTS
Since the study is concerned with syntactic deviations made by the students of CBSE as well as UP Board of Education at junior level in Uttar Pradesh, the respondents of the present study are supposed to be students getting their schooling in 6^{th}, 7^{th} and 8^{th} standard. The researcher categorizes them into two groups. (A) the students getting their schooling in 6^{th}, 7^{th} and 8^{th} standards from UP Board (B) the students getting their schooling in 6^{th}, 7^{th} and 8^{th} standards from CBSC. The respondents or participants either from CBSE or UP Board of Education are generally native speakers of Hindi. Sometimes, the students belonging to the schools affiliated to CBSE may be of different language and cultural background. The respondents have already completed 5 to 7 years of their schooling and are supposed be familiar enough with English at the level of communication; written or spoken.

OBJECTIVES OF THE STUDY
The objectives of the study have been mentioned as follows:
1. To indentify the syntactic deviations made by the two groups of learners (students from CBSE as well as UP Board of Education) separately.
2. To categories the deviations or errors into different groups according to their nature.

3. To prepare spread-sheets and to put the errors in the sheets according to their nature for the purpose of analysis.
4. To represent the deviations statistically to compare the learners from CBSE as well as UP Board of Education so that the researcher may asses their competence in the use of appropriate sentences in a given context.
5. To make analysis of the syntactic deviations according to the model or framework chosen by the researcher.
6. To suggest a set of guidelines and solutions regarding L2 teaching-learning process so that teachers may bring some modifications in dealing with their L2 class.

THE HYPOTHESES

The researcher hypothesizes that Students getting their schooling in CBSE schools make less errors than that of UP Board of Education i.e. the students of CBSE Board are at better level of transitional competence in the usage of English in a given context.

DATA COLLECTION PROCEDURE

The researcher, for the study, has used questionnaire cum written test as tool for the collection of the data. The questionnaire is divided into two parts. The first part of the questionnaire deals with some questions seeking general information about the respondent such as name, college, gender and class etc.

The second part of the questionnaire comprises a set of five descriptive questions. All these questions are open-ended and context-specific in nature. To answer these questions, the students or learners require writing paragraphs that imply use of appropriate tense, verb forms, auxiliaries, modals and S-V agreement in a specific context. The respondents have freedom to develop alternative means of expression and to avoid the structures in which they feel difficulty. Using such questions, the researcher, however, made an attempt to restrict students to respond in particular areas to meet the requirement of the data. Each section of the questionnaire used for collecting data has been described briefly as follows:

In question 1, respondents were asked to write about their daily routine in their school to know whether the students or learners are familiar with the structures used in the context or not i.e. the question demands the respondents to provide information using the present imperfect tense.

In question 2, respondents were asked to write about their aim after completion of their education. The question requires the respondents to use modals that are implied to represent future time.

In question 3, the researcher has given a picture to describe it. The respondents were given the picture to describe to know whether they are familiar with the progressive aspects of present imperfect tense or not because the question demands the respondents to use structures with progressive aspect of present imperfect tense.

In question 4, four topics; *A Visit to a Zoo, Mother Teresa, An Interesting Cricket Match, Your Memorable Day,* have been given to the respondents to write a short note on any one of them. The response to the question demands the use of past simple tense i.e. the main objective the question is to find out how competent the students are in the use past simple tense.

In question 5, the respondents have been asked to write an application to their principal to get four days leave so that the respondents may use different types of grammatical structures in different tenses.

RESEARCH FRAMEWORK

To interpret the data the researcher has taken the model given by J. C. Richard titled 'A Non-Contrastive Approach to Error Analysis'. J. C. Richard used the model to interpret syntactic deviations in English produced by speakers of Japanese, Chinese, Burmese, French, Czech, Polish, Tagalog, Maori, Maltese, and the major Indian and West African languages. Dulay & Burt (1974) also used a different model from J. C. Richard to interpret them in his article titled "You cannot Learn without Goofing".

INTERPRETATION OF THE DATA

The researcher visited six intermediate colleges affiliated to CBSE and UP Board of Education and collected 180 test scripts; 90 test scripts from CBSE Board of Education and rest of the 90 test scripts from UP Board of Education. The researcher selected only those sentences from the test scripts to make interpretation, which did not confirm to the norms in the use of the tenses, auxiliaries, modals, S-V agreement and verb forms in English language. As the study is only concerned with the deviations made in the five areas mentioned above, the

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researcher does not take the syntactic deviations for interpretation in the use of adverbial phrase, prepositions, articles, noun phrase and pronouns etc.

The researcher examined 3600 sentences from each board of education; CBSE as well as UP Board of Education and found that respondents from CBSE used 3098 correct sentences out of 3600 sentences observed from their test scripts while they deviated in 502 sentences. As already stated, the researcher also examined 3600 sentences from the test scripts of the respondents belonged to UP Board of Education and found that respondents used 1945 correct sentences out of 3600 sentences observed from their scripts while they got deviated in 1655 sentences. The different data collected through observation and testing students from the two boards of education have been also presented in tabulated form as follows:

Q.N.	No. Of Total Sentences	No. Of Correct Sentences in CBSE	No. Of Correct Sentences in UP	No. Of Deviated Sentences in CBSE	No. Of Deviated Sentences in UP
Q-1	850	720	440	130	410
Q-2	900	788	470	112	430
Q-3	560	458	315	102	245
Q-4	640	552	325	88	315
Q-5	650	580	395	70	255
Total	3600	3098	1945	502	1655

Table-1.1

In the table mentioned above, the researcher has presented miscellaneous information; question number, number of total sentences, number of correct sentences in CBSE, number of correct sentences in the UP Board of Education, number of deviated sentences in CBSE and number of deviated sentences in the UP Board of Education in all of the five questions given in the questionnaire. The set of information reflects that students or learners belonging to CBSE deviated in only 502 sentences out of 3600 sentences from their norms i.e. the learners used only 13.94% deviated sentences in their response, while students belonged to UP Board of Education deviated in 1655 sentences out of 3600 i.e. the learners used 45.97% deviant sentences in their response. The researcher also analyze the nature of the deviant sentences using the framework taken for the present study and found that the learners belonged to both groups deviated because of developmental errors or intralingual errors i.e. the deviations took place because of interference within the language itself. For example, we can go through some of the deviant sentences mentioned as follows:

- I **am come** school at 8:00 o’ clock in morning.

(For ‘come’)

- I **am get** ready to my school.

(For ‘get’)

- I **am attend** the assembly at 9am.

(For ‘attend’)

- I **am attend** the class.

(For ‘attend’)

- I **am study** at 2 o’clock to 3o’clock.

(For ‘study’)

In the sentences mentioned above, the learners used auxiliary “am” with first form of lexical verb where only first form of lexical verbs are required. Learners therefore used the auxiliary ‘am’ in an environment where it was not required. Learner used auxiliary ‘am’ as a marker of present imperfect tense. Such sentences can be seen as examples of ‘false concepts hypothesized’ (Richards 1974: 178). Such types of intralingual errors refer to the developmental errors which come from faulty comprehension of the distinctions of the target language rules.

In the response to the first question, the respondents are supposed to use ‘present simple tense’ as the question asks about their daily routine in school but respondents (mostly of the students belonged to UP Board of Education) deviated because of incorrect use of tenses or incorrect use of tense marker. In the observation, the researcher found such sentences mentioned as follows:

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- We **are studying** Physics which is one of favourite subject.

(For 'study')

- 3rd of S.st. which **has been** taught by ma'am.

(For 'is')

- In morning I **woke up**7:30am

(For 'wake up')

- We all **gathered** in the assembly for praying to God.

(For 'gather')

- Then we **goes** our homes.

(For 'go')

- I **does worship** in my temple.

(For 'worship')

- In assembly some student **speaks** thought.

(For 'speak')

- The teacher **explain** all things.

(For 'explain')

- Class teacher **take** attendance after that we are study.

(For 'take')

In the first two sentences, learners used 'present progressive and present perfect tense' where 'present simple' was required because given context demands it and in the next two sentences, learners used past tense in the place of present simple. In these structures, learners violated the rules in the use of different forms of the tenses. Such intralingual errors can be categorized as 'the ignorance of rule restrictions' (Richards 1974: 175).

In the next three sentences (sentence no-5, 6 and 7), learners used lexical verbs with "-s/-es" inflection, that is, learners treated the first person singular and third person plural as the third person singular (as he, she and it etc.) whereas there should be use of present form of lexical verbs. In all these sentences, learners deviated from the norm of subject-verb agreement because of their experiences of other structures in the target language. Learners here extended the use of third person "–s/-es" inflection and in a restricted environment. Such intralingual errors can be taken as examples of 'overgeneralization of target language rules' (Richards 1974: 174). In the last two sentences, learners used present form of lexical verb without "s/-es" inflection. Learners treated the subjects of these sentences as third person plural and used with neutral form of lexical verb. Such sentences can be considered as 'incomplete application of target language rules' (ibid: 177) because such sentences represent the degree of development of the rules required to produce acceptable utterances.

In some of the deviant sentences, learners omitted the use of auxiliaries "is" and "are". Learners produced unique type of sentence constructions without the use of auxiliaries or verb. Such sentences did not reflect the learners' native language structures and were not found in the L1 acquisition data of target language. These sentences had no connection to the forms of either L1 or L2 of the learners. These sentences can be considered as the perfect examples of Dulay and Burt's unique goofs (Dulay & Burt 1974: 115). For example, let's see the sentences given below.

- Then second period # Maths, 3rd Science, 4th Computer/ Art, 5th Hindi, 6th S.ST and 7th Sanskrit/G.K.

(For 'is')

- My Ist period # of science which my class teacher teaches.

(For 'is')

The analysis of the data reflects that respondents belonged to CBSE are more competent in the use of the language in a given context as the respondents from CBSE used 86.06% correct sentences while they deviated

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in only 13.97% sentences. The respondents from UP Board of Education used 54.03% correct sentences and deviated in 45.97% of the sentences. The study therefore proves the hypothesis true that *students getting their schooling in the schools affiliated to CBSE deviate from the norms of language less than that of UP Board of Education i.e. students of CBSE Board reach at better level of transitional competence in English.*

CONCLUSION

The researcher, in the present study, identified categorized and interpreted different types of developmental errors or intralingual errors such as *false concept hypothesis, ignorance of rule restrictions, overgeneralizations, incomplete application of rules and unique goofs.* These types of syntactic deviations indicate that the errors are not because of mother tongue but from within the language itself, which shows the developmental stage of the learners in L2. The result of the study shows that the learners belonged to UP Board are poor. They also find using appropriate sentence difficult in a given context, while construct grammatically correct sentences in many cases. There may be many reasons for such differences found between the performances of the two groups of the learners. One of the reasons may be teaching grammar exclusively. As teaching of grammar exclusively does not give contexts to the learners to use them appropriately, the method has always been questionable. In many cases, learners can perform well in grammar and constructing sentences if they are tested separately, but they find it difficult when they need to use them in real situations. This gap can be minimized through modifying teaching materials and methods and making it more functional in nature. L2 teaching-learning process, therefore, in UP Board of Education need to be reviewed and modified so that the learners may be competent enough in the usage of L2 instead of only knowing about it.

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MANUSCRIPT SUBMISSION

GUIDELINES FOR CONTRIBUTORS

1. Manuscripts should be submitted preferably through email and the research article / paper should preferably not exceed 8 – 10 pages in all.
2. Book review must contain the name of the author and the book reviewed, the place of publication and publisher, date of publication, number of pages and price.
3. Manuscripts should be typed in 12 font-size, Times New Roman, single spaced with 1" margin on a standard A4 size paper. Manuscripts should be organized in the following order: title, name(s) of author(s) and his/her (their) complete affiliation(s) including zip code(s), Abstract (not exceeding 350 words), Introduction, Main body of paper, Conclusion and References.
4. The title of the paper should be in capital letters, bold, size 16" and centered at the top of the first page. The author(s) and affiliations(s) should be centered, bold, size 14" and single-spaced, beginning from the second line below the title.

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1Author Designation, Department, Organization, City, email id

2Author Designation, Department, Organization, City, email id

3Author Designation, Department, Organization, City, email id

5. The abstract should summarize the context, content and conclusions of the paper in less than 350 words in 12 points italic Times New Roman. The abstract should have about five key words in alphabetical order separated by comma of 12 points italic Times New Roman.
6. Figures and tables should be centered, separately numbered, self explained. Please note that table titles must be above the table and sources of data should be mentioned below the table. The authors should ensure that tables and figures are referred to from the main text.

EXAMPLES OF REFERENCES

All references must be arranged first alphabetically and then it may be further sorted chronologically also.

- **Single author journal article:**

Fox, S. (1984). Empowerment as a catalyst for change: an example for the food industry. *Supply Chain Management*, *2*(3), 29–33.

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S. Neelamegham," Marketing in India, Cases and Reading, Vikas Publishing House Pvt. Ltd, III Edition, 2000.

- **Edited book having one editor:**

Raine, A. (Ed.). (2006). *Crime and schizophrenia: Causes and cures.* New York: Nova Science.

- **Edited book having more than one editor:**

Greenspan, E. L., & Rosenberg, M. (Eds.). (2009). *Martin's annual criminal code:Student edition 2010*. Aurora, ON: Canada Law Book.

- **Chapter in edited book having one editor:**

Bessley, M., & Wilson, P. (1984). Public policy and small firms in Britain. In Levicki, C. (Ed.), *Small Business Theory and Policy* (pp. 111–126). London: Croom Helm.

- **Chapter in edited book having more than one editor:**

Young, M. E., & Wasserman, E. A. (2005). Theories of learning. In K. Lamberts, & R. L. Goldstone (Eds.), *Handbook of cognition* (pp. 161-182). Thousand Oaks, CA: Sage.

- **Electronic sources should include the URL of the website at which they may be found, as shown:**

Sillick, T. J., & Schutte, N. S. (2006). Emotional intelligence and self-esteem mediate between perceived early parental love and adult happiness. *E-Journal of Applied Psychology*, *2*(2), 38-48. Retrieved from http://ojs.lib.swin.edu.au/index.php/ejap

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Central Bank of India (2005). *Income Recognition Norms Definition of NPA.* Retrieved August 10, 2005, from http://www.centralbankofindia.co.in/ home/index1.htm, viewed on

7. The submission implies that the work has not been published earlier elsewhere and is not under consideration to be published anywhere else if selected for publication in the journal of Indian Academicians and Researchers Association.

8. Decision of the Editorial Board regarding selection/rejection of the articles will be final.