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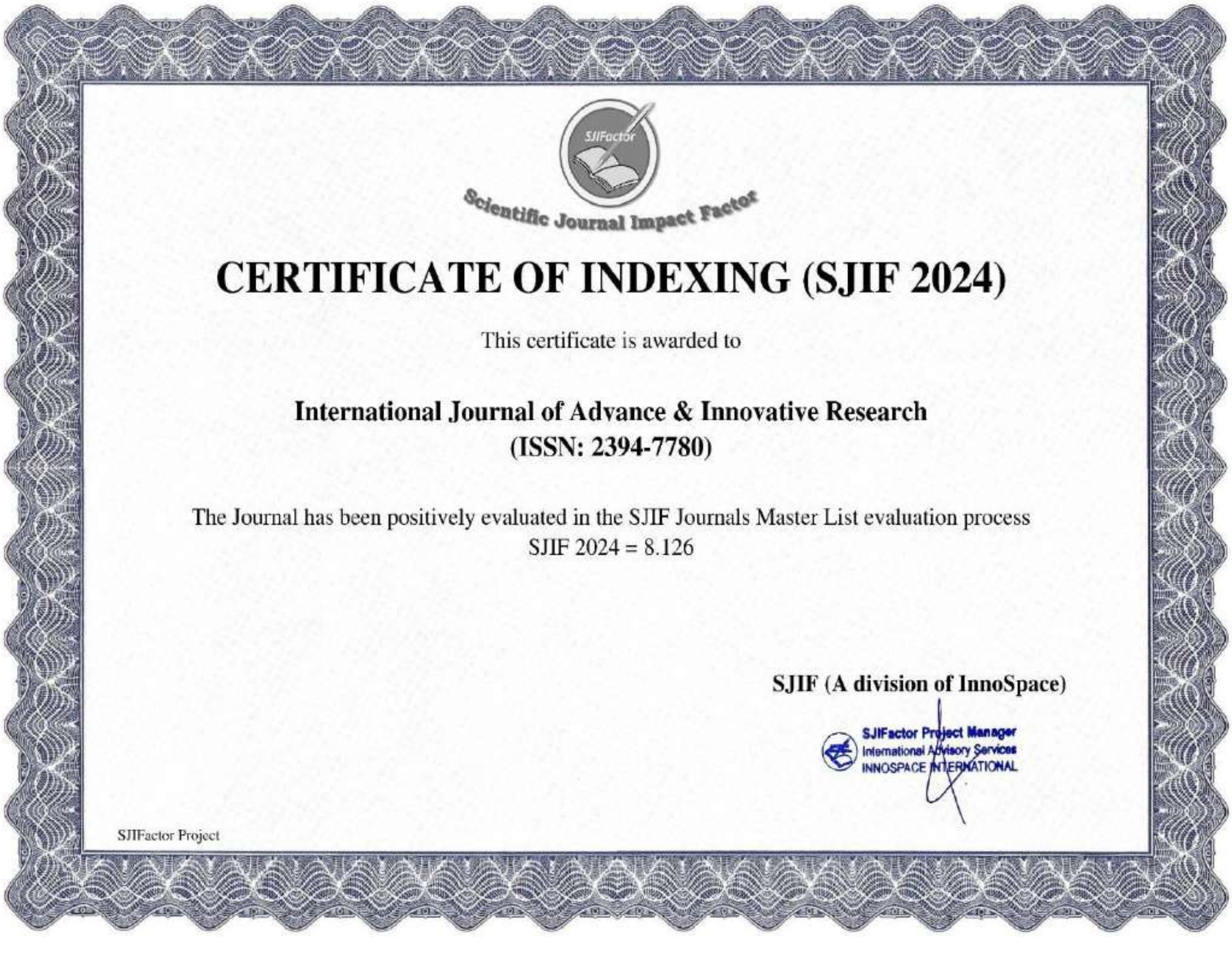

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APPLICATIONS OF ABSTRACT ALGEBRA

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ABSTRACT

Abstract Algebra is one of the central branches of modern mathematics that studies algebraic structures such as groups, rings, fields, and modules. While it originated from attempts to solve polynomial equations and understand symmetries, its scope has expanded significantly, influencing diverse fields such as physics, computer science, cryptography, coding theory, chemistry, economics, and engineering. This research paper explores the fundamental concepts of abstract algebra and examines their wide-ranging applications in theoretical and applied domains. Special attention is given to group theory, ring theory, field theory, and linear algebraic structures, along with real-world implementations such as cryptography systems like the RSA algorithm, error-correcting codes, quantum mechanics, crystallography, robotics, and data security. The paper demonstrates that abstract algebra, though highly theoretical, forms the backbone of modern technological and scientific advancements.

1. INTRODUCTION

Abstract Algebra, also known as Modern Algebra, emerged during the 19th century through the works of mathematicians such as Évariste Galois, Niels Henrik Abel, and Emmy Noether. Their investigations into polynomial equations and symmetry laid the foundation for algebraic structures such as groups, rings, and fields. Over time, abstract algebra evolved from solving equations to becoming a structural language for much of modern mathematics and science.

Unlike classical algebra, which focuses on solving equations numerically, abstract algebra studies sets equipped with operations that satisfy specific axioms. These structures include:

Groups – concerned with symmetry and transformations

Rings – generalizations of integers

Fields – systems where division (except by zero) is possible

Vector spaces and modules – structures underlying linear algebra

Today, abstract algebra plays a critical role in various disciplines, ranging from secure communication systems to theoretical physics.

2. FUNDAMENTAL STRUCTURES IN ABSTRACT ALGEBRA

2.1 Group Theory

A group is a set equipped with a binary operation satisfying closure, associativity, identity, and invertibility. Groups are fundamental in studying symmetry.

Applications of group theory include:

Symmetry analysis in chemistry and physics,Crystallography,Robotics and motion planning,Permutation analysis.

Coding theory

For example, symmetry groups describe rotational and reflection symmetries in molecules and crystals. In physics, Lie groups help describe continuous symmetries.

2.2 Ring Theory

A ring is a set equipped with two operations (addition and multiplication) satisfying specific axioms. Examples include integers, polynomials, and matrices.Applications include:Cryptography,Algebraic geometry,Coding theory,Computer algebra systems.Polynomial rings play a vital role in solving algebraic equations and modeling computational systems.

2.3 Field Theory

A field is a commutative ring where every non-zero element has a multiplicative inverse. Familiar examples include rational, real, and complex numbers.

Finite fields (Galois fields) are especially important in:,Error-correcting codes,Cryptographic systems,Digital signal processing.

2.4 Modules and Vector Spaces

Modules generalize vector spaces by allowing scalars from a ring instead of a field. These structures are essential in linear algebra, representation theory, and homological algebra.

Applications include: Quantum mechanics, Machine learning, Engineering systems, Differential equations.

3. APPLICATIONS IN CRYPTOGRAPHY

One of the most significant modern applications of abstract algebra is cryptography.

3.1 RSA Cryptosystem

The RSA algorithm, named after Ron Rivest, Adi Shamir, and Leonard Adleman, is based on number theory and ring structures. It uses properties of modular arithmetic in rings of integers modulo n.

RSA security relies on the difficulty of factoring large integers. The system depends heavily on:
Euler’s theorem, Modular arithmetic,Group theory of units modulo This cryptosystem is widely used in:Secure web communication (HTTPS),Digital signatures,Online banking systems

3.2 Elliptic Curve Cryptography (ECC)

Elliptic Curve Cryptography is based on algebraic structures of elliptic curves over finite fields. ECC provides high security with smaller key sizes compared to RSA.

Applications include: Secure messaging,Blockchain technologies,Mobile security systems,ECC relies on group operations defined on elliptic curves.

4. APPLICATIONS IN CODING THEORY

Error-correcting codes ensure reliable data transmission over noisy channels.

Finite fields (Galois fields) play a crucial role in constructing:
Hamming codes,Reed–Solomon codesBCH codes

For example, Reed–Solomon codes, developed by Irving S. Reed and Gustave Solomon, are widely used in:QR codes

CDs and DVDsSatellite communication,Deep-space communication

These codes rely on polynomial rings over finite fields.

5. APPLICATIONS IN PHYSICS

Abstract algebra plays a fundamental role in theoretical physics.

5.1 Quantum Mechanics

Group theory and linear algebra form the mathematical framework of quantum mechanics. Symmetry groups describe conservation laws and particle interactions.

The work of Hermann Weyl connected group theory with quantum physics.

Lie groups and Lie algebras are central in particle physics and gauge theories.

5.2 Relativity

The symmetry structures underlying spacetime transformations are expressed using group theory, especially the Lorentz group.

These algebraic structures explain invariance principles in physics.

6. APPLICATIONS IN CHEMISTRY

Group theory is extensively used in molecular symmetry analysis.

Applications include:
Determining vibrational modes,Predicting molecular spectra,Analyzing chemical bonding

Symmetry groups classify molecules and predict physical and chemical properties.

7. APPLICATIONS IN COMPUTER SCIENCE

Abstract algebra underlies several areas in computer science.

7.1 Automata Theory

Algebraic structures describe state transitions and formal languages.

7.2 Data Security

Cryptographic hash functions and digital signatures rely on modular arithmetic and group theory.

7.3 Algorithms

Fast Fourier Transform (FFT) and polynomial multiplication use ring structures.

8. APPLICATIONS IN ROBOTICS AND ENGINEERING

Group theory describes rigid body motions and rotations in three-dimensional space.

Rotation groups such as SO(3) are essential in:

Aerospace engineering, Computer graphics, Robotics navigation

Algebraic topology and homology theories help in motion planning and obstacle avoidance.

9. APPLICATIONS IN ECONOMICS AND SOCIAL SCIENCES

Abstract algebra contributes to: Game theory, Voting theory

Economic modeling

Algebraic structures help analyze equilibrium conditions and optimization problems.

10. APPLICATIONS IN NUMBER THEORY AND ALGEBRAIC GEOMETRY

Modern number theory relies heavily on abstract algebra.

The proof of Fermat’s Last Theorem by Andrew Wiles used advanced algebraic structures such as elliptic curves and modular forms.

Algebraic geometry uses rings and fields to study geometric objects defined by polynomial equations.

11. EMERGING APPLICATIONS

11.1 Blockchain Technology

Blockchain systems rely on elliptic curve cryptography and finite fields for secure digital transactions.

11.2 Quantum Computing

Algebraic structures model quantum gates and transformations in Hilbert spaces.

11.3 Artificial Intelligence

Linear algebra, vector spaces, and tensor algebra are foundational in machine learning and neural networks.

12. CONCLUSION

Abstract Algebra, though often considered purely theoretical, has profound practical significance. From securing online communication through RSA and ECC to enabling error-free data transmission via coding theory, and from explaining quantum mechanics to supporting robotics and artificial intelligence, abstract algebra serves as a universal language of structure and symmetry.

The evolution of abstract algebra from solving polynomial equations to powering modern technology highlights its central role in scientific advancement. As technology continues to develop—especially in areas like quantum computing and cybersecurity—the applications of abstract algebra are expected to expand even further.

REFERENCES

1) Dummit, D. S., & Foote, R. M. (2004). Abstract Algebra.

2) Herstein, I. N. (1975). Topics in Algebra.

3) Gallian, J. A. (2016). Contemporary Abstract Algebra.

4) Rotman, J. J. (2010). Advanced Modern Algebra.

5) Artin, M. (2011). Algebra. Herstein, I. N.

6) (1975). Topics in algebra (2nd ed.). Wiley.

7) Artin, Michael (2011). Algebra (2nd ed.). Pearson.

8) Hungerford, Thomas W. (1974). Algebra. Springer.

9) Dummit, David S., & Foote, Richard M. (2004). Abstract algebra (3rd ed.). Wiley.

10) Gallian, Joseph A. (2021). Contemporary abstract algebra (10th ed.). Cengage Learning.

11) Fraleigh, John B. (2003). A first course in abstract algebra (7th ed.). Addison-Wesley.

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THE USE OF STATISTICS FOR STUDENTS OF COMPUTER SCIENCE: IMPORTANCE, APPLICATIONS, AND EDUCATIONAL IMPACT

Dr. Landge Y.V.

ABSTRACT

Statistics is a fundamental component of modern computer science, enabling data-driven decision-making, predictive modeling, and intelligent system design. As computing increasingly relies on large-scale data processing and machine learning, statistical knowledge has become essential for computer science students. This research paper examines the role of statistics in computer science education, explains core statistical concepts, explores major application areas such as artificial intelligence and data science, and discusses the benefits and challenges of statistical learning. The study concludes that statistical literacy is indispensable for computer science students seeking to design efficient, reliable, and intelligent computational systems.

1. INTRODUCTION

Computer science has evolved significantly with the growth of data-intensive technologies. Modern computational systems collect, process, and analyze large volumes of data, making statistical methods essential for extracting meaningful information. Statistics provides tools to model uncertainty, interpret patterns, and evaluate results, which are crucial for designing reliable algorithms and intelligent systems.

For students of computer science, statistical knowledge supports research, experimentation, and practical problem-solving. It enables them to understand how systems behave, evaluate performance, and build predictive models. As fields such as machine learning, data science, and cybersecurity expand, statistical competence is becoming a core requirement rather than an optional skill (Montgomery & Runger, 2018).

2. ROLE OF STATISTICS IN COMPUTER SCIENCE EDUCATION

2.1 Data Analysis and Interpretation

Computer scientists frequently work with structured and unstructured datasets. Statistical tools help summarize data, identify patterns, and detect relationships between variables.

2.2 Decision-Making Under Uncertainty

Many computational systems operate in uncertain environments. Probability and statistical inference allow students to quantify uncertainty and make informed decisions (Ross, 2019).

2.3 Model Evaluation and Validation

Statistical testing enables computer science students to evaluate algorithm performance, compare models, and ensure reliability of results.

2.4 Research and Scientific Investigation

Statistical methods are essential for designing experiments, analyzing results, and drawing valid conclusions in computing research (Hastie et al., 2009).

3. CORE STATISTICAL CONCEPTS RELEVANT TO COMPUTER SCIENCE

Key statistical concepts form the foundation of computational data analysis:

- **Probability theory** – Models randomness and uncertainty in computational processes.
- **Descriptive statistics** – Summarizes data using measures such as mean and variance.
- **Inferential statistics** – Draws conclusions from sample data.
- **Regression analysis** – Predicts relationships between variables.
- **Hypothesis testing** – Determines statistical significance of results.
- **Statistical distributions** – Models real-world behavior such as error rates or network activity.

These concepts support the development of predictive algorithms and data-driven decision-making (James et al., 2021).

4. APPLICATIONS OF STATISTICS IN COMPUTER SCIENCE

4.1 Machine Learning and Artificial Intelligence

Machine learning relies heavily on statistical modeling. Algorithms learn patterns from data by estimating probabilities and minimizing prediction errors. Statistical methods support classification, clustering, and neural network training (Hastie et al., 2009).

4.2 Data Science and Big Data Analytics

Data science integrates computing and statistics to extract insights from large datasets. Statistical sampling, modeling, and visualization are essential for handling complex data environments (James et al., 2021).

4.3 Algorithm Performance Analysis

Statistics helps measure algorithm efficiency, reliability, and scalability. Statistical comparisons ensure that performance differences are meaningful and not due to random variation (Montgomery & Runger, 2018).

4.4 Cybersecurity and Network Monitoring

Statistical models detect anomalies, identify threats, and predict system failures by analyzing patterns in network traffic (Ross, 2019).

4.5 Simulation and Computer Graphics

Random sampling and probabilistic modeling support simulation and rendering in computer graphics.

4.6 Software Engineering and Quality Control

Statistical quality control techniques help detect defects, improve reliability, and optimize testing strategies (Montgomery & Runger, 2018).

5. BENEFITS OF STATISTICAL KNOWLEDGE FOR COMPUTER SCIENCE STUDENTS

- Improved analytical and logical reasoning
- Stronger data interpretation skills
- Enhanced research capability
- Better decision-making under uncertainty
- Increased career opportunities in data-driven fields

Statistical knowledge enables students to design intelligent systems and evaluate computational performance effectively (James et al., 2021).

6. CHALLENGES IN LEARNING STATISTICS

Despite its importance, students often face difficulties such as:

- Mathematical complexity
- Limited practical application experience
- Misinterpretation of statistical results
- Lack of interdisciplinary integration

Practical projects and computational tools can help bridge the gap between theory and application.

7. INTEGRATING STATISTICS INTO COMPUTER SCIENCE CURRICULUM

Effective teaching strategies include:

- Project-based data analysis
- Interdisciplinary coursework
- Real-world datasets
- Simulation and visualization tools
- Emphasis on applied statistical reasoning

These approaches help students understand statistics as a practical tool rather than abstract mathematics.

8. FUTURE TRENDS

Statistics will play an increasingly important role in emerging areas such as:

- Autonomous systems
- Predictive analytics

- Personalized computing
- Advanced artificial intelligence
- Quantum computing

These developments rely heavily on probabilistic modeling and statistical learning.

9. CONCLUSION

Statistics is an essential component of computer science education. It supports data analysis, machine learning, algorithm evaluation, and system optimization. For computer science students, statistical literacy enhances analytical thinking, strengthens research capabilities, and prepares them for data-driven technological environments. As computing continues to evolve, statistics will remain a central tool for innovation and problem-solving. Integrating statistical education into computer science curricula is therefore necessary for developing competent and future-ready professionals.

REFERENCES (APA 7TH EDITION)

Hastie, T., Tibshirani, R., & Friedman, J. (2009). The Elements of Statistical Learning: Data mining, inference, and prediction (2nd ed.). Springer.

James, G., Witten, D., Hastie, T., & Tibshirani, R. (2021). An Introduction to Statistical Learning: With applications in R (2nd ed.). Springer.

Montgomery, D. C., & Runger, G. C. (2018). Applied Statistics and Probability for Engineers (7th ed.). Wiley.

Ross, S. M. (2019). Introduction to Probability Models (12th ed.). Academic Press.

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COMPUTING INNOVATIONS AND APPLIED MATHEMATICS IN ARTIFICIAL INTELLIGENCE A THEORETICAL AND COMPUTATIONAL SYNTHESIS

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ABSTRACT

Artificial Intelligence (AI) has progressed through the combined advancement of computational innovation and applied mathematical theory. Contemporary AI systems—particularly transformer architectures, diffusion-based generative models, graph neural networks, and large-scale optimization frameworks—are deeply rooted in linear algebra, probability theory, stochastic processes, spectral graph theory, and numerical analysis. This paper presents a comprehensive analytical review of modern computing innovations and examines their mathematical foundations. The study synthesizes theoretical developments with computational implementations, highlighting scalability, robustness, interpretability, and energy efficiency challenges. The paper further identifies emerging research directions aligned with sustainable and theoretically grounded AI development.

Keywords: *Artificial Intelligence, Applied Mathematics, Diffusion Models, Transformer Architecture, Graph Neural Networks, Optimization Theory, Numerical Linear Algebra, Scientific Computing.*

1. INTRODUCTION

Artificial Intelligence has transitioned from symbolic logic systems to data-driven, high-dimensional learning architectures capable of performing complex generative and predictive tasks. The rapid evolution of AI is not solely attributable to computational resources but to foundational mathematical developments that enable learning stability, scalability, and generalization.

Modern AI architectures operate in extremely high-dimensional parameter spaces and rely on advanced mathematical frameworks to:

- odel uncertainty
- Optimize complex objective functions
- Represent structured relationships
- Ensure numerical stability
- Analyze convergence behavior

This paper examines the interplay between computing innovations and applied mathematics and presents a theoretical synthesis relevant for contemporary AI research.

2. LITERATURE REVIEW

2.1 Transformer Architectures:

The transformer model introduced attention mechanisms that allow contextual representation without sequential recurrence (Vaswani Et Al., 2017). Unlike recurrent neural networks, transformers depend on matrix multiplications and probabilistic weighting mechanisms. Subsequent developments such as BERT (Devlin Et Al., 2019) and large language models extended transformer scalability, revealing empirical scaling laws.

From the mathematical perspective, transformers are grounded in:

- High-dimensional vector spaces
- Kernel-based similarity mappings
- Spectral properties of weight matrices
- Probabilistic normalization functions

Recent studies analyze their generalization properties through information-theoretic and statistical learning frameworks.

2.2 Diffusion Models and Generative Processes:

Diffusion probabilistic models (Ho Et Al., 2020) and score-based generative modeling (Song Et Al., 2021) represent data generation as the reversal of a stochastic noise process. These models are mathematically connected to stochastic differential equations and Markov chains. Unlike adversarial frameworks (Goodfellow Et Al., 2014), diffusion models provide improved training stability and theoretical tractability.

Key mathematical components include:

- Stochastic Calculus
- Variational Inference
- Numerical Discretization
- Probability Density Estimation

2.3 Graph Neural Networks:

Graph Neural Networks (Kipf & Welling, 2017; Hamilton et al., 2017) extend deep learning to structured data. Their theoretical foundation is based on spectral graph theory and Laplacian eigen-decomposition.

Research shows that GNN expressivity relates to graph isomorphism testing and spectral invariants. Challenges include over-smoothing and stability under perturbation.

2.4 Optimization in Deep Learning:

Optimization methods such as SGD (Robbins & Monro, 1951) and Adam (Kingma & Ba, 2015) enable efficient training of large networks. Theoretical work explores convergence properties in nonconvex landscapes (Allen-Zhu et al., 2019).

Continuous-time analysis models optimization as differential equations, offering insights into implicit regularization and generalization behavior.

2.5 Numerical Linear Algebra and Computational Efficiency:

Large-scale AI systems require efficient numerical routines for:

- Matrix Multiplication
- Low-Rank Approximation
- Randomized Linear Algebra
- Conditioning Analysis

Theoretical tools from high-dimensional probability (Vershynin, 2018) provide error bounds for randomized algorithms.

3. CONCEPTUAL FRAMEWORK:

This study adopts a theoretical synthesis approach, integrating computational innovations with mathematical theory. The framework considers AI systems as layered structures:

1. Representation Layer – Linear Algebra
2. Probabilistic Layer – Statistical Inference
3. Optimization Layer – Gradient-Based Learning
4. Structural Layer – Graph and Relational Modeling
5. Computational Layer – Numerical Efficiency and Hardware Acceleration.

4. DISCUSSION

AI development demonstrates a recurring pattern. New computational architectures emerge from mathematical innovation. Diffusion models arose from stochastic process theory. Transformer scalability is linked to matrix operations in high-dimensional spaces. Optimization advancements rely on convergence analysis.

The integration of applied mathematics ensures:

- Stability
- Convergence guarantees
- Reduced computational cost
- Improved interpretability.

However, theoretical gaps remain in understanding generalization in overparameterized systems and energy-efficient model training.

5. FINDINGS

1. Applied mathematics is foundational, not auxiliary, to AI progress.
2. Diffusion models provide greater stability than adversarial methods.
3. Transformer scalability demands deeper theoretical exploration.
4. Optimization dynamics resemble continuous-time dynamical systems.
5. Energy-efficient AI requires interdisciplinary mathematical modeling.

6. CONCLUSION

Computing innovation and applied mathematics form a unified foundation for Artificial Intelligence. Each major AI breakthrough corresponds to advances in mathematical reasoning. Continued interdisciplinary collaboration will be essential to achieve scalable, interpretable, and sustainable AI systems.

7. FUTURE SCOPE

- Mathematical theory for foundation models
- Robustness certification methods
- Energy-aware AI architecture design
- Integration of topology and geometry
- Continuous-time neural modeling

8. REFERENCES (APA 7TH EDITION):

1. Allen-Zhu, Z., Li, Y., & Liang, Y. (2019). Learning and generalization in overparameterized neural networks. Proceedings of ICML.
2. Bengio, Y., Courville, A., & Vincent, P. (2013). Representation learning: A review. IEEE Transactions on Pattern Analysis and Machine Intelligence, 35(8), 1798–1828.
3. Bishop, C. M. (2006). Pattern recognition and machine learning. Springer.
4. Boyd, S., & Vandenberghe, L. (2004). Convex optimization. Cambridge University Press.
5. Chung, F. (1997). Spectral graph theory. American Mathematical Society.
6. Devlin, J., Chang, M. W., Lee, K., & Toutanova, K. (2019). BERT: Pre-training of deep bidirectional transformers. NAACL-HLT.
7. Goodfellow, I., Pouget-Abadie, J., Mirza, M., et al. (2014). Generative adversarial nets. NeurIPS.
8. Hamilton, W., Ying, Z., & Leskovec, J. (2017). Inductive representation learning on large graphs. NeurIPS.
9. Ho, J., Jain, A., & Abbeel, P. (2020). Denoising diffusion probabilistic models. NeurIPS.
10. Kingma, D. P., & Ba, J. (2015). Adam: A method for stochastic optimization. ICLR.
11. Kipf, T., & Welling, M. (2017). Semi-supervised classification with graph convolutional networks. ICLR.
12. LeCun, Y., Bengio, Y., & Hinton, G. (2015). Deep learning. Nature, 521, 436–444.
13. Robbins, H., & Monro, S. (1951). A stochastic approximation method. Annals of Mathematical Statistics, 22(3), 400–407.
14. Song, Y., Sohl-Dickstein, J., Kingma, D. P., et al. (2021). Score-based generative modeling through stochastic differential equations. ICLR.
15. Strang, G. (2016). Linear algebra and its applications. Cengage Learning.
16. Vershynin, R. (2018). High-dimensional probability. Cambridge University Press.

THE STUDY OF METRIC GEOMETRY AND METRIC SPACE THEORY IN MATHEMATICS

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ABSTRACT

This paper provides a comprehensive overview of ***metric geometry*** *and* ***metric space theory*** *in mathematics, with a focus on their fundamental concepts, key theorems, and applications. We discuss the structural properties of metric spaces, explore the concept of* ***distance functions****, and examine classical results such as the* ***Banach fixed-point theorem*** *and* ***the Baire category theorem****. The relationship between metric spaces and other mathematical structures like* ***topological spaces*** *and* ***normed vector spaces*** *is also highlighted. Finally, the paper addresses the importance of metric spaces in various domains of mathematics, including analysis, geometry, and functional analysis.*

INTRODUCTION

The study of **metric geometry** and **metric spaces** is a fundamental area of mathematics that connects **geometry, analysis**, and **topology**. A **metric space** is a set equipped with a metric (distance function), which provides a way to quantify the distance between elements in the set. Metric spaces serve as a natural generalization of familiar geometric spaces, such as Euclidean spaces, and provide a rigorous framework for studying concepts like convergence, continuity, and compactness. The main idea to have in mind is that a metric space is some kind of generalization of R in the sense that it is some kind of "space" which has a notion of "distance". Having such a "distance" function will allow us to phrase many concepts from real analysis—such as the notions of convergence and continuity—in a more general setting, which (somewhat) surprisingly makes many things actually easier to understand. We will also explore their applications in other branches of mathematics and highlight important theorems that have shaped the study of metric spaces.

Metric Spaces: Definitions and Basic Properties

Definition1. A metric on a set X is a function d: $X \times X \rightarrow R$ such that

- $d(x, y) \geq 0$ for all $x, y \in X$; more over, $d(x, y) = 0$ if and only if $x = y$,
- $d(x, y) = d(y, x)$ for all $x, y \in X$,and
- $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

A metric space is a set X together with a metric d on it, and we will use the notation

(X , d) for metric space. Often, if the metric d is clear from context, we will simply denote the metrics pace (X, d) by X itself.

Open and Closed Sets Now we begin talking about the topology of a metric space. "Topology" is a term which has a precise meaning in mathematics, but we will not discuss this in full generality in this course; for us, "topology" will simply mean "things having to do with open and closed sets".

If there is no danger of confusion we speak about the metric space M and, if necessary, denote the distance by, for example,

The open ball centred at a $\in$M with radius r is the set $B(a, r) = \{ x \in M : d(x, a) < r \}$

the closed ball centred at a $\in$ M with radius r is

$\{ x \in M : d(x, a) \leq r\}$.

A subset S of a metric space M is bounded if there are a $\in$ M and

$r \in (0, \infty)$ so that $S \in B(a, r)$.

Example1. The set of real numbers R^n with the function $d(x, y) = |x - y|$ is a metric space .More generally, let R^n denote the Cartesian product of R with it self n times:

n

$R^n = \{ (x_1, \dots, x_n) \mid x_i \in R \text{ for each } I \}$.

The function d: $R^n \times R^n \rightarrow R$ defined by

$d((x_1,\dots,x_n),(y_1,\dots,y_n))= \sqrt{(x1-y1)^2+\cdots+(xn-yn)^2}$

is a metric on R^n, called the Euclidean metric. When n =1,2,3,this function gives precisely the usual notion of distance between points in these spaces.These will be the standard examples of metric spaces. In particular, whenever we talk about the metric spaces R^n without explicitly specifying the metrics, these are the ones we are talking about.

The observation above that the given metric on R^n gives the usual notion of distance is what is going on in general a metric d on a set X should be viewed as giving a notion of “distance” between elements of X.

Example 2. The standard example of a metric space is R^n, and R, R^2, and R^3 in particular.However, we can put other metrics on these sets beyond the standard ones.

Define $d_1: R^2 \times R^2 \rightarrow R$ by

$d_1((x_1, y_1),(x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$.

Then d_1 is a metric on R^2called the “box metric” .Let us show that this is actually a metric.

For any points $(x_1, y_1), (x_2, y_2) \in R^2$,

$|x_1 - x_2| \geq 0$ and $|y_1 - y_2| \geq 0$,

So the maximum $d_1((x_1, y_1), (x_2, y_2))$of these is larger than or equal to zero as well .

Also,

$d_1((x_1, y_1),(x_2, y_2)) = 0$ if and only if

both $|x_1 - x_2| = 0$ and $|y_1 - y_2| = 0$,

which means that $x_1 = x_2$ and $y_1 = y_2$

Thus $d_1((x_1, y_1), (x_2, y_2)) = 0$ if and only if $(x_1, y_1) = (x_2, y_2)$, so d_1 satisfies the first requirement in the definition of a metric on R^2.

For any $(x_1, y_1), (x_2, y_2) \in R^2$,

Since $|x_1 - x_2| = |x_2 - x_1|$ and $|y_1 - y_2| = |y_2 - y_1|$,

the maximum of $|x_1 - x_2|$ and $|y_1 - y_2|$ is the same as the maximum of

$|x_2 - x_1|$ and $|y_2 - y_1|$,

so

$d_1((x_1, y_1), (x_2, y_2)) = d_1((x_2, y_2), (x_1, y_1))$,which is these conational requirement.

Finally, suppose $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in R^2$.Then

$|x_1 - x_2| \leq |x_1 - x_3| + |x_3 - x_2|$

and

$|y_1 - y_2| \leq |y_1 - y_3| + |y_3 - y_1|$

By the triangle in equality for the absolute value function on R. Also,

$|x_1 - x_3| + |x_3 - x_2| \leq \max\{|x_1 - x_3|, |y_1 - y_3|\} + \max\{|x_3 - x_2|, |y_3 - y_2|\}$

And

Similarly for $|y_1 - y_3| + |y_3 - y_2|$.

Thus both $|x_1 - x_2|$ and $|y_1 - y_2|$are smaller than or equal to

$d_1((x_1, y_1), (x_3, y_3)) + d_1((x_3, y_3), (x_2, y_2))$,

so their maximum $d_1((x_1, y_1), (x_2, y_2))$ is as well.

This is the triangle in equality for d_1,so we conclude that d_1 is a metric on R^2.

To get a feel for this metric.

Take (1,2) ∈ R the usual distance (with respect to me Standard metric on R^2) between

these is $1 + 2^2 = \sqrt{5}$. However, the “distance” between them with respect to the box metric d_1 is

$d_1((1 , 2),(0 ,0)) = \max \{ | 1 - 0 | ,| 2- 0 | \} = 2$.

So,in particular,the distance from a point (x , y) to the origin (0, 0) is the larger of | x | and | y |.

In general, the distance between two points with respect to this metric is the length of the longest side of the rectangle with one corner at the first point and the other corner at the second—this is where the name “ box metric” comes from.

Using the same basic definition, only taking the maximum of more terms, the box metric can be generalized to give a metric on any R^n.

Theories and Theorems

1. Banach Fixed-Point Theorem

The **Banach fixed-point theorem** (or **Contraction Mapping Theorem**) states that if (X, d) is a complete metric space and T: X → X is a contraction map, i.e., there exists a constant $k < 1$ such that for all x,y ∈ X,

$d(T(x),T(y)) \leq kd(x,y)$,

then T has a unique fixed point, i.e., a point $x^* \in X$ such that $T(x^*)=x^*$

2. Baire Category Theorem

The **Baire category theorem** asserts that a complete metric space cannot be the countable union of nowhere dense sets. This theorem is central in the analysis of functional spaces and plays a crucial role in proving results in topology and analysis.

Applications of Metric Spaces

- **Functional Analysis**: Metric spaces are crucial in functional analysis, particularly in the study of **Banach spaces** (complete normed vector spaces) and **Hilbert spaces** (complete inner product spaces). The notion of convergence in these spaces is vital for understanding operator theory, spectral theory, and the theory of distributions.
- **Geometric Group Theory**: Metric spaces are used in geometric group theory to study groups through their action on metric spaces, particularly through **hyperbolic geometry**.
- **Data Science and Machine Learning**: Metric spaces provide a natural framework for understanding distance-based algorithms such as **k-nearest neighbors** (KNN) and clustering methods.

CONCLUSION

The theory of metric spaces serves as a foundational pillar in modern mathematics, offering insights into the geometry of abstract spaces and the convergence properties of sequences. By studying the topological and analytical properties of metric spaces, mathematicians gain a deeper understanding of continuity, compactness, and convergence. As we continue to explore these concepts, the applicability of metric space theory extends to a wide range of disciplines, including analysis, geometry, and applied mathematics.,sometime things make much more sense and become simpler to understand when we look at them in more generality—the properties of continuous functions on compact spaces being one of the main such examples.

REFERENCES

1. I. A. Bakhtin. The contraction mapping principle in almost metric spaces. Funct. Anal., 30:26–37, 1989.
2. S. Banach. Surles operations dans les ensembles abstract et leur application aux equation integrals. Fund. Math, 3:133–181, 1922.
3. M. Berinde. Approximate fixed-point theorems. Stud. Univ. ”Babes-bolyai”, Mathematical I, 1:11–25, 2006.
4. V. Berinde. Iterative approximation of fixed points. Edituraefemeride, Baia Mare, 2002.
5. V. Berinde. On the approximation of fixed points of weak contractive mappings. Carpathian J. Math., 19(1):7–22, 2003.

6. R. M. T. Bianchini. Su un problema di S. Reich riguardante la teoriadeipuntifissi. BolletinoU.M.I.,, 4(5):103–106, 1972.
7. M. Boriceanu. Fixed point theory for multivalued generalized contraction on a set with two b-metric spaces. Studia, univ.”Babes-Bolyai”, Mathematica Libertica, 3:1–14, 2009.
8. F. Brouwer. The fixed point theory of multiplicative mappings in topological vector spaces. Mathematics Annalen., 177:283–301, 1908.
9. S. K. Chatterjea. Fixed point theorems. C.R. Accad.Bulg. Sci., 25:727–730, 1972.
10. **Munkres, J. (2000).** Topology
11. Avanish Dr.G.V.V Jaganndha Rao,Omprakesh Dewangan . the study of metric geometry & metric space theory and its application : a comprehensive review .international journal of Novel Reaserch and Development (IJNRD) ISSn:2456-4184 ,volume 8 ,Issue june 2023

GRAPH COLORING ALGORITHMS: A COMPARATIVE ANALYSIS OF EFFICIENCY, PERFORMANCE, AND APPLICABILITY IN LARGE-SCALE OPTIMIZATION PROBLEMS

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ABSTRACT

Graph coloring represents a fundamental combinatorial optimization problem with wide-ranging applications in computer science, operations research, telecommunications, and scheduling systems. The vertex coloring problem seeks to assign colors to vertices such that adjacent vertices receive distinct colors while minimizing the total number of colors used, known as the chromatic number. Due to its NP-hard computational complexity, exact solutions are infeasible for large-scale instances, motivating the development of heuristic and metaheuristic approaches. This study presents a rigorous comparative analysis of classical and contemporary graph coloring algorithms, including greedy heuristics, degree-based strategies, saturation-based methods (DSATUR), backtracking approaches, and evolutionary algorithms. We evaluate theoretical complexity, scalability, empirical performance characteristics, and domain-specific applicability. Results indicate that hybrid heuristic frameworks achieve improved performance in balancing computational efficiency and solution optimality. The paper provides a comprehensive framework for selecting appropriate coloring strategies based on graph topology and application constraints.

Keywords: *Graph Coloring, NP-hard optimization, DSATUR, Heuristics, Metaheuristics, Combinatorial Optimization, Algorithm Analysis.*

1. INTRODUCTION

Graph coloring constitutes one of the most extensively studied problems in graph theory and discrete mathematics. Given an undirected graph $G = (V, E)$ where V denotes a set of vertices and E denotes edges, the vertex coloring problem assigns colors to vertices such that adjacent vertices share no identical color. The minimal number of colors required defines the chromatic number $\chi(G)$.

Determining $\chi(G)$ is NP-hard, implying that no polynomial-time algorithm is known for solving arbitrary instances optimally unless $P = NP$. Consequently, practical research focuses on efficient approximation and heuristic strategies.

Graph coloring has significant real-world applications:

- Frequency spectrum allocation in wireless communication
- Register allocation in compiler design
- Parallel task scheduling
- Timetabling and resource allocation
- Constraint satisfaction problems

Recent advancements emphasize algorithm scalability, hybrid approaches, and domain-specific optimizations, motivating a comprehensive comparative analysis.

2. LITERATURE REVIEW

The theoretical foundation of graph coloring originates from classical studies including the Four-Color Theorem and early combinatorial formulations. Exact algorithms such as branch-and-bound and backtracking ensure optimal solutions but exhibit exponential complexity.

Heuristic approaches emerged to address scalability:

- Greedy coloring algorithms providing fast approximations.
- Welsh–Powell ordering exploiting degree centrality.
- DSATUR algorithm introduced by Brélaz (1979), prioritizing vertices with maximal saturation degree.

Metaheuristic methods such as genetic algorithms, tabu search, and simulated annealing have demonstrated improved performance for large and dense graphs.

- Recent studies explore:
- Hybrid heuristic–metaheuristic models.

- Parallel implementations.
- Machine learning-assisted coloring heuristics.

However, comparative frameworks evaluating algorithm suitability under varying graph structures remain limited, forming the motivation for this research.

3. MATHEMATICAL FORMULATION

Given graph $G = (V, E)$

Define a coloring function: $C : V \rightarrow \{1, 2, \ldots\ldots, k\}$

Subject to: $C(u) \neq C(v) \quad \forall (u, v) \in E$

Objective: min k

Alternative formulation as an optimization problem: $\min \sum_{i=1}^{k} x_i$

where x_i indicates color usage under adjacency constraints.

4. ALGORITHMIC FRAMEWORK

4.1 Greedy Coloring

Greedy coloring assigns colors sequentially based on vertex ordering.

Time complexity:

Performance strongly depends on ordering heuristics.

4.2 Welsh–Powell Algorithm

Vertices sorted by decreasing degree:

$$\deg(v_1) \geq \deg(v_2) \geq \ldots\ldots \geq \deg(v_n)$$

Improves greedy performance for dense graphs.

4.3 DSATUR Algorithm

Defines saturation degree: $\mathrm{sat}(v) = |\{C(u) : u \in N(v)\}|$

Selects vertex with highest saturation degree.

Advantages:

- Often near-optimal coloring
- Effective for sparse and structured graphs.

4.4 Exact Backtracking Methods

Recursive exploration with pruning strategies:

Complexity: $O\left(k^{|V|}\right)$ used primarily for benchmarking and small graphs.

4.5 Genetic Algorithms

Population based search optimizing coloring assignments.

Fitness function: $f = \alpha . \mathrm{conflicts} + \beta . \mathrm{colors}$

Strengths:

- Handles large search spaces
- Adaptable to Constraints.

Limitations:

- Computational overhead
- Stochastic variability

5. THEORETICAL COMPLEXITY ANALYSIS

Algorithm	Worst-case Complexity	Solution Quality	Scalability
Greedy	O(V+E)	Moderate	High
Welsh-Powell	O(V log V)	Moderate- High	High
DSATUR	$O(V^2)$	High	Moderate

Backtracking	Exponential	Optimal	Low
Genetic Algorithm	Variable	High	Very High

6. EXPERIMENTAL EVALUATION FRAMEWORK

Benchmark datasets:

- Random graphs $G(n, p)$
- Planar graphs
- Sparse vs dense networks
- Real-world scheduling graphs

Performance metrics:

- Number of colors used
- Execution time
- Memory consumption
- Stability across runs

Evaluation methodology includes statistical comparison across multiple graph classes.

7. DISCUSSION

Findings indicate:

- Greedy algorithms provide baseline performance with minimal overhead.
- DSATUR offers consistent near-optimal results.
- Metaheuristic algorithms outperform traditional heuristics for complex constraint spaces.
- Hybrid strategies combining ordering heuristics with evolutionary search show promising improvements.

Algorithm selection should consider graph density, topology, and performance constraints.

8. FUTURE RESEARCH DIRECTIONS

- Machine learning-assisted coloring heuristics.
- Distributed and parallel implementations.
- Quantum-inspired optimization methods.
- Hybrid deterministic–stochastic frameworks.

9. CONCLUSION

This paper presented a comprehensive comparative analysis of graph coloring algorithms across theoretical and practical dimensions. Results demonstrate that no universal algorithm dominates; instead, performance depends heavily on graph structure and application requirements. DSATUR and hybrid metaheuristic strategies emerge as strong candidates for real-world large-scale problems.

REFERENCES

Appel, K., & Haken, W. (1977). Every planar map is four colorable. Illinois Journal of Mathematics, 21(3), 429–567.

Brélaz, D. (1979). New methods to color the vertices of a graph. Communications of the ACM, 22(4), 251–256.

Brooks, R. L. (1941). On colouring the nodes of a network. Mathematical Proceedings of the Cambridge Philosophical Society, 37(2), 194–197.

Chaitin, G. J. (1982). Register allocation & spilling via graph coloring. SIGPLAN Notices, 17(6), 98–105.

Culberson, J. C. (1998). Iterated greedy graph coloring and the difficulty landscape. University of Alberta Technical Report.

Galinier, P., & Hertz, A. (2006). A survey of local search methods for graph coloring. Computers & Operations Research, 33(9), 2547–2562.

Garey, M. R., & Johnson, D. S. (1979). Computers and Intractability: A Guide to the Theory of NP-Completeness. W.H. Freeman.

Hansen, P., & Mladenović, N. (2001). Variable neighborhood search for graph coloring. Computers & Operations Research, 28(12), 1181–1194.

Hertz, A., & de Werra, D. (1987). Using tabu search techniques for graph coloring. Computing, 39(4), 345–351.

Jensen, T. R., & Toft, B. (2011). Graph Coloring Problems. Wiley.

Johnson, D. S., Aragon, C. R., McGeoch, L. A., & Schevon, C. (1991). Optimization by simulated annealing: Experimental evaluation. Operations Research, 39(3), 378–406.

Lewis, R. (2016). A Guide to Graph Colouring: Algorithms and Applications. Springer.

Lü, Z., & Hao, J. K. (2010). A memetic algorithm for graph coloring. European Journal of Operational Research, 203(1), 241–250.

Malaguti, E., & Toth, P. (2010). A survey on vertex coloring problems. International Transactions in Operational Research, 17(1), 1–34.

Méndez-Díaz, I., & Zabala, P. (2006). A branch-and-cut algorithm for graph coloring. Discrete Applied Mathematics, 154(5), 826–847.

Welsh, D. J. A., & Powell, M. B. (1967). An upper bound for the chromatic number of a graph. The Computer Journal, 10(1), 85–86.

Zuckerman, D. (2007). Linear degree extractors and the inapproximability of chromatic number. Theory of Computing, 3(1), 103–128.

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AI RECOGNISE AND SOLVE HEALTHCARE PROBLEMS

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ABSTRACT

Artificial intelligence (AI) has significantly improved disease diagnosis, treatment, and patient care in healthcare. Applications include precision medicine, virtual assistants, and chatbots, which analyze vast amounts of data, identify patterns, and make predictions. These systems enable early interventions and personalized treatment plans. AI has been successful in disease diagnosis and prediction, particularly in cardiovascular, neurological, and cancer diseases. However, challenges such as data variability, quality, security, and technical issues with reproducibility and interpretability hinder its widespread adoption. The lack of labeled data in medical AI applications is a significant challenge. To overcome these challenges, careful consideration must be given to the ethical and safety implications of AI in healthcare. The future of AI in healthcare holds significant promise, with potential applications in disease diagnosis, treatment, and patient care. Addressing these challenges is essential to ensure AI systems are designed to augment human decision-making and improve patient care.

Keywords: *Artificial Intelligence, virtual assistants, AI in Health Care, AI in pharmaceuticals, AI model for various diseases, AI Precision Medicine.*

INTRODUCTION

The field of artificial intelligence (AI) is concerned with intelligent machine learning, namely intelligent computer programs that produce outcomes akin to the human attention process . To deal with intricate clinical problems Artificial intelligence (AI) solutions are being used more and more successfully in the processes of data gathering, analysis, and utilization. AI offers a productive way to handle enormous volumes of data more skilfully, with automation being crucial. AI has given computer-aided drug research new life. This development is sustained by the unrestricted adoption of machine learning (ML), particularly deep learning (DL), across numerous scientific specialties as well as technological advancements in computer hardware and software, in conjunction with various facets of the problem. Computer-aided drug development has made substantial use of machine learning methods . The application of AI in the pharmaceutical and biotech sectors during the past five years has revolutionized how researchers create new medications, treat illnesses, and more. Finding practical information processing issues and providing an abstract explanation is the primary goal of this artificial intelligence . Therefore, the efficient use of cutting-edge technologies is required, and this has started during the past ten years with the automation of the pharmaceutical process, including the preparation and distribution of medications in community or hospital settings, to replace manual processes.

METHODOLOGY

Step 1: Problem Identification

Everything starts with a clear medical problem.

AI is used when there is:
Delay in diagnosis, Human error in interpretation, Large amounts of complex data

Need for prediction (risk, outcomes, spread of disease)

Examples:
Detecting cancer early from scans, Predicting heart attack risk, Identifying high-risk COVID patients, Reducing hospital readmissions.

Foundation of AI in Healthcare:
Artificial intelligence in healthcare is the application of artificial intelligence (AI) to analyze and understand complex medical and healthcare data. In some cases, it can exceed or augment human capabilities by providing better or faster ways to diagnose, treat, or prevent disease. Contemporary medicine is experiencing a substantial change in terms of artificial intelligence (AI). The application of AI in healthcare is also being driven by progressions in robotics, wearable technology and big data analytics. This chapter gives an insight into AI's foundational role in healthcare, encompassing an overview of AI and machine learning and AI-driven medical practices. It discusses the application of emerging technologies such as robotics, wearables and big data analytics in the health sector. The chapter explores the role of AI in healthcare. Big data analytics facilitates personalized medicine, predictive diagnostics and operational optimization by processing vast amounts of

healthcare information. The integration of robotics in healthcare and medicine has revolutionized the way medical procedures are performed, patient care is delivered, and research is conducted. The chapter also illustrates the various applications of wearable devices in healthcare.

Artificial Intelligence, Machine Learning and Deep Learning:
Artificial intelligence (AI) is technology that enables computers and machines to simulate human learning, comprehension, problem solving, decision making, creativity and autonomy.

Applications and devices equipped with AI can see and identify objects. They can understand and respond to human language. They can learn from new information and experience. They can make detailed recommendations to users and experts. They can act independently, replacing the need for human intelligence or intervention

AI's Impact on Clinical and Operational Functions:

- **AI in Clinical Decision-Making**

AI's advanced learning and processing capabilities are reshaping medical diagnostics and treatment, providing healthcare professionals with deeper insights into patient data. AI algorithms analyse vast amounts of medical data to identify patterns and correlations that might elude human analysis. In oncology, for instance, AI can sift through radiographic images, genetic information, and patient histories to detect early-stage cancer. Similarly, AI models predict heart attacks and strokes in cardiology by analysing ECG patterns and other vital signs. These AI systems enhance diagnostic accuracy and enable more personalised and effective treatment plans, ultimately improving patient outcomes.

AI's role in personalised medicine is also notable. By integrating and analysing various types of health data—clinical records, lab results, lifestyle information, and environmental factors—AI helps create more refined and comprehensive treatment plans. For chronic diseases like diabetes, AI can analyse data from wearable devices, diet logs, and blood sugar readings to recommend personalised lifestyle and medication adjustments. This capability allows for better disease management and improved patient care.

AI in Hospital Operations and Management:
Integrating AI into hospital operations enhances efficiency, reduces costs, and improves patient care. AI optimises logistics and resource management by predicting inventory needs, managing facilities more efficiently, and optimising resource allocation. Predictive inventory management ensures that hospitals maintain optimal stock levels, reducing wastage and ensuring the availability of critical supplies. AI also contributes to facility management by controlling HVAC systems efficiently and predicting maintenance needs, minimising downtime and repair costs. AI's role in automating administrative tasks is transformative. AI systems manage patient data, automate billing and claims processing, and optimise scheduling systems. This reduces the workload on healthcare staff and improves overall service delivery. For example, AI-driven scheduling systems analyse patterns in appointment bookings and cancellations to optimise scheduling, reducing wait times and improving patient flow. AI also enhances data security and compliance by monitoring data access patterns and ensuring administrative processes comply with healthcare regulations.

AI in Medical Imaging and Diagnostics:
AI's integration into medical imaging and diagnostics significantly enhances accuracy and efficiency. In radiology and pathology, AI algorithms analyse medical images to detect abnormalities such as tumours, fractures, and signs of diseases with high precision. AI systems highlight subtle findings that may be overlooked by human eyes, serving as invaluable tools for radiologists. For instance, AI can measure tumour size and growth over time, providing crucial information for treatment planning.

AI also excels in integrating diagnostic data from various sources, providing a more comprehensive diagnostic insight. This is particularly valuable in diagnosing complex conditions like autoimmune diseases, where symptoms are ambiguous. AI systems combine information from imaging, laboratory tests, and patient histories to offer more accurate and faster diagnoses, ultimately improving patient outcomes. Additionally, hardware accelerators like Tensor Processing Units (TPUs) and Field-Programmable Gate Arrays (FPGAs) enhance the performance of AI-powered diagnostics, enabling faster and more efficient processing of large medical datasets.

Electronic Health Records (EHRs):
An Electronic Health Record (EHR) is a collection of various medical records that get generated during any clinical encounter or events. With rise of self-care and homecare devices and systems, nowadays meaningful healthcare data get generated 24x7 and also have long-term clinical relevance. The purpose of collecting medical records, as much as possible, are manifold – better and evidence based care, increasingly accurate and

faster diagnosis that translates into better treatment at lower costs of care, avoid repeating unnecessary investigations, robust analytics including predictive analytics to support personalized care, improved health

The Future : Ethics, Challenges, and Integration:

The future of technology is no longer defined only by what we can build, but by how responsibly we choose to build it. As emerging innovations become more powerful and deeply woven into our lives, the ethical implications of their development, deployment, and long-term impact demand urgent attention.

From artificial intelligence (AI) and data-driven systems to autonomous vehicles and biotechnology, technological progress is accelerating at an unprecedented pace. Yet, alongside these advances, critical ethical challenges are surfacing—ones that, if left unaddressed, could erode public trust, cause real harm, and deepen existing inequalities.

This article explores the key ethical challenges shaping the future of technology and outlines the strategies needed to ensure innovation benefits humanity as a whole.

Building trust in healthcare AI requires more than after-the-fact adjustments; it calls for weaving ethical considerations directly into the fabric of AI systems from the start. An “Ethical by Design” approach ensures that core principles—such as fairness, safety, privacy, and accountability—are not retrofitted but form the foundation of an AI system’s architecture, algorithms, and operational protocols .Rather than viewing ethics as an external checkpoint, developers, clinicians, and policymakers should treat it as a guiding compass that informs technical decisions, clinical integration, and policy formation.

In this model, developers collaborate closely with ethicists, legal experts, and healthcare professionals to anticipate potential harm or biases early in the design process. Techniques from responsible AI frameworks, such as continuous impact assessments, formalized documentation, and user-focused explainability tools, support proactive decision-making [93,108,109]. For example, a surgical robot could be engineered with clearly defined accountability layers and operational constraints that safeguard patients, ensuring a human surgeon remains “in the loop” for critical decisions [93]. Similarly, AI-driven diagnostic tools can be designed with embedded fairness checks and transparency features, enabling clinicians and patients to understand how predictions are made and to identify instances of bias or error.

Operationalizing “Ethical by Design” also entails going beyond surface-level compliance. It involves implementing iterative audits, rigorous validation protocols, and clear escalation paths for addressing misconduct or errors. Ethical frameworks become living documents that adapt to evolving technology and data landscapes, thus ensuring that AI systems remain aligned with core human values as they advance .

CONCLUSION

Thus, this integrated approach not only strengthens trust and acceptance of AI in healthcare but also enhances clinical outcomes by making ethical integrity, patient well-being, and responsible innovation inseparable from the development and deployment lifecycle.

REFERENCES

1) Foundation of Artificial Intelligence in Healthcare: author Abhishek Kumar, Priya Battta, Sachin Ahuja, Pramod dingh Rathore
2) Artificial Intelligence in Healthcare.Yu. K.H.;Beam; Kohane I. S.
3) Best Practice , Ethical Challenges and regulatory Frameworks for AI Integration; R. Vettriselvan; Palanivel Rathinasabapathi Velmurugan.
4) Berwick DM, Nolan TW, Whittington J. The Triple Aim: Care, health, and cost. Health Affairs 2008
5) Bodenheimer T, Sinsky C. From triple to quadruple aim: care of the patient requires care of the provider. Ann Fam Med 2014
6) Feeley D. The Triple Aim or the Quadruple Aim? Four Points to Help Set Your Strategy. Institute for Healthcare Improvement, 2017]
7) The Health Foundation, Nuffield Trust, The King's Fund . The health care workforce in England: make or break? The King's Fund, 2018.
8) World Health Organization. Working for health and growth: Investing in the health workforce. WHO, 2016
9) Satya Nadella announces strategic collaboration with Novartis. You Tube, 2019.

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RESEARCH REVIEW ON ARTIFICIAL INTELLIGENCE: EVOLUTION, TECHNIQUES, APPLICATIONS, AND CHALLENGES

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ABSTRACT

Artificial Intelligence (AI) refers to the simulation of human intelligence in machines designed to think and act like humans. This paper provides a comprehensive overview of AI, tracing its historical development, foundational techniques, major application domains, and the ethical challenges it presents. Emphasis is placed on recent advancements, real-world applications, and future prospects.

***Keywords**: Asymmetric Authentication, Attacks, Key Exchange, Routing, Security, Wireless Network*

1. INTRODUCTION

Artificial Intelligence (AI) has emerged as one of the most influential and transformative fields of computer science in recent decades. It focuses on the development of intelligent systems capable of performing tasks that traditionally require human intelligence, such as learning, reasoning, problem-solving, perception, and natural language understanding. With rapid advancements in computational power, data availability, and algorithmic efficiency, AI has evolved from a theoretical concept into a practical technology that impacts almost every aspect of modern life.

The term Artificial Intelligence was first introduced by John McCarthy in 1956 during the Dartmouth Conference, which marked the formal beginning of AI as a research discipline. Early AI systems were primarily rule-based and relied heavily on symbolic reasoning. Although these systems showed promise in constrained environments, their performance was limited when dealing with real-world complexity. Over time, the focus of AI research shifted toward data-driven approaches, leading to the development of machine learning and, more recently, deep learning techniques.

In the present era, AI systems are capable of processing vast amounts of data, identifying complex patterns, and making autonomous decisions with minimal human intervention. Technologies such as machine learning, neural networks, natural language processing, and computer vision have enabled AI applications in diverse domains including healthcare, finance, education, transportation, agriculture, and cyber security. For example, AI-based diagnostic systems assist medical professionals in disease detection, while intelligent recommendation systems enhance user experience in digital platforms.

Despite its remarkable progress, the widespread adoption of AI also raises significant challenges and concerns. Issues related to data privacy, algorithmic bias, lack of transparency, ethical responsibility, and potential job displacement has become central topics in AI research and policy discussions. As AI systems increasingly influence critical decision-making processes, it is essential to ensure that these technologies are developed and deployed in a responsible, fair, and explainable manner.

This paper aims to provide a comprehensive overview of Artificial Intelligence by discussing its fundamental concepts, core technologies, major applications, and associated challenges. Additionally, the paper highlights ethical considerations and future research directions, emphasizing the need for balanced innovation and governance. Through this study, the paper seeks to enhance understanding of AI's role in shaping the future of technology and society.

2. HISTORICAL BACKGROUND

The conceptual foundation of AI began in the mid-20th century:

- **1950s:** Alan Turing's question “Can machines think?” and the Turing Test.
- **1956:** Dartmouth Conference formalizes AI as a field.
- **1960–1980:** Early symbolic AI, expert systems.
- **1990–2010:** Rise of machine learning and data-driven models.
- **2012 onwards:** Deep learning breakthroughs (e.g., ImageNet), leading to current AI resurgence.

3. CORE TECHNIQUES IN AI

3.1 Machine Learning (ML)

ML enables systems to learn from data without explicit programming.

- **Supervised Learning** (classification, regression)
- **Unsupervised Learning** (clustering, dimensionality reduction)
- **Reinforcement Learning** (learning via rewards/punishments)

3.2 Deep Learning

A subset of ML using artificial neural networks:

- Convolutional Neural Networks (CNNs)
- Recurrent Neural Networks (RNNs)
- Transformer-based models (e.g., GPT, BERT)

3.3 Natural Language Processing (NLP)

Enables machines to understand and generate human language. Includes tasks like sentiment analysis, machine translation, and language generation.

3.4 Computer Vision

Focuses on interpreting visual information from the world — object detection, image segmentation, face recognition, etc.

4. MAJOR APPLICATIONS OF AI

4.1 Healthcare

- Disease diagnosis (e.g., cancer detection from imaging)
- Personalized treatment recommendations
- Drug discovery acceleration

4.2 Autonomous Systems

- Self-driving cars
- Robotics in manufacturing and logistics

4.3 Finance

- Fraud detection
- Algorithmic trading
- Credit scoring

4.4 Education

- Adaptive learning platforms
- Automated grading systems
- Personalized curriculum planning

4.5 Smart Cities & IoT

- Traffic optimization
- Energy management
- Public safety surveillance

5. CHALLENGES AND ETHICAL CONSIDERATIONS

5.1 Data Privacy

AI systems rely on large datasets, raising concerns about user privacy and consent.

5.2 Bias and Fairness

Biased training data can lead to discriminatory outcomes in lending, hiring, or law enforcement.

5.3 Job Displacement

Automation threatens certain job sectors, necessitating workforce reskilling.

5.4 Accountability

Who is responsible when AI systems fail? Legal and regulatory frameworks are evolving slowly.

5.5 Transparency

Black-box models make it difficult to explain AI decisions — a problem in critical domains like healthcare.

6. FUTURE DIRECTIONS

- **Explainable AI (XAI):** Making models interpretable.
- **AI Governance:** Regulatory policies and ethical guidelines.
- **AI + Edge Computing:** Bringing AI to low-power, real-time devices.
- **AI in Climate Modeling:** Predictive tools for environmental sustainability.

7. CONCLUSION

Artificial Intelligence has become a powerful and indispensable component of modern technological development. From its early beginnings as a theoretical concept to its current role in real-world applications, AI has demonstrated an exceptional ability to enhance efficiency, accuracy, and decision-making across a wide range of domains. The continuous evolution of AI techniques such as machine learning, deep learning, natural language processing, and computer vision has enabled machines to perform complex tasks that were once considered exclusive to human intelligence.

This paper has presented a comprehensive overview of Artificial Intelligence by discussing its fundamental concepts, historical development, core technologies, and practical applications. The analysis highlights how AI is transforming critical sectors including healthcare, education, finance, agriculture, and transportation by improving productivity, optimizing resources, and supporting data-driven decisions. These advancements indicate that AI has the potential to significantly contribute to economic growth and societal development.

However, alongside its benefits, Artificial Intelligence also introduces several challenges and ethical concerns. Issues related to data privacy, algorithmic bias, transparency, accountability, and workforce displacement require careful consideration. The increasing reliance on AI systems in sensitive and decision-critical environments makes it essential to ensure fairness, reliability, and explainability. Addressing these concerns through proper regulatory frameworks, ethical guidelines, and responsible design practices is crucial for building public trust in AI technologies.

Looking toward the future, the focus of AI research is expected to shift toward more human-centric and explainable systems that can collaborate effectively with humans rather than replace them. The development of Explainable AI, robust governance models, and interdisciplinary research will play a vital role in ensuring the sustainable and ethical growth of AI. By balancing innovation with responsibility, Artificial Intelligence can be harnessed as a transformative tool that supports human welfare and contributes positively to global progress.

In conclusion, Artificial Intelligence represents both an opportunity and a responsibility. When developed and applied thoughtfully, AI has the capacity to reshape society in meaningful and beneficial ways. Continued research, ethical awareness, and collaborative efforts among researchers, policymakers, and industry stakeholders are essential to fully realize the potential of Artificial Intelligence while minimizing its risks.

REFERENCES

[1] S. Russell and P. Norvig, Artificial Intelligence: A Modern Approach, 4th ed. Pearson Education, 2021.

[2] I. Goodfellow, Y. Bengio, and A. Courville, Deep Learning. Cambridge, MA, USA: MIT Press, 2016.

[3] A. M. Turing, “Computing machinery and intelligence,” Mind, vol. 59, no. 236, pp. 433–460, Oct. 1950.

[4] J. McCarthy, “What is artificial intelligence?” Stanford University, Tech. Rep., 2007.

[5] Y. LeCun, Y. Bengio, and G. Hinton, “Deep learning,” Nature, vol. 521, no. 7553, pp. 436–444, May 2015.

[6] D. Silver et al., “Mastering the game of Go with deep neural networks and tree search,” Nature, vol. 529, pp. 484–489, Jan. 2016.

[7] N. J. Nilsson, Artificial Intelligence: A New Synthesis. San Francisco, CA, USA: Morgan Kaufmann, 1998.

[8] N. Bostrom, Superintelligence: Paths, Dangers, Strategies. Oxford, U.K.: Oxford University Press, 2014.

[9] T. Mitchell, Machine Learning. New York, NY, USA: McGraw-Hill, 1997.

[10] IEEE Computational Intelligence Society, “IEEE Transactions on Artificial Intelligence,” IEEE, 2022.

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FIRST ORDER NONLINEAR RANDOM DIFFERENTIAL EQUATION IN SEPARABLE BANACH ALGEBRA

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ABSTRACT

In this work, a class of first-order nonlinear random ordinary differential equations in a separable Banach algebra is studied. Determining the existence and attractivity of random solutions for the equations under consideration is the primary goal. We investigate and analyze, under appropriate assumptions, an initial value problem related to first-order nonlinear random ordinary differential equations. To guaranty the presence of random solutions, suitable hypotheses are formulated using fixed point techniques in conjunction with the Caratheodory and Lipschitz criteria. Additionally, under the Caratheodory condition and some monotonicity assumptions, the presence of extremal solutions is obtained. Within the suggested framework, the attractivity behavior of the produced random solutions is also covered. An illustrative example is presented to verify and demonstrate the applicability of the established theoretical results.

Keyword: *fixed point theorem, nonlinear ordinary differential equation, existence of solution to ordinary differential equation, Banach Algebra, nonlinear integral equation, random operator.*

AMS (MOS) Subject Classification: *47H10, 34A34, 34A12, 47B48, 45G20, 60H50.*

1. INTRODUCTION

Differential and integral equations are fundamental tools in both pure and applied mathematics, playing a key role in modeling physics, engineering, and biological systems [15] [14]. Theories of differential and integral equations have rapidly developed using methods from topology, functional analysis, and fixed point theory, particularly in Banach spaces and nonlinear operator frameworks [3] [18]. These equations are especially relevant in applied fields such as engineering, mechanical vibrations, and mathematical physics, where they are used to describe complex dynamic systems [17] [14]. Nonlinear differential and integral equations have been extensively studied in the literature, focusing on various aspects of the solutions such as existence, uniqueness, extremal solutions, and numerical approximations [1] [2] [10] [12] [16]. The initial value problem of ordinary differential equations have been studied in the literature on bounded as well as unbounded intervals of real line for different aspects of the solutions. For examples Banas and Dhage [3], Burton and Furumochi [4], Dhage [8] , B.C. Dhage, S.V. Badgire and J. Henderson [5], . Nonlinear random integral equation in banach algebras have also been studied by Dhage [9] [3] [6], X. Hu and Yan [12] for various aspects of solutions. In this paper we discuss the first order nonlinear random differential equations for existence and attractivity of solutions as well as for existence of the extremal solutions in separable Banach algebra using Lipschitz, Caratheodory and certain monotonicity condition and results are presented with use of examples.

2. STATEMENT OF PROBLEM

We consider the following first order nonlinear random differential equation (FNRDE):

$$D\left(\frac{x(t,\ \upsilon)}{\mathcal{J}(t, x(t,\upsilon),\upsilon)}\right) = \mathcal{K}(t, x(t,\upsilon),\upsilon)$$

$$x(0,\upsilon) = 0 \qquad 2.1$$

For all $\upsilon \in \Omega, t \in R_+$ and $x \in BM\ (R_+, R)$

Where, $\mathcal{J}: R_+ \times R \times \Omega \to R - \{0\}; \mathcal{K}: R_+ \times R \times \Omega \to R$

By the solution of first order nonlinear random differential equation FNRDE (2.1) we have a function $x \in BM\ (R_+, R)$ where BM is bounded measurable functions such that:

i) The function $t \to \left(\frac{x}{\mathcal{J}(t,x(t,\upsilon),\upsilon)}\right)$ is bounded measurable for each $x \in R$

ii) x satisfies (2.1)

3. PRELIMINARIES

This section is devoted to collecting notions, definitions hypothesis and preliminary tools which will be needed in sequel.

Definition 3.1 [13]: Let $(\Omega, \mathcal{A})$ be a measurable space and let S be a separable Banach space with σ −algebra β_S of all Borel subsets of S. A mapping $f: \Omega \to S$ is called measurable if for any Borel subset $\mathfrak{B}$ of S,

$$f^{-1}(\mathfrak{B}) = \{\upsilon \in \Omega : f(\upsilon) \in \mathfrak{B}\} \in \mathcal{A}$$

Definition 3.2 [13] [18]: A mapping $\mathcal{T}: \Omega \times S \to S$ is called random operator if $\upsilon \to \mathcal{T}(\upsilon, s)$ is measurable for all $s \in S$ and we denote it by

$$\mathcal{T}(\upsilon)s = \mathcal{T}(\upsilon, s).$$

Definition 3.3 [7] [13]: A random variable $\xi : \Omega \to X$ is called random fixed point of a random operator $\mathcal{T} : \Omega \times S \to S$, if $\mathcal{T}(\upsilon)\xi(\upsilon) = \xi(\upsilon)$ for every $\upsilon \in \Omega$.

Definition 3.5[7]: A random operator $\mathcal{T} : \Omega \times S \to S$ is called totally bounded if for any bounded subset D of S, $\mathcal{T}(\upsilon)(D)$ is totally bounded subset of S for every $\upsilon \in \Omega$.

Definition 3.6[7]: A random operator $\mathcal{T} : \Omega \times S \to S$ is called compact random operator, if $\overline{\mathcal{T}(\upsilon)(S)}$ is compact subset of S where $\overline{\mathcal{T}(\upsilon)(S)}$ is closure of $\mathcal{T}(\upsilon)(S)$.

Definition 3.7[7]: A random operator $\mathcal{T}(\upsilon)$ is called completely continuous on Banach space S, if it is continuous and totally bounded on S.

Note that every compact operator is totally bounded, but the converse may not be true.

Definition 3.8: A class of measurable functions $\{s_n(\upsilon)\}$ is said to be equicontinuous class, if for every $\varepsilon > 0$ there exist $\delta > 0$ such that $\|s_n(t_1,\upsilon) - s_n(t_2,\upsilon)\| < \varepsilon$ whenever $|t_1 - t_2| < \delta$ for all $n = 1,2,3, \ldots$

Definition 3.9 [7]: A random operator $\mathcal{T} : \Omega \times S \to S$ is called a $\mathfrak{D}$-Lipschitzician, if there exist a nondecreasing continuous function $\varphi: \Omega \times R_+ \to R_+$ such that for every $\upsilon \in \Omega$

$$\|\mathcal{T}(\upsilon)s_1 - \mathcal{T}(\upsilon)s_2\| \leq \varphi_\upsilon(\|s_1 - s_2\|)$$

For all $s_1, s_2 \in S$, where $\varphi_\upsilon(r) = \varphi(\upsilon, r)$ with $\varphi(\upsilon, 0) = 0$. Here some special case

$\varphi_\upsilon(r) = \alpha(\upsilon)r$, for $\alpha(\upsilon) > 0$ for every $\upsilon \in \Omega$ then the random operator $\mathcal{T}(\upsilon)$ is called Lipschitz with Lipschitz constant $\alpha(\upsilon)$, for $\upsilon \in \Omega$. In particular for each $\upsilon \in \Omega$, $\alpha(\upsilon) < 1$ then random operator $\mathcal{T}(\upsilon)$ is known as a contraction with contraction constant $\alpha(\upsilon)$. Further if

$\varphi_\upsilon(r) = \varphi(\upsilon, r) < r$, For $r > 0$ and for every $\upsilon \in \Omega$ then $\mathcal{T}(\upsilon)$ is known as nonlinear $\mathfrak{D}$-contraction random operator on S.

Theorem 3.10 [7]: Let $\mathcal{D}$ be a closed and bounded subset of a separable Banach algebra S and let

$\mathcal{P}(\upsilon), \mathcal{Q}(\upsilon) : \Omega \times \mathcal{D} \to S$be two random operators satisfying for each $\upsilon \in \Omega$,

(a) $\mathcal{P}(\upsilon)$ is $\mathfrak{D}$-Lipschitz with $\mathfrak{D}$-function φ

(b) $\mathcal{Q}(\upsilon)$ is completely continuous, and

(c) $\mathcal{P}(\upsilon)x\,\mathcal{Q}(\upsilon)x \in \mathcal{D}$ for each $x \in \mathcal{D}$.

Then the random equation

$$\mathcal{P}(\upsilon)x\,\mathcal{Q}(\upsilon)x = x$$

has a random solution whenever $M(\upsilon)\varphi_\upsilon(r) < r, r > 0$, for each $\upsilon \in \Omega$

where $M(\upsilon) = \|\mathcal{Q}(\upsilon)(\mathcal{D})\|$.

Corollary 3.11 [7]: Let $\mathcal{D}$ be a closed, convex and bounded subset of a separable Banach algebra S and let $\mathcal{P}(\upsilon), \mathcal{Q}(\upsilon) : \Omega \times \mathcal{D} \to S$ be two random operators satisfying for each

$\upsilon \in \Omega$,

(a) $\mathcal{P}(\upsilon)$ is Lipschitz with Lipschitz constant $\alpha(\upsilon)$

(b) $\mathcal{Q}(\upsilon)$ is continuous and compact,

(c) $\mathcal{P}(\upsilon)x\,\mathcal{Q}(\upsilon)x \in \mathcal{D}$ for each $x \in \mathcal{D}$.

Then the random equation

$$\mathcal{P}(\upsilon)x\,\mathcal{Q}(\upsilon)x = x$$

has a random solution and the set of such solutions is compact whenever $\alpha(\upsilon)M(\upsilon) < 1$ for each $\upsilon \in \Omega$, where $M(\upsilon) = \|\mathcal{Q}(\upsilon)(\mathcal{D})\|$.

4. EXISTENCE THEORY

We seek random solutions of FNRDE (2.1) in the space $BM(R_+, R)$ of bounded measurable and real valued function on R_+ . We equip the space $BM(R_+, R)$ with the supremum norm$\|.\|$defined by

$$\|x\| = \sup_{t \in R_+} |x(t)|,$$

for $t \in R_+$. Clearly $BM(R_+, R)$ become a Banach space with respect to above norm. By

$L^1(R_+, R)$ is defined by

$$\|x\|_{L^1} = \int_0^t |x(t)|\ ds\,.$$

Definition 4.1 [7]: A mapping $\beta: R_+ \times R \times \Omega \to R$ is said to satisfy condition of $L(\upsilon)$-caratheodory if

i) $(t, \upsilon) \to \beta(t, x, \upsilon)$ is measurable for each $x \in R$

ii) $x \to \beta(t, x, \upsilon)$ is continuous for each $t \in R_+$, $\upsilon \in \Omega$ Furthermore of $L(\upsilon)$-caratheodory function $\beta: R_+ \times R \times \Omega \to R$ is called $L^1(\upsilon)$-caratheodory if

iii) There exists measurable and bounded function $q: \Omega \to L^1(R_+)$ such that $|\beta(t, x, \upsilon)| \le q(t, \upsilon)$ a.e., $t \in R_+$ for all $x \in R$.

We need following some hypothesis for existence solution of FNRDE (2.1)

(H_0) The function $\mathcal{J}: R_+ \times R \times \Omega \to R - \{0\}$ is measurable.

(H_1) The function $(t, \upsilon) \to \mathcal{K}(t, x, \upsilon)$ is measurable for all $t \in R_+$and $x \in R$.

(H_2) The function $\mathcal{J}: R_+ \times R \times \Omega \to R - \{0\}$ is continuous function and there exist a bounded function $\alpha: R_+ \times \Omega \to R_+ \times \Omega$ such that

$$|\mathcal{J}(t, x(t,\upsilon), \upsilon) - \mathcal{J}(t, y(t,\upsilon), \upsilon)| \le \alpha(t,\upsilon)|x(t,\upsilon) - y(t,\upsilon)|$$

For all $t \in R_+$, with $x, y \in R$, $|\alpha(t,\upsilon)| < 1$.

(H_3) There exists function $\phi: R_+ \times \Omega \to R_+ \times \Omega$ such that for each $\upsilon \in \Omega$

$|\mathcal{J}(t,x,\upsilon)| \le \phi(t,\upsilon)$, a.e. $t \in R_+$ and $\lim\limits_{t\to\infty} \phi(t,\upsilon) = 0$.

(H_4) The function $\mathcal{K}$ is $L^1(\upsilon)$-caratheodory function.

(H_5) There exists a function $q: \Omega \to L^1(R_+, R)$ with $q(t, \upsilon) > 0$ a.e. for $t \in R_+$ and $\upsilon \in \Omega$ such that $|\mathcal{K}(t, x, \upsilon)| \le q(t, \upsilon)$ a.e. for $t \in R_+$.

Remark: The function $\mathcal{V}: R_+ \times \Omega \to R \times \Omega$ defined by $\mathcal{V}(t,\upsilon) = \int_0^t q(s,\upsilon)ds$ is bounded on R_+ with bound $K_1 = \sup\{\mathcal{V}(t,\upsilon): t \in R_+ \& \upsilon \in \Omega\} > 0$.

Lemma 4.2: If $\mathcal{J}, \mathcal{K}$ satisfies FNRDE (2.1) then x is the solution of the FNRDE (2.1) if and only if it is the solution of integral equation

$$x(t,\upsilon) = \mathcal{J}(t, x(t,\upsilon), \upsilon) \int_0^t \mathcal{K}(s, x(s, \upsilon), \upsilon)\ ds \tag{4.1}$$

For $t \in R_+$and $\upsilon \in \Omega$.

Proof: FNRDE (2.1) is

$$D\left(\frac{x(t, \upsilon)}{\mathcal{J}(t, x(t,\upsilon), \upsilon)}\right) = \mathcal{K}(t, x(t,\upsilon), \upsilon)$$

$$x(0,\upsilon) = 0$$

Integrating equation (2.1) from 0 to t we have

$$\mathrm{ID}\left(\frac{x(t,\ \upsilon)}{\mathcal{J}(t,x(t,\upsilon),\upsilon)}\right) = \mathrm{I}\mathcal{K}(t,x(t,\upsilon),\upsilon)$$

$$\left(\frac{x(t,\ \upsilon)}{\mathcal{J}(t,x(t,\upsilon),\upsilon)}\right)_0^t = \int_0^t \mathcal{K}(s,x(s,\upsilon),\upsilon)\ ds$$

$$\left(\frac{x(t,\ \upsilon)}{\mathcal{J}(t,x(t,\upsilon),\upsilon)}\right) - \left(\frac{x(0,\ \upsilon)}{\mathcal{J}(0,x(0,\upsilon),\upsilon)}\right) = \int_0^t \mathcal{K}(s,x(s,\upsilon),\upsilon)\ ds$$

$\left(\frac{x(t,\upsilon)}{\mathcal{J}(t,x(t,\upsilon),\upsilon)}\right) = \int_0^t \mathcal{K}(s,x(s,\upsilon),\upsilon)\ ds$ (4.2)

$\therefore\ x(t,\upsilon) = \mathcal{J}(t,x(t,\upsilon),\upsilon)\int_0^t \mathcal{K}(s,x(s,\ \upsilon),\ \upsilon)\ ds$ (4.3)

Conversely,

Differentiate equation (4.2) we get

$$\mathrm{D}\left(\frac{x(t,\ \upsilon)}{\mathcal{J}(t,x(t,\upsilon),\upsilon)}\right) = \mathrm{D}\int_0^t \mathcal{K}(s,x(s,\upsilon),\upsilon)\ ds$$

$$\mathrm{D}\left(\frac{x(t,\ \upsilon)}{\mathcal{J}(t,x(t,\upsilon),\upsilon)}\right) = \mathrm{D}\ \frac{1}{\Gamma(1)}\int_0^t (t-s)^{1-1}\mathcal{K}(s,x(s,\upsilon),\upsilon)\ ds$$

$$\mathrm{D}\left(\frac{x(t,\ \upsilon)}{\mathcal{J}(t,x(t,\upsilon),\upsilon)}\right) = \mathrm{D\ I}\ \mathcal{K}(t,x(t,\upsilon),\upsilon)$$

$$\mathrm{D}\left(\frac{x(t,\ \upsilon)}{\mathcal{J}(t,x(t,\upsilon),\upsilon)}\right) = \mathcal{K}(t,x(t,\upsilon),\upsilon)$$

Put t = 0 in equation (4.3) then

$x(0,\upsilon) = 0.$

Theorem 4.3: Assume that the hypothesis (H_0) – (H_5) hold and if $\|\alpha(\ \upsilon)\|\ K_1 < 1$ then FNRDE (2.1) has random solutions in R_+ and moreover, the random solutions are locally attractive on R_+ .

Proof: Let $S = BM(R_+, R)$ be a measurable Banach Algebra and define a subset $\mathcal{D}$ of S as

$$\mathcal{D} = \{x \in S :\ \|x\| \le r\}$$

Where, $r = \|\phi(\ \upsilon)\|K_1$.

Clearly $\mathcal{D}$ is a closed convex and bounded subset of S.

Now we define the two operators $\mathcal{P}$ and $\mathcal{Q}$ on $\mathcal{D}$ i.e. $\mathcal{P},\mathcal{Q} :\ \Omega \times \mathcal{D}\ \to S$ defined by

$$\mathcal{P}(\ \upsilon)x(t) = \mathcal{J}(t,x(t,\upsilon),\upsilon)$$

$$\mathcal{Q}(\ \upsilon)x(t) = \int_0^t \mathcal{K}(s,x(s,\upsilon),\upsilon)\ ds$$

For all $t \in R_+$ and $\upsilon \in \Omega$

Then NRIE (4.1) is transformed in to the random operator equation

$$x(t,\upsilon) = \mathcal{P}(\ \upsilon)x(t)\mathcal{Q}(\ \upsilon)x(t)$$

For all $t \in R_+$ and $\upsilon \in \Omega$

We shall show that the operators $\mathcal{P}(\upsilon)$ & $\mathcal{Q}(\upsilon)$ satisfies all conditions of corollary (3.11) on $\mathcal{D}$. This will be done in following steps.

Step I: First we show that $\mathcal{P}(\upsilon)$& $\mathcal{Q}(\upsilon)$ are random operators on $\mathcal{D}$.

Since, by hypothesis H_0 a function $\mathcal{J}(t, x, \upsilon)$ is measurable in υ for all $t \in R_+$ and $x \in R$ then the function $\upsilon \rightarrow \mathcal{P}(\upsilon)x$ is measurable for all $x \in R$. Hence $\mathcal{P}(\upsilon)$ is random operator on $\mathcal{D}$.

By hypothesis H_1 the function $(t, \upsilon) \rightarrow \mathcal{K}(t, x, \upsilon)$ is measurable for all $x \in R$. We know that Riemann integral as a limit of a finite sum of measurable function is again measurable.

Therefore, the function $\upsilon \rightarrow \int_0^t \mathcal{K}(s, x(s, \upsilon), \upsilon)\, ds$ is measurable.

Hence $\mathcal{Q}(\upsilon)$ is random operator on $\mathcal{D}$.

Step II: Next we show that $\mathcal{P}(\upsilon)$ is a Lipschitz random operator on $\mathcal{D}$

Let $x, y \in \mathcal{D}$

$$|\mathcal{P}(\upsilon)x(t) - \mathcal{P}(\upsilon)y(t)| = |\mathcal{J}(t, x(t, \upsilon), \upsilon) - \mathcal{J}(t, y(t, \upsilon), \upsilon)|$$
$$\leq \alpha(t, \upsilon)|x(t, \upsilon) - y(t, \upsilon)|$$

Taking the maximum over t in the above inequality, we obtain

$$\|\mathcal{P}(\upsilon)x(t) - \mathcal{P}(\upsilon)y(t)\| \leq \|\alpha(\upsilon)\|\|x(\upsilon) - y(\upsilon)\|$$

For all $t \in R_+$, $x, y \in \mathcal{D}$ and $\upsilon \in \Omega$. This shows that $\mathcal{P}(\upsilon)$ is a Lipschitzician random operator on $\mathcal{D}$ with Lipschitz constant $\|\alpha(\upsilon)\|$.

Step III: Now show that $\mathcal{Q}(\upsilon)$ is continuous

Let $\{x_n\}$ be convergent sequence of points in $\mathcal{D}$ converging to point x in $\mathcal{D}$, then it is enough to prove that

$$\lim_{n\to\infty} \mathcal{Q}(\upsilon)x_n(t) = \mathcal{Q}(\upsilon)x(t), \quad t \in R_+$$

By Lebesgue dominated converging theorem we obtain

$$\lim_{n\to\infty} \mathcal{Q}(\upsilon)x_n(t) = \lim_{n\to\infty}\left(\int_0^t \mathcal{K}(s, x_n(s, \upsilon), \upsilon)\, ds\right)$$
$$= \int_0^t (\lim_{n\to\infty} \mathcal{K}(s, x_n(s, \upsilon), \upsilon))\, ds$$
$$= \int_0^t \mathcal{K}(s, x(s, \upsilon), \upsilon)\, ds$$
$$= \mathcal{Q}(\upsilon)x(t)$$

For every $t \in R_+$ and $\upsilon \in \Omega$

Hence the random operator $\mathcal{Q}(\upsilon)$ is continuous on $\mathcal{D}$.

Step IV: To show that $\mathcal{Q}(\upsilon)$ is compact operator on $\mathcal{D}$.

It suffices to show that $\mathcal{Q}(\upsilon)(\mathcal{D})$ is uniformly bounded and equicontinuous set in S, for each

$\upsilon \in \Omega$.

First we show that $\mathcal{Q}(\upsilon)(\mathcal{D})$ is uniformly bounded for each $\upsilon \in \Omega$.

Let $x \in \mathcal{D}$ be arbitrary thus by hypothesis H_4, $\mathcal{K}$ is $L^1(\upsilon)$-caratheodory

$$|\mathcal{Q}(\upsilon)x(t)| = \left|\int_0^t \mathcal{K}(s, x(s, \upsilon), \upsilon)\, ds\right|$$

$$\leq \int_0^t |\mathcal{K}(s, x(s, \upsilon), \upsilon)|\, ds$$

$$\leq \int_0^t q(s, \quad \upsilon) ds$$

$$\leq \nu(t, \upsilon)$$

Taking supremum all over t

$$\|\mathcal{Q}(\upsilon)x\| \leq K_1$$

This shows that $\mathcal{Q}(\upsilon)(\mathcal{D})$ is uniformly bounded subset of S for each $\upsilon \in \Omega$

Now we show that $\mathcal{Q}(\upsilon)(\mathcal{D})$ is equicontinuous set in S for each $\upsilon \in \Omega$

Let $t, \tau \in R_+$ then for any $x \in \mathcal{D}$

$$|\mathcal{Q}(\upsilon)x(t) - \mathcal{Q}(\upsilon)x(\tau)| = \left| \int_0^t \mathcal{K}(s, x(s, \upsilon), \upsilon)\, ds - \int_0^\tau \mathcal{K}(s, x(s, \upsilon), \upsilon)\, ds \right|$$

$$\leq \left| \int_0^t q(s, \upsilon) ds - \int_0^\tau q(s, \upsilon) ds \right|$$

$$\leq |\mathcal{V}(t, \upsilon) - \mathcal{V}(\tau, \upsilon)|$$

Since $\mathcal{V}(t, \upsilon) = \int_0^t q(s, \upsilon) ds$ and $\mathcal{V}$ is uniformly continuous function R_+.

Hence

$$|\mathcal{Q}(\upsilon)x(t) - \mathcal{Q}(\upsilon)x(\tau)| \rightarrow 0$$

as $t \rightarrow \tau$ for all $t, \tau \in R_+$ and $\upsilon \in \Omega$.

Therefore $\mathcal{Q}(\upsilon)(\mathcal{D})$ is equicontinuous set in S. Hence $\mathcal{Q}(\upsilon)$ is compact operator on $\mathcal{D}$.

Step V: Now

$$\|\mathcal{P}(\upsilon)x\mathcal{Q}(\upsilon)x\| = \|\mathcal{P}(\upsilon)x\|\|\mathcal{Q}(\upsilon)x\|$$

$$\leq \left\{\sup_{t \in R_+} |\mathcal{P}(\upsilon)x(t)|\right\}\left\{\sup_{t \in R_+} |\mathcal{Q}(\upsilon)x(t)|\right\}$$

$$\leq \left\{\sup_{t \in R_+} |\mathcal{J}(t, x(t, \upsilon), \upsilon)|\right\}\left\{\sup_{t \in R_+} \left| \int_0^t \mathcal{K}(s, x(s, \upsilon), \upsilon) ds \right|\right\}$$

$$\leq \left\{\sup_{t \in R_+} |\mathcal{J}(t, x(t, \upsilon), \upsilon)|\right\}\left\{\sup_{t \in R_+} \int_0^t |\mathcal{K}(s, x(s, \upsilon), \upsilon)|\, ds\right\}$$

$$\leq \left\{\sup_{t \in R_+} |\mathcal{J}(t, x(t, \upsilon), \upsilon)|\right\}\left\{\sup_{t \in R_+} \int_0^t q(s, \upsilon) ds\right\}$$

$$\leq \left\{\sup_{t \in R_+} |\mathcal{J}(t, x(t, \upsilon), \upsilon)|\right\}\left\{\sup_{t \in R_+} \mathcal{V}(t, \upsilon)\right\}$$

$$\leq \|\phi(\upsilon)\| K_1$$

$$\leq r$$

for each $x \in S$ with $\|x\| \leq r$, then $x = \mathcal{P}(\upsilon)x\, \mathcal{Q}(\upsilon)x \in \mathcal{D}$.

Also we have

$$
\begin{aligned}
\mathrm{M}(\upsilon) &= \|\mathcal{Q}(\upsilon)(\mathcal{D})\| \\
&= \sup_{x\in\mathcal{D}}\{\|\mathcal{Q}(\upsilon)x\| : x\in\mathcal{D}\} \\
&= \sup_{x\in\mathcal{D}}\left\{\sup_{t\in R_+}|\mathcal{Q}(\upsilon)x(t)|\right\} \\
&= \sup_{x\in\mathcal{D}}\left\{\sup_{t\in R_+}\left|\int_0^t \mathcal{K}(s,x(s,\upsilon),\upsilon)ds\right|\right\} \\
&\le \sup\left\{\sup_{t\in R_+}\int_0^t |\mathcal{K}(s,x(s,\upsilon),\upsilon)|\,ds\right\} \\
&\le \sup\left\{\sup_{t\in R_+}\int_0^t q(s,\upsilon)ds\right\} \\
&\le \sup\left\{\sup_{t\in R_+} v(t,\upsilon)\right\} \\
&\le K_1
\end{aligned}
$$

And therefore $\|\alpha(\upsilon)\|M(\upsilon) \le \|\alpha(\upsilon)\|K_1 < 1$, for each $\upsilon \in \Omega$.

Thus all the conditions of corollary (3.11) follows, hence random operator equation $\mathcal{P}(\upsilon)x\mathcal{Q}(\upsilon)x = x$ has random solutions. This further implies that the random differential equation (2.1) has random solutions on R_+.

Step VI: Finally, we show that the locally attractivity of the solutions.

Let x & y be any two solutions of the integral equation (2.1) in $\mathcal{D}$, then we have

$$
\begin{aligned}
&|x(t,\upsilon) - y(t,\upsilon)| \\
&= \left|\mathcal{J}(t,x(t,\upsilon),\upsilon)\int_0^t \mathcal{K}(t,x(s,\upsilon),\upsilon)ds - \mathcal{J}(t,y(t,\upsilon),\upsilon)\int_0^t \mathcal{K}(t,y(s,\upsilon),\upsilon)ds\right| \\
&\le \left|\mathcal{J}(t,x(t,\upsilon),\upsilon)\int_0^t q(s,\upsilon)ds\right| + \left|\mathcal{J}(t,y(t,\upsilon),\upsilon)\int_0^t q(s,\upsilon)ds\right| \\
&\le \{|\mathcal{J}(t,x(t,\upsilon),\upsilon)| + |\mathcal{J}(t,y(t,\upsilon),\upsilon)|\}\,\mathcal{V}(t,\upsilon) \\
&\le 2\phi(t,\upsilon)\mathcal{V}(t,\upsilon)
\end{aligned}
$$

Taking the limit superior in the above inequality as t → ∞ yields

$$\lim_{t\to\infty}\sup|x(t,\upsilon) - y(t,\upsilon)| \le 2\lim_{t\to\infty}\sup\,[\phi(t,\upsilon)\mathcal{V}(t,\upsilon)]$$

$$\lim_{t\to\infty}\sup|x(t,\upsilon) - y(t,\upsilon)| = 0$$

Since $\lim_{t\to\infty}\sup\phi(t,\upsilon) = 0$

Consequently, the FNRDE (2.1) has random solutions and all the solutions are locally attractive on R_+.

5. EXISTENCE OF EXTREMAL RANDOM SOLUTIONS:

In this part, we established the existence of extremal solutions under a few reasonable monotonicity requirements.

Definition 5.1 [5]: A non-empty closed subset $\varkappa$ of a separable Banach algebra S is called cone if

(i) $\varkappa + \varkappa \subseteq \varkappa$

(ii) $\delta\varkappa \subset \varkappa$ for all $\delta \in R_+$

(iii) $\{-\varkappa\} \cap \varkappa = \{0\}$

Where 0 is zero element of S.

We define order relation $\leq$ in S with the help of a cone $\varkappa$ as follows

Let $x, y \in S$ then $x \leq y$ iff $y - x \in \varkappa$, the Banach algebra S with order relation $\leq$ is called ordered Banach algebra and it is denoted by $(S, \|.\|, \leq)$.

A cone $\varkappa$ in S is known as normal if the norm $\|.\|$ is defined on S is monotone increasing on $\varkappa$, i.e. if $x, y \in \varkappa$ with $x \leq y$ then there exist a constant $N > 0$ such that

$$\|x\| \leq N\|y\|$$

It is known that if the cone $\varkappa$ is normal in S then every order-bounded set in S is norm- bounded set in S. The detail properties of cone appear in Guo and Lakshimikantham [11].

It is known that if the cone $\varkappa$ in S is normal, then every order interval is norm bounded set in S. Let $a, b: \Omega \to S$ be two measurable functions then by $a \leq b$ we mean $a(\upsilon) \leq b(\upsilon)$ for all $\upsilon \in \Omega$. In this case the random order interval $[a, b]$ is defined to be a set in S given by

$$[a, b] = \{x \in S : a(\upsilon) \leq x \leq b(\upsilon), \text{for all } \upsilon \in \Omega\}$$

$$= \bigcap_{\upsilon\in\Omega} [a(\upsilon), b(\upsilon)]$$

Definition 5.2 [5]: A random operator $\mathcal{T}: \Omega \times S \to S$ is called monotone non-decreasing if for any $x, y \in \varkappa$, $x \leq y$ implies $\mathcal{T}(\upsilon)x \leq \mathcal{T}(\upsilon)y$ for all $\upsilon \in \Omega$.

Definition 5.3 [5]: Define order relation $\leq$ in continuous real valued space $C(R_+, R)$ by a cone $\varkappa$ in $C(R_+, R)$ by

$$\varkappa = \{x \in c(R_+, R): x \geq 0\}$$

Clearly cone $\varkappa$ is normal in $C(R_+, R)$.

Lemma 5.4 [11]: Let $\mathcal{u}_1, \mathcal{u}_2, \mathcal{v}_1, \mathcal{v}_2 \in K$ be such that $\mathcal{u}_1 \leq \mathcal{u}_2$ and $\mathcal{v}_1 \leq \mathcal{v}_2$then $\mathcal{u}_1\mathcal{u}_2 \leq \mathcal{v}_1\mathcal{v}_2$. For any $\mathcal{u}, \mathcal{v} \in S = C(R_+, R)$ be such that$\mathcal{u} \leq \mathcal{v}$ then the order interval $[\mathcal{u}, \mathcal{v}]$ is a set in S defined by

$[\mathcal{u}, \mathcal{v}] = \{x \in S : \mathcal{u} \leq x \leq \mathcal{v}\}$

It is known that if the cone $\varkappa$ in S is normal, then every order interval is norm bounded set in S

Theorem 5.5 [5]: Let $\mathcal{P}$, $\mathcal{Q} : \Omega \times S \to S$ be two monotones non decreasing random operators satisfying for each $\upsilon \in \Omega$

(a) $\mathcal{P}(\upsilon)$ is Lipschitz with the Lipschitz constant $\alpha(\upsilon)$

(b) $\mathcal{Q}(\upsilon)$ is completely continuous

(c) There exist two continuous functions $a, b : \Omega \to S$ s. t..

$$a(\upsilon) \leq \mathcal{P}(\upsilon)a \;\; \mathcal{Q}(\upsilon)a \quad \& \quad \mathcal{P}(\upsilon)b \;\; \mathcal{Q}(\upsilon)b \;\leq b(\upsilon)$$

Further cone $\varkappa$ in S is normal then the random equation

$$\mathcal{P}(\upsilon)x\,\mathcal{Q}(\upsilon)x = x$$

has the least random solutions x_m and the greatest random solution x_M in $[a, b]$ whenever

$\alpha(\upsilon)M(\upsilon) < 1$ for each $\upsilon \in \Omega$,

Where, $M(\upsilon) = \|\,\mathcal{Q}(\upsilon)([a, b])\|$

Moreover, $x_m(\upsilon) = \lim\limits_{n\to\infty} x_n(\upsilon)$; $\quad x_M(\upsilon) = \lim\limits_{n\to\infty} y_n(\upsilon)$

Where, $x_{n+1}(\upsilon) = \mathcal{P}(\upsilon)x_n\,\mathcal{Q}(\upsilon)x_n,\;\; n \;\geq 0$

With $x_0(\upsilon) = a(\upsilon)$

$$y_{n+1}(\upsilon) = \mathcal{P}(\upsilon)y_n\,\mathcal{Q}(\upsilon)y_n, \qquad n \;\geq 0$$

With $y_0(\upsilon) = b(\upsilon)$

Definition 5.6: A measurable function $a: \Omega \to BC(R_+, R)$ is called lower random solution of the FNRDE (2.1) if for each $\upsilon \in \Omega$

$$D\left(\frac{a(t)}{\mathcal{J}(t, a(t,\upsilon), \upsilon)}\right) \le \mathcal{K}(t, a(t,\upsilon), \upsilon) \qquad \text{a.e. } t \in R_+$$

With $$a(0) = 0$$

For all $t \in R_+$ and $\upsilon \in \Omega$. Similarly, a measurable function $b: \Omega \to BC(R_+, R)$ is called upper random solution of FNRDE (2.1) if for each $\upsilon \in \Omega$

$$\mathcal{K}(t, b(t,\upsilon), \upsilon) \le D\left(\frac{b(t)}{\mathcal{J}(t, b(t,\upsilon), \upsilon)}\right) \quad \text{a.e.} t \in R_+$$

With $$b(0) = 0$$

Definition 5.7 [5]: A random solution x_M to the FNRDE (2.1) is said to be maximal if for any other random solution x we have $x(t,\upsilon) \le x_M(t,\upsilon)$ for all $t \in R_+$ and $\upsilon \in \Omega$.

Similarly, a random solution x_m to the FNRDE (2.1) is said to be minimal if for any other random solution x we have $x_m(t,\upsilon) \le x(t,\upsilon)$ for all $t \in R_+$ and $\upsilon \in \Omega$.

We consider the following hypothesis

(B_0) The function $\mathcal{J}: R_+ \times R \times \Omega \to R - \{0\}$ is measurable.

(B_1) The function $(t,\ \upsilon) \to \mathcal{K}(t, x,\ \upsilon)$ is measurable for all $t \in R_+$ and $x \in R$.

(B_2) The function $\mathcal{J}: R_+ \times R \times \Omega \to R - \{0\}$ is continuous function and there exists a bounded function $\alpha: R_+ \times \Omega \to R_+ \times \Omega$ such that.

$$|\mathcal{J}(t, x(t,\upsilon), \upsilon) - \mathcal{J}(t, y(t,\upsilon), \upsilon)| \le \alpha(t,\upsilon)|x(t,\upsilon) - y(t,\upsilon)|$$

For all $t \in R_+$, with $x, y \in R$

(B_3) For each $\upsilon \in \Omega$ the function $\mathcal{J}(t, x,\ \upsilon)$ is monotone non-decreasing for

$$t \in R_+ , x \in R \text{ a.e.}$$

(B_4) The function $\mathcal{K}$ is $L^1(\upsilon)$-caratheodory function.

(B_5) The FNRDE (2.1) has a lower random solution a and upper random solution b with $a \le b$

Remark: Note that if hypothesis B_1, B_4, H_4 holds then there exist constant $K_1 > 0$ such that

$$K_1 = \sup\{\mathcal{V}(t,\upsilon) : t \in R_+ \& \ \upsilon \in \Omega\}$$

Theorem 5.8: Assume that the hypothesis B_0-B_5 hold then FNRDE (2.1) has minimal random solution x_m and maximal random solution x_M in $[a, b]$ whenever

$$\|\alpha(\upsilon)\| K_1 < 1$$

Moreover $x_m(\upsilon) = \lim_{n\to\infty} x_n(\upsilon)$; $x_M(\upsilon) = \lim_{n\to\infty} y_n(\upsilon)$

Where,

$$x_{n+1}(t,\upsilon) = \mathcal{J}(t, x_n(t,\upsilon), \upsilon) \int_0^t \mathcal{K}(s, x_n(s,\upsilon), \upsilon)\, ds \quad \text{a.e.}$$

For all $t \in R_+$ with $x_0 = a$

$$y_{n+1}(t,\upsilon) = \mathcal{J}(t, y_n(t,\upsilon), \upsilon) \int_0^t \mathcal{K}(s, y_n(s,\upsilon), \upsilon)\, ds \quad \text{a.e.}$$

For all $t \in R_+$ with $y_0 = b$

Proof: Let $S = BC(R_+, R)$ be separable Banach space. Define the operator $\mathcal{P}, \mathcal{Q} : \Omega \times S \to S$ as

$$\mathcal{P}(\upsilon)x(t) = \mathcal{J}(t, x(t,\upsilon), \upsilon)$$

$$\mathcal{Q}(\upsilon)x(t) = \int_0^t \mathcal{K}(s, x(s,\upsilon), \upsilon)\, ds$$

Then the random integral equation equivalent to the random operator

$$x(t,\upsilon) = \mathcal{P}(\upsilon)x(t)\ \ \mathcal{Q}(\upsilon)x(t)$$

For all $t \in R_+$

Show that random operator $\mathcal{P}(\upsilon)$ and $\mathcal{Q}(\upsilon)$ satisfies theorem (5.5). In the theorem (4.3) we have proved that $\mathcal{P}(\upsilon)$ is Lipschitz and $\mathcal{Q}(\upsilon)$ is completely continuous random operator,

Here Lipschitz constant of $\mathcal{P}(\upsilon)$ is $\|\alpha(\upsilon)\|$ and $M(\upsilon) = \|\ \mathcal{Q}(\upsilon)([a,b])\|$

To prove that $\mathcal{P}(\upsilon)$ and $\mathcal{Q}(\upsilon)$ are monotone non-decreasing on $[a,b]$

Let $x, y \in [a,b]$ be such that $x \leq y$ then

$$\mathcal{P}(\upsilon)x(t) = \mathcal{J}(t, x(t,\upsilon), \upsilon)$$
$$\leq \mathcal{J}(t, y(t,\upsilon), \upsilon)$$
$$\leq \mathcal{P}(\upsilon)y(t)$$

For all $t \in R_+$

Similarly,

$$\mathcal{Q}(\upsilon)x(t) = \int_0^t \mathcal{K}(s, x(s,\upsilon), \upsilon) ds$$
$$\leq \int_0^t \mathcal{K}(s, y(s,\upsilon), \upsilon)\, ds$$
$$\leq\ \mathcal{Q}(\upsilon)y(t)$$

For all $t \in R_+$

Thus, random operators $\mathcal{P}(\upsilon)$ and $\mathcal{Q}(\upsilon)$ are monotone non-decreasing operators on $[a,b]$ and by hypothesis B_5, a is lower random solution to FNRDE (2.1) we have

$$a(t,\upsilon)\ \leq \mathcal{J}(t, a(t,\upsilon), \upsilon) \int_0^t \mathcal{K}(s, a(s,\upsilon), \upsilon) ds$$

$$a(t,\upsilon)\ \leq \mathcal{P}(\upsilon)a(t)\ \ \mathcal{Q}(\upsilon)a(t)$$

For all $t \in R_+$ and $\upsilon \in \Omega$

b is upper random solution of FNRDE (2.1) we have

$$\mathcal{J}(t, b(t,\upsilon), \upsilon) \int_0^t \mathcal{K}(s, b(s,\upsilon), \upsilon) ds\ \leq b(t,\upsilon)$$

$$\mathcal{P}(\upsilon)b(t)\ \ \mathcal{Q}(\upsilon)b(t) \leq b(t,\upsilon)$$

For all $t \in R_+$ and $\upsilon \in \Omega$

$$\therefore \mathcal{P}(\upsilon)x(t)\ \ \mathcal{Q}(\upsilon)x(t) \in [a,b]$$

For all $x \in [a,b]$

Now,

$$M(\upsilon) = \|\ \mathcal{Q}(\upsilon)([a,b])\|$$

$$
\begin{aligned}
&= \sup_{x\in[a,b]} \{\| \mathcal{Q}(\upsilon)x\| : x \in [a,b]\} \\
&= \sup_{x\in[a,b]} \left\{ \sup_{t\in R_+} | \mathcal{Q}(\upsilon)x(t)| \right\} \\
&= \sup_{x\in[a,b]} \left\{ \sup_{t\in R_+} \left| \int_0^t \mathcal{K}(s, x(s,\upsilon), \upsilon) ds \right| \right\} \\
&\le \sup \left\{ \sup_{t\in R_+} \int_0^t |\mathcal{K}(s, x(s,\upsilon), \upsilon)|\, ds \right\} \\
&\le \sup \left\{ \sup_{t\in R_+} \int_0^t q(s,\upsilon) ds \right\} \\
&\le \sup \left\{ \sup_{t\in R_+} \mathcal{V}(t,\upsilon) \right\} \\
&\le K_1
\end{aligned}
$$

And therefore $\|\alpha(\upsilon)\| M(\upsilon) = \|\alpha(\upsilon)\| K_1 < 1$

For each $\upsilon \in \Omega$

Hence by theorem (5.5) FNRDE (2.1) has a minimal random solution x_m and a maximal random solution x_M in $[a,b]$

Moreover,

$$x_m(\upsilon) = \lim_{n\to\infty} x_n(\upsilon) \quad ;$$

$$x_M(\upsilon) = \lim_{n\to\infty} y_n(\upsilon)$$

Where $\{x_{n+1}(\upsilon)\}$ and $\{y_{n+1}(\upsilon)\}$ are defined by theorem.

6. EXAMPLE

Let $\upsilon = (-\infty, -1)$ be equipped with the usual σ-algebra consisting of Lebesgue measurable subset of $(-\infty, -1)$ and let $S = BM\,(R_+, R)$. Given measurable function $x: \Omega \to S$ consider the following FNRDE

$$D\left(\frac{x(t,\upsilon)}{\frac{e^{-t}x(t,\upsilon)}{5\,\upsilon\left(1 + x^2(t,\upsilon)\right)}} \right) = \frac{\upsilon\, e^{-t}x^2(t,\upsilon)}{7(1+\upsilon^2)(1+x^2(t,\upsilon))} \tag{6.1}$$

With $x(0,\upsilon) = 0$

Solution: Here,

$$\mathcal{J}(t,x,\upsilon) = \frac{e^{-t}x(t,\upsilon)}{5\,\upsilon\left(1 + x^2(t,\upsilon)\right)}$$

$$\mathcal{K}(t,x,\upsilon) = \frac{\upsilon\, e^{-t}x^2(t,\upsilon)}{7(1+\upsilon^2)(1+x^2(t,\upsilon))}$$

Clearly the map $\upsilon \to \mathcal{J}(t,x,\upsilon)$ and $(t,\upsilon) \to \mathcal{K}(t,x,\upsilon)$ also measurable a.e. for $t \in R_+$ and $\upsilon \in \Omega$. The function $\mathcal{K}$ is caratheodory on $R_+ \times R \times \Omega$.

To hypothesis H_2,

$$|\mathcal{J}(t,x(t,\upsilon),\upsilon)-\mathcal{J}(t,y(t,\upsilon),\upsilon)| = \left|\frac{e^{-t}x(t,\upsilon)}{5\,\upsilon\left(1\,+\,x^2(t,\upsilon)\right)}-\frac{e^{-t}y(t,\upsilon)}{5\,\upsilon\left(1\,+\,y^2(t,\upsilon)\right)}\right|$$

$$=\left(\frac{e^{-t}}{5\,\upsilon}\right)\left|\frac{x(t,\upsilon)}{\left(1\,+\,x^2(t,\upsilon)\right)}-\frac{y(t,\upsilon)}{\left(1\,+\,y^2(t,\upsilon)\right)}\right|$$

$$\le\left(\frac{e^{-t}}{5\,\upsilon}\right)|x(t,\upsilon)-y(t,\upsilon)|$$

Here, $\alpha(t,\upsilon)=\frac{e^{-t}}{5t\,\upsilon}$

$$\|\alpha(\,\upsilon)\| = \sup\left|\frac{e^{-t}}{5\,\upsilon}\right| \le \frac{1}{5} < 1$$

Therefore $\|\alpha(\,\upsilon)\| < 1$ and $|\mathcal{J}(t,x(t,\upsilon),\upsilon)| = \left|\frac{e^{-t}x(t,\upsilon)}{5\,\upsilon\left(1\,+\,x^2(t,\upsilon)\right)}\right| \le \left|\frac{e^{-t}}{5\,\upsilon}\right|$

Hence, $\lim\limits_{t\to\infty}\phi(t,\upsilon) = \lim\limits_{t\to\infty}\left|\frac{e^{-t}}{5\,\upsilon}\right| = 0$

$$\lim_{t\to\infty}\ \mathcal{J}(t,x(t,\upsilon),\upsilon) = \lim_{t\to\infty}\ \frac{e^{-t}x(t,\upsilon)}{5\,\upsilon\left(1\,+\,x^2(t,\upsilon)\right)} = 0$$

Now to hypothesis H_4,

$$|\mathcal{K}(t,x(t,\upsilon),\upsilon)| = \left|\frac{\upsilon\,e^{-t}x^2(t,\upsilon)}{7(1+\upsilon^2)(1+x^2(t,\upsilon))}\right| \le \frac{e^{-t}}{7}$$

$$q(t,\upsilon) = \frac{e^{-t}}{7}$$

Now,

$$\mathcal{V}(t,\upsilon) = \int_0^t q(s,\upsilon)ds$$

$$= \int_0^t \frac{e^{-s}}{7}ds$$

$$= \frac{1}{7}\int_0^t e^{-s}\,ds = \frac{1}{7}[1-e^{-t}]$$

$$K_1 = \sup\{\mathcal{V}(t,\upsilon): t\in R_+ \,\&\, \upsilon\in\Omega\}$$

$$= \sup\left\{\frac{1}{7}[1-e^{-t}]: t\in R_+ \,\&\, \upsilon\in\Omega\right\}$$

$$= \frac{1}{7} > 0$$

Therefore, $\|\alpha(\,\upsilon)\|K_1 < 1$

Hence by theorem (4.3) FNRDE (6.1) has random solutions on R_+.

REFERENCES

[1] Tamer A. Abassy. Improved adomian decomposition method. Computers Mathematics with Applications, 59(1): 42–54, 2010.

[2] Saeid Abbasbandy. Extended newton's method for a system of nonlinear equations by modified adomian decomposition method. Applied Mathematics and Computation, 170(1): 648–656, 2005.

[3] J´ozef Bana´s and Bapurao C Dhage. Global asymptotic stability of solutions of a functional integral equation. Nonlinear Analysis: Theory, Methods & Applications, 69(7): 1945–1952, 2008.

[4] T A Burton and Tetsuo Furumochi. A note on stability by schauder's theorem. Funkcialaj Ekvacioj Serio Internacia, 44(1): 73–82, 2001.

[5] B C Dhage, S V Badgire, and Henderson. Second order nonlinear random differential equations. Differential Equations and Applications, 2(1): 53–65, 2010.

[6] B. C. Dhage and S. S. Bellale. Local asymptotic stability for nonlinear quadratic functional integral equations. Electronic Journal of Qualitative Theory of Differential Equations, 10: 1-13, 2008.

[7] BC Dhage. Random fixed point theorems in banach algebras with applications to random integral equations. Tamkang journal of Mathematics, 34(1): 29–44, 2003.

[8] BC Dhage. Existence theory for first order functional random differential equations. Nonlinear Functional Analysis and Applications, 21(2): 227–246, 2016.

[9] BC Dhage. A random version of a schaefer type fixed point theorem with applications to functional random integral equations. Nonlinear Functional Analysis and Applications, 21(3): 389–403, 2016.

[10] Kai Diethelm and NJ Ford. The analysis of fractional differential equations.Lecture notes in mathematics, 2004: 2010.

[11] D. Guo and V. Lakshmikantham. Nonlinear Problems in Abstract Cones. Developmental Psychology Series. Academic Press, 1988.

[12] Xiaoling Hu and Jurang Yan. The global attractivity and asymptotic stability of solution of a nonlinear integral equation. Journal of Mathematical Analysis and Applications, 321(1): 147–156, 2006.

[13] Chayut Kongban, Poom Kumam, Juan Mart´ınez-Moreno, Antonio Francisco Rold´an, and Rafael L´opez de Hierro. On random fixed point for generalizations of suzuki's type with application to stochastic dynamic programming. Results in Fixed Point Theory and Applications, 27: 1-19,2019.

[14] Alfred H. Mishi, Arigu I. Sabari, Deborah A. Amos, Chinedu F. Egbogu, Charity A.Kuje, and John O. Ojosipe. Application of differential equations in physics. Global Scientific Journal, 8(9): 757–773, 2020.

[15] Sahar Mohsen and Ahmed Hussain. A move recent review of the integral equations and their applications. Journal of Physics: Conference Series, 1818:012170, 03 2021.

[16] Santanu Saha Ray and Prakash Kumar Sahu. Novel methods for solving linear and nonlinear integral equations. Chapman and Hall/CRC, 2018.

[17] A. A. Shabana. Solution of the Vibration Equations, Springer US, New York, NY, 27–53 1991.

[18] Neetu Vishwakarma. Common random fixed point theorem for compatible random multivalued operators. IOSR Journal of Mathematics, 3: 39–43, 2012.

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DIGITAL IMAGE PROCESSING VIA REMOTE SENSING

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ABSTRACT

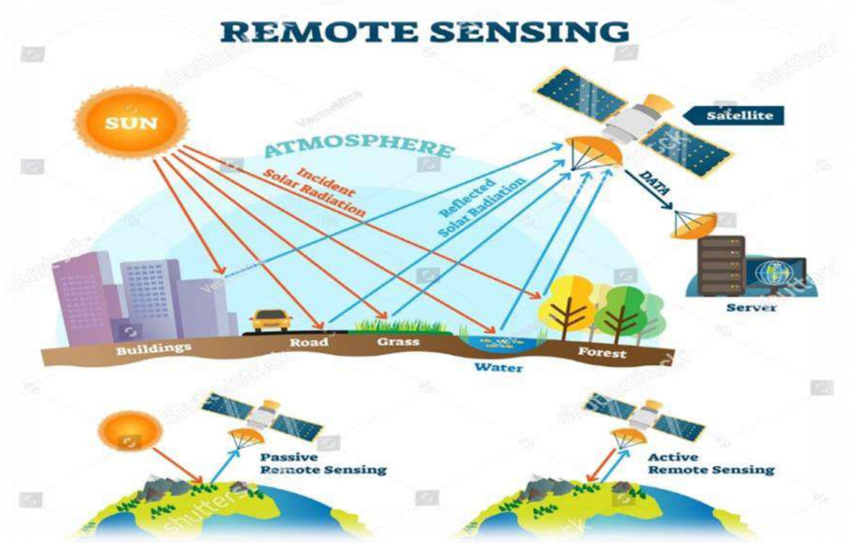
Digital Image Processing (DIP) has become an essential tool for analyzing and interpreting data obtained from remote sensing systems. Remote sensing technologies use satellite and airborne sensors to collect information about the Earth's surface without direct physical contact. The large amount of image data generated from these systems requires efficient processing techniques for meaningful interpretation. This research paper discusses the fundamental concepts of digital image processing and its application in remote sensing. It highlights image acquisition, enhancement, and analysis techniques used to extract useful information from satellite imagery. The study also explains different types of image resolutions and their role in remote sensing analysis. Furthermore, the paper examines the challenges associated with processing large volumes of remote sensing data and emphasizes the importance of advanced computational techniques for effective analysis. The findings indicate that digital image processing plays a vital role in environmental monitoring, disaster management, resource exploration, and land-use analysis.

Keywords: *Digital Image Processing, Remote Sensing, Satellite Imagery, Image Enhancement, Image Resolution*

1. INTRODUCTION

Remote sensing is a technology used to collect information about the Earth's surface without direct physical contact. It relies on sensors mounted on satellites or aircraft to capture images and other data related to land, water, vegetation, and atmospheric conditions. With the rapid advancement of satellite technology, enormous volumes of image data are generated daily. These images must be processed and analyzed to extract meaningful information.

Digital Image Processing (DIP) refers to the use of computer algorithms to manipulate and analyze digital images. It involves techniques such as image enhancement, filtering, segmentation, and feature extraction. These methods help improve image quality and enable the identification of patterns and objects in images.



In remote sensing applications, digital image processing plays a critical role in converting raw satellite data into useful information. It helps researchers, scientists, and decision-makers analyze environmental changes, monitor natural disasters, and manage natural resources effectively. Therefore, the integration of digital image processing with remote sensing has become a fundamental component of modern geospatial analysis.

2. LITERATURE REVIEW

Several researchers have studied the role of digital image processing in remote sensing applications. Gonzalez and Woods (2018) emphasized the importance of digital image processing techniques in improving image quality and extracting useful features from satellite images. Their work provides a comprehensive overview of image processing methods including filtering, segmentation, and pattern recognition.

Jensen (2016) discussed the fundamental principles of remote sensing and explained how satellite sensors capture electromagnetic radiation from the Earth’s surface. The study highlighted the importance of spatial, spectral, radiometric, and temporal resolution in remote sensing imagery.

Richards and Jia (2013) focused on image analysis techniques used in remote sensing data interpretation. Their research explained classification methods, image enhancement techniques, and the role of machine learning algorithms in remote sensing image analysis.

Recent research has also emphasized the use of deep learning techniques for semantic segmentation and object detection in satellite imagery. These advanced approaches significantly improve the accuracy of land-use classification, urban mapping, and environmental monitoring.

Overall, existing studies confirm that digital image processing is essential for interpreting remote sensing data and extracting meaningful information from complex datasets.

3. METHODOLOGY

The methodology of digital image processing in remote sensing generally follows a structured workflow involving several stages. These stages include image acquisition, preprocessing, enhancement, analysis, and interpretation.

3.1 Image Acquisition

Image acquisition is the first step in remote sensing. Satellite sensors capture electromagnetic radiation reflected or emitted from the Earth’s surface. These sensors convert the captured signals into digital images that can be processed by computer systems.

3.2 Image Preprocessing

Preprocessing involves correcting distortions and noise present in raw satellite images. Techniques such as radiometric correction, geometric correction, and noise removal are applied to improve image quality.

3.3 Image Enhancement

Image enhancement techniques improve the visual appearance of images and highlight important features. Methods such as contrast enhancement, histogram equalization, and edge enhancement help reveal hidden information in satellite images.

3.4 Image Analysis

Image analysis involves extracting meaningful information from processed images. Techniques such as segmentation, classification, and feature extraction are used to identify objects, land cover patterns, and environmental changes.

3.5 Interpretation and Application

After processing and analysis, the extracted information is interpreted by experts to support decision-making in fields such as agriculture, environmental monitoring, and disaster management.

4. IMAGE CHARACTERISTICS AND RESOLUTION IN REMOTE SENSING

Remote sensing images are characterized by different types of resolution that determine the quality and usability of the data.

4.1 Spatial Resolution

Spatial resolution refers to the size of the smallest object that can be detected in an image. Higher spatial resolution provides more detailed images.

4.2 Spectral Resolution

Spectral resolution represents the ability of a sensor to detect different wavelengths of electromagnetic radiation. Higher spectral resolution allows better identification of materials and land cover types.

4.3 Radiometric Resolution

Radiometric resolution indicates the sensitivity of a sensor to detect slight variations in energy levels. It determines how many shades of brightness can be represented in an image.

4.4 Temporal Resolution

Temporal resolution refers to how frequently a satellite revisits and captures images of the same location on Earth. Higher temporal resolution is useful for monitoring environmental changes over time.

5. APPLICATIONS OF DIGITAL IMAGE PROCESSING IN REMOTE SENSING

Digital image processing has numerous applications in remote sensing, including:

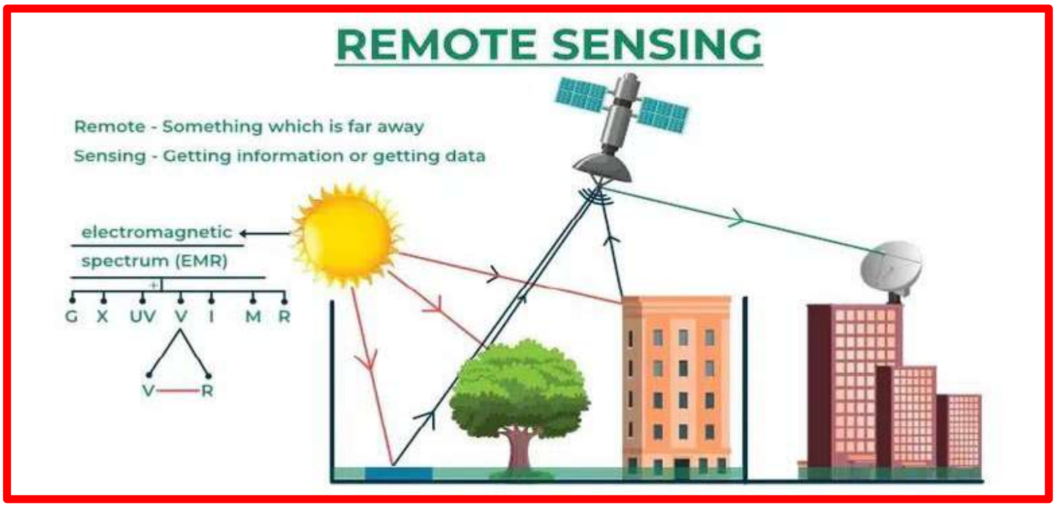
Environmental Monitoring
Satellite imagery helps track deforestation, desertification, and climate change.

Disaster Management
DIP assists in assessing damage caused by floods, earthquakes, and hurricanes.

Agriculture
Remote sensing images help monitor crop health, soil moisture, and irrigation management.

Urban Planning
Digital image analysis helps monitor urban growth and land-use changes.

Resource Exploration
Remote sensing helps identify mineral deposits, oil reserves, and water resources.



6. CHALLENGES IN REMOTE SENSING IMAGE PROCESSING

Despite its advantages, several challenges exist in processing remote sensing data:

- Large volume of satellite data
- Complex data formats and storage requirements
- High computational costs
- Difficulty in interpreting high-dimensional data
- Need for advanced algorithms and machine learning techniques

To overcome these challenges, researchers are increasingly using artificial intelligence, cloud computing, and big data technologies.

7. CONCLUSION

Digital Image Processing plays a crucial role in the effective utilization of remote sensing data. By applying computational techniques to satellite images, it becomes possible to extract meaningful information about the Earth's surface and environmental conditions. This study highlighted the importance of digital image processing techniques such as image enhancement, segmentation, and classification in remote sensing applications. It also discussed various types of image resolution and their impact on image interpretation. The integration of advanced technologies such as machine learning and artificial intelligence is further enhancing the capabilities of remote sensing systems. As satellite data continues to grow rapidly, efficient image processing methods will become increasingly important for environmental monitoring, disaster management, and sustainable development.

REFERENCES

Gonzalez, R. C., & Woods, R. E. (2018). Digital Image Processing (4th ed.). Pearson Education.

Jensen, J. R. (2016). Introductory Digital Image Processing: A Remote Sensing Perspective. Pearson.

Richards, J. A., & Jia, X. (2013). Remote Sensing Digital Image Analysis. Springer.

Schowengerdt, R. A. (2007). Remote Sensing: Models and Methods for Image Processing. Academic Press.

Lillesand, T., Kiefer, R. W., & Chipman, J. (2015). Remote Sensing and Image Interpretation. Wiley.

GeeksforGeeks. (2023). Digital Image Processing Tutorial.
https://www.geeksforgeeks.org/electronics-engineering/digital-image-processing-tutorial/

SPIE Digital Library. (2023). Remote Sensing and Image Processing Resources.
https://ebooks.spiedigitallibrary.org

A DISTRIBUTION - THEORETIC APPROACH TO UNIVALENT ANALYTIC FUNCTIONS IN THE UNIT DISK

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ABSTRACT

This paper develops a theoretical framework connecting univalent analytic functions with generalized distribution series generated by probability measures on the unit interval. By constructing analytic functions through distribution-based coefficient structures, we establish sufficient conditions for univalence in the unit disk using classical criteria such as the Noshiro–Warschawski theorem and coefficient bounds inspired by the Bieberbach conjecture (proved by Louis de Branges in 1985). We introduce a new class of distribution-induced analytic functions, derive coefficient inequalities, provide extremal constructions, and discuss implications for geometric function theory and probabilistic function modeling.

Keywords: Univalent functions, analytic functions, distribution series, coefficient bounds, geometric function theory.

1. INTRODUCTION

The theory of univalent (injective) analytic functions plays a central role in geometric function theory. Let

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

Denote the open unit disk. A function $f : \mathbb{D} \to \mathbb{C}$ is univalent if it is analytic and injective.

The study of normalized univalent functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

Was revolutionized by the Bieberbach conjecture (1916), resolved by Louis de Branges, which established that

$$|a_n| \leq n$$

In parallel, probability theory often represents distributions via generating functions:

$$G(z) = \sum_{n=0}^{\infty} p_n z^n$$

Where $\{p_n\}$ is a probability distribution.

This paper investigates analytic functions whose Taylor coefficients arise from distribution sequences and establishes conditions under which such functions are univalent.

2. PRELIMINARIES

2.1 Normalized Univalent Functions

Let $\mathcal{S}$ denote the class of functions:

$$\mathcal{S} = \left\{ f(z) = z + \sum_{n=2}^{\infty} a_n z^n : f \text{ analytic and univalent in } \mathbb{D} \right\}$$

Important subclasses include:

- Starlike functions $\mathcal{S}^*$
- Convex functions $\mathcal{K}$

2.2 Distribution Series

Let $\{p_n\}_{n=1}^{\infty}$ satisfy:

$$p_n \geq 0, \quad \sum_{n=1}^{\infty} p_n = 1$$

Define the distribution series generating function

$$P(z) = \sum_{n=1}^{\infty} p_n z^n$$

We construct a related analytic function:

$$f(z) = z + \sum_{n=2}^{\infty} \alpha_n p_{n-1} z^n$$

Where $\{\alpha_n\}$ is a real sequence controlling geometric growth.

3. DISTRIBUTION-INDUCED UNIVALENT FUNCTION CLASS

Definition 3.1

Let $\mathcal{S}_D(\alpha)$ denote the class of functions:

$$f(z) = z + \sum_{n=2}^{\infty} \alpha_n p_{n-1} z^n$$

Where:

1. $\{p_n\}$ is a probability distribution,
2. $\alpha_n \geq 0$,
3. $\alpha_n\, p_{n-1} \leq$ n.

4. MAIN RESULTS

Theorem 4.1 (Coefficient-Based Univalence Criterion)

Let

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

With

$a_n = \alpha_n p_{n-1}$

If

$$\sum_{n=2}^{\infty} n|a_n| \leq 1$$

Then f is univalent in $\mathbb{D}$.

Proof :

Consider

$$f'(z) = 1 + \sum_{n=2}^{\infty} n a_n z^{n-1}$$

For

$$|f'(z) - 1| \leq \sum_{n=2}^{\infty} n|a_n|$$

If

$$\sum_{n=2}^{\infty} n|a_n| \leq 1$$

Then

$$\operatorname{Re} f'(z) > 0$$

By the Noshiro–Warschawski theorem, f is univalent in $\mathbb{D}$.

Corollary 4.2

If

$$\sum_{n=2}^{\infty} n\alpha_n p_{n-1} \leq 1$$

Then $f \in \mathcal{S}_D(\alpha)$

5. SPECIAL CASE: GEOMETRIC DISTRIBUTION

Let

$$p_n = (1-q)q^{n-1}, \quad 0 < q < 1$$

Then

$$a_n = \alpha_n(1-q)q^{n-2}$$

If $\alpha_n = n$, then

$$\sum_{n=2}^{\infty} n^2(1-q)q^{n-2}$$

converges for all $0 < q < 1$

Thus for sufficiently small , the univalence condition holds.

6. GROWTH AND DISTORTION ESTIMATES

For $f \in \mathcal{S}_D(\alpha)$

$$|f(z)| \leq |z| + \sum_{n=2}^{\infty} \alpha_n p_{n-1}|z|^n$$

If $\alpha_n \leq C$, then

$$|f(z)| \leq |z| + C|z|^2$$

Similarly, distortion bounds follow from classical results in $\mathcal{S}$.

7. EXTREMAL PROBLEMS

Define the functional:

$$J(f) = a_2 = \alpha_2 p_1$$

Maximization under the constraint

$$\sum_{n=2}^{\infty} n\alpha_n p_{n-1} \leq 1$$

Occurs when all mass concentrates at $n = 2$, giving:

$$\max J(f) = \frac{1}{2}$$

Connections to Geometric Function Theory

This construction relates distribution series to:

- Subordination principles
- Convex combinations of analytic mappings
- Probabilistic interpretations of coefficient bounds The framework provides a bridge between:
- Probability generating functions
- Univalent function theory
- Coefficient extremal problems

8. CONCLUSION

We introduced a new class of analytic functions constructed from probability distribution series and established sufficient coefficient conditions ensuring univalence. The results show that probabilistic weight structures can generate subclasses of univalent functions under appropriate growth control.

Future directions include:

- Sharp coefficient bounds
- Radius problems
- Multivalent extensions
- Connections with fractional calculus operators

REFERENCES

1. De Branges L. (1985). A proof of the Bieberbach conjecture. Acta
2. Duren P. L. (1983). Univalent functions. Springer-Verlag.
3. Pommerenke, C. (1975). Univalent Functions.
4. Goodman, A. W. (1983). Univalent Functions.
5. Kaplan W. (1952). Close-to-convex schlicht functions. Michigan Mathematical Journal
6. Kazimoglu S. (2021). Partial sums of the Miller–Ross function. Turkish Journal of Science
7. Littlewood J. E. (1925). On inequalities in the theory of functions.
8. Nehari Z. (1952). Conformal mapping. McGraw–Hill.
9. Sakaguchi K. (1959). On a certain univalent mappings. Journal of the Mathematical Society of Japan
10. Nisar K. S., Baleanu, D. & Qurashi M. A. (2016). Fractional calculus and application of generalized Struve function

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LOCALLY ATTRACTIVE SOLUTION FOR NONLINEAR QUADRATIC DIFFERENTIAL EQUATION IN BANACH ALGEBRA

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ABSTRACT

In this paper, we study the existence the solution and existence the locally attractive solution for an arbitrary order quadratic differential equation in Banach algebras under Lipschitz and Caratheodory conditions using a hybrid fixed point theorem.

1. INTRODUCTION

The theory of fractional calculus (that is fractional order differential and integral equation) has newly received a lot of attention and establishes a meaningful branch of nonlinear analysis [9,10,14,15] Number of research monographs and research papers has appeared to contribute to integrals and differential equation of fractional order which now constitutes a significant branch of nonlinear analysis [8, 10, 20, 3]. In the last few decades, however fractional differentiation proved very useful in various fields of applied sciences and engineering [11,17,22,25] and [5]. Fractional differential equation rise in the mathematical modelling of system and process occurring scientific disciplines such as physics, chemistry, biology, economics, signal and image processing, feedback amplifier and electric circuits [6.7.23,24]. Numerous research papers and monographs devoted to differentialand integral equation of fractional order have.

In this chapter we study the existence of locally attractive solution and existence of extremal solution of the following fractional order quadratic differential equation.

$$D^2\left(\frac{x(t)-h(t,x(t))}{f(t,x(t))}\right)=g(t,x(t)) \text{ a.e. } t\in\mathbb{R}_+ \qquad (1)$$

$$x(0)=x_0 \in\mathbb{R}_+$$

where $t\in\mathbb{R}_+=[0,\infty)$ and $0<\xi<1$, $g:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$, $f:\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $h:\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are functions which satisfy special assumptions.

2. PRELIMINARIES

In this section we give the definitions, notation, hypothesis and preliminary tools, which will be needed in the sequel.

Let $\mathbb{X}=AC(\mathbb{R}_+,\mathbb{R})$ be the space of absolutely continuous function on $\mathbb{R}_+$ and S be a subset of $\mathbb{X}$. Let a mapping $\mathbb{A}:\mathbb{X}\to\mathbb{X}$ be an operator and consider the following operator equation in $\mathbb{X}$, namely,

$$x(t)=(\mathbb{A}x)(t), \text{ for all } t\in\mathbb{R}_+ \qquad (2.1)$$

Below we give some different characterization of the solutions for operator equation (2.1) on $\mathbb{R}_+$. We need the following definitions.

Definition2. 1[12, 13, 22, 23, 25]: We say that solution of the equation (2.1) is locally attractive if there exists a closed ball $\overline{B_r(0)}$ in the space $AC(\mathbb{R}_+,\mathbb{R})$for some $x_0\in AC(\mathbb{R}_+,\mathbb{R})$and for some real number $r>0$ such that for arbitrary solution $x=x(t)$ and $y=y(t)$ of equation (2.1) belonging to $\overline{B_r(0)}\cap S$ we have that, $\lim_{t\to\infty}(x(t)-y(t))=0$

Theorem 2.2: **[4,16,18,23,25]:** Let S be a non-empty, convex, closed and bounded subset of the Banach space $\mathbb{X}$ and let $\mathbb{A},\mathbb{C}:\mathbb{X}\to\mathbb{X}$ and $\mathbb{B}:S\to\mathbb{X}$ are two operators satisfying:

a) $\mathbb{A}$ and $\mathbb{C}$ are Lipschitzian with lipschitz constants ζ,η respectively.

b) $\mathbb{B}$ is completely continuous, and

c) $x=\mathbb{A}x\mathbb{B}y+\mathbb{C}, x\in S$ for all $y\in S$

d) $\zeta M+\eta<1$ where $M=\|\mathbb{B}(s)\|=\sup\{\|\mathbb{B}x\|:x\in S\}$

Then the operator equation $x=\mathbb{A}x\mathbb{B}x+\mathbb{C}x$ has a solution in S

3. EXISTENCE THEORY

Definition 3.1[22, 23, 24]: A mapping $\sigma: \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is Caratheodory if:

i) $t \to \sigma(t, x, y)$ is measurable for each $x, y \in \mathbb{R}$ and

ii) $(x, y) \to \sigma(t, x, y)$is continuous almost everywhere for $t \in \mathbb{R}_+$.

Furthermore, a Caratheodary function σ is $\mathcal{L}^1$ −Caratheodary if:

iii) For each real number $r > 0$ there exists a function $h_r \in \mathcal{L}^1(\mathbb{R}_+, \mathbb{R})$ such that$|\sigma(t, x, y)| \le h_r(t)$ a.e. $t \in \mathbb{R}_+$ for all $x \in \mathbb{R}$ with $|x|_r \le r$ and $|y|_r \le r$.

Finally, a caratheodary function σ is $\mathcal{L}^1_{\mathbb{X}}$ −caratheodary if:

iv) There exists a function $h \in \forall\mathcal{L}^1(\mathbb{R}_+, \mathbb{R})$ such that $|\sigma(t, x, y)| \le h(t)$, a.e. $t \in \mathbb{R}_+$ for all $x, y \in \mathbb{R}$

For convenience, the function h is referred to as a bound function for σ.

4 MAIN RESULT

4.1 Existence the solution of SNQDE (1)

Lemma 4.1: Suppose that $\xi \in (0,1)$ and the function f, g satisfying SNQDE (1.1) then x is the solution of the SNQDE (1.1) if and only if it is the solution of integral equation

$$x(t) = h\big(t, x(t)\big) + f\big(t, x(t)\big)\left[\frac{x_0 - h(0, x_0)}{f(0, x_0)} + \int_0^t (t-s) g\big(s, x(s)\big) ds\right], t \in \mathbb{R}_+ \quad (4.1)$$

Proof: Integrating equation (1.1) to second order, we get

$$I^2 D^2 \left(\frac{x(t) - h\big(t, x(t)\big)}{f\big(t, x(t)\big)}\right) = I^2 g\big(t, x(t)\big)$$

$$\left[\frac{x(t) - h\big(t, x(t)\big)}{f\big(t, x(t)\big)}\right]_0^t = \int_0^t \int_0^t g\big(s, x(s)\big) ds\, ds$$

$$\frac{x(t) - h\big(t, x(t)\big)}{f\big(t, x(t)\big)} - \frac{x(0) - h\big(0, x(0)\big)}{f\big(0, x(0)\big)} = \int_0^t \int_0^t g\big(s, x(s)\big) ds\, ds$$

Since $\int_0^t f(t) dt^n = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s) ds$, Where $n = 0,1,2,3, \dots\dots\dots$

$$\frac{x(t) - h\big(t, x(t)\big)}{f\big(t, x(t)\big)} - \frac{x(0) - h\big(0, x(0)\big)}{f\big(0, x(0)\big)} = \frac{1}{(2-1)!} \int_0^t (t-s) g\big(s, x(s)\big) ds$$

$$\frac{x(t) - h\big(t, x(t)\big)}{f\big(t, x(t)\big)} = \frac{x_0 - h(0, x_0)}{f(0, x_0)} + \int_0^t (t-s) g\big(s, x(s)\big) ds$$

$$x(t) = h\big(t, x(t)\big) + f\big(t, x(t)\big)\left(\frac{x_0 - h(0, x_0)}{f(0, x_0)} + \int_0^t (t-s) g\big(s, x(s)\big) ds\right)$$

Conversely differentiate (4.1) twice w.r.to t, we get,

$$x(t) = h\big(t, x(t)\big) + f\big(t, x(t)\big)\left[\frac{x_0 - h(0, x_0)}{f(0, x_0)} + \int_0^t (t-s) g\big(s, x(s)\big) ds\right]$$

Can be written as,

$$\frac{x(t) - h\big(t, x(t)\big)}{f\big(t, x(t)\big)} = \left[\frac{x_0 - h(0, x_0)}{f(0, x_0)} + \int_0^t (t-s) g\big(s, x(s)\big) ds\right]$$

Differentiating this equation

$$D\left[\frac{x(t) - h\big(t, x(t)\big)}{f\big(t, x(t)\big)}\right] = D\left[\frac{x_0 - h(0, x_0)}{f(0, x_0)} + \int_0^t \int_0^t g\big(s, x(s)\big) ds\right]$$

$$D\left[\frac{x(t)-h\big(t,x(t)\big)}{f\big(t,x(t)\big)}\right]=\int_0^t g\big(s,x(s)\big)ds$$

Again, differentiating above equation

$$D^2\left[\frac{x(t)-h\big(t,x(t)\big)}{f\big(t,x(t)\big)}\right]=g\big(s,x(t)\big)$$

hence x is the solution of the SNQDE (1.1) if and only if it is the solution of integral equation $x(t)=h\big(t,x(t)\big)+f\big(t,x(t)\big)\left[\frac{x_0-h(0,x_0)}{f(0,x_0)}+\int_0^t(t-s)g\big(s,x(s)\big)ds\right]t\in\mathbb{R}_+$

We shall study the existence of solution for the (SNQDE) (1.1) under the following general assumptions:

($\mathcal{H}_1$) The function $f:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}-\{0\}$ is continuous and bounded with bound $\mathbb{F}=\sup_{(t,x)}\left|f\big(t,x(t)\big)\right|$, there exist a bounded function $\zeta:\mathbb{R}_+\to\mathbb{R}$ with bound $\|\zeta\|$ such that, $\left|f\big(t,x(t)\big)-f\big(t,y(t)\big)\right|\leq\xi(t)|x(t)-y(t)|,\forall x,y\in\mathbb{R}$

($\mathcal{H}_2$) The function $h:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ is measurable in (t,s) for any $x\in\mathbb{R}$ and continuous in x for almost all (t,s) and $\mathbb{H}=\sup_{(t,x)}|h(t,x)|$, there exist a bounded function $\eta:\mathbb{R}_+\to\mathbb{R}$ with bound $\|\eta\|$ such that,

$\left|h\big(t,x(t)\big)-h\big(t,y(t)\big)\right|=\eta(t)|x(t)-y(t)|,\forall x,y\in\mathbb{R}$ and it vanishes at infinity.

($\mathcal{H}_3$) The function $g:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ continuous and Caratheodory then there exist a function $p:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ with bound $\|p\|$ such that $\left|g\big(t,x(t)\big)\right|\leq p(t,s)\ \forall t\in\mathbb{R}_+$.

($\mathcal{H}_4$) The function $\rho_1:\mathbb{R}_+\to\mathbb{R}_+$ defined by the formula $\rho_2(t)=\int_0^t(t-s)g\big(s,x(s)\big)ds$ bounded on $\mathbb{R}_+$ and $\lim_{t\to\infty}\rho(t)=0$ that is vanishing at infinity.

Remark (4.1): Note that the hypothesis **($\mathcal{H}_1$)**$-$**($\mathcal{H}_4$)** hold then there exist a constant function $\mathcal{K}_2>0$ such that $\mathcal{K}_1=\sup_{t\geq0}\rho_2(t)$

Theorem (4.2): Assume that the hypothesis **($\mathcal{H}_1$)**$-$**($\mathcal{H}_4$)** holds, further if $\|\zeta\|\left(\left|\frac{x_0-h(0,x_0)}{f(0,x_0)}\right|+\mathcal{K}_2\right)+\|\eta\|<1$ then SNQDE (1.1) has locally attractive solution on the Banach space $\mathbb{X}=AC(\mathbb{R}_+,\mathbb{R})$.

Proof: Define a non- empty, convex, closed and bounded subset S of Banach space $\mathbb{X}=AC(\mathbb{R}_+,\mathbb{R})$ as $S=\{x\in\mathbb{X}:\|x\|\leq r\}$, where r satisfies the inequality $\|\zeta\|\left(\left|\frac{x_0-h(0,x_0)}{f(0,x_0)}\right|+\mathcal{K}_2\right)+\|\eta\|<r$.

Now we define the operators $\mathbb{A}:\mathbb{X}\to\mathbb{X}$ and $\mathbb{B}:S\to\mathbb{X}$ and $\mathbb{C}:\mathbb{X}\to\mathbb{X}$ by,

$\mathbb{A}x(t)=f(t,x(t),),t\in\mathbb{R}_+$ (4.2)

$\mathbb{B}x(t)=\frac{x_0-h(0,x_0)}{f(0,x_0)}+\int_0^t(t-s)g\big(s,x(s)\big)ds,t\in\mathbb{R}_+$ (4.3)

$\mathbb{C}x(t)=h\big(t,x(t)\big),t\in\mathbb{R}_+$ (4.4)

The SNQDE is equivalent to the operator equation

$x(t)=\mathbb{A}x(t)\mathbb{B}x(t)+\mathbb{C}x(t),\forall t\in\mathbb{R}_+$ (4.5)

Now, we will prove that, the operators $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ satisfy all the axioms of theorem (2.1).

Step I: To show that $\mathbb{A}$ and $\mathbb{C}$ are Lipschitzian on $\mathbb{X}$.

For that, let $x,y\in\mathbb{X}$, then by hypothesis **($\mathcal{H}_1$)**, for $t\in\mathbb{R}_+$ we have,

$$\begin{aligned}|\mathbb{A}x(t)-\mathbb{A}y(t)|&=\left|f\big(t,x(t)\big)-f\big(t,y(t)\big)\right|\\&\leq\zeta(t)|x(t)-y(t)|\end{aligned}$$

After taking supremum over t, we get

$\|\mathbb{A}x-\mathbb{A}y\|\leq\|\zeta\|\|x-y\|$ for all $x,y\in\mathbb{R}$.

Therefore, the operator $\mathbb{A}$ is lipschitzian with lipschitz constant $\|\zeta\|$.

Now to show $\mathbb{C}$ is Lipschitzian on $\mathbb{X}$ for any $x,y\in\mathbb{X}$we have

$$|\mathbb{C}x(t)-\mathbb{C}y(t)| = \left|h\big(t,x(t)\big)-h\big(t,y(t)\big)\right|$$
$$\le \eta(t)|x(t)-y(t)|$$

Taking supremum over t, we obtain

$\|\mathbb{C}x-\mathbb{C}y\| \le \|\eta\|\|x-y\|$, for all $x, y \in \mathbb{R}$

Therefore, the operator $\mathbb{C}$ is lipschitzian with lipschitz constant $\|\eta\|$.

Step II: To show that $\mathbb{B}$ is completely continuous on $\mathbb{X}$.

This can be achieved by showing that $\mathbb{B}$ is continuous, uniformly bounded and equicontinuous.

Let $\{x_n\}$ be a sequence in S such that $\{x_n\} \to x$.Then by dominated convergence theorem,

$$\lim_{n\to\infty} \mathbb{B}x_n(t) = \lim_{n\to\infty}\left\{\frac{x_0-h(0,x_0)}{f(0,x_0)}+\int_0^t (t-s)g\big(s,x_n(s)\big)\,ds\right\}$$
$$=\frac{x_0-h(0,x_0)}{f(0,x_0)}+\int_0^t (t-s)g\big(s,x(s)\big)\,ds = \mathbb{B}x(t),, \forall t \in \mathbb{R}_+$$

This shows that $\mathbb{B}x_n$ converges to $\mathbb{B}x$ pointwise on S.

Next to show sequence $\{\mathbb{B}x_n\}$ is a uniformly convergent in S.

Let $t_1, t_2 \in \mathbb{R}_+$be arbitrary with $t_1 < t_2$ then

$$|\mathbb{B}x_n(t_2)-\mathbb{B}x_n(t_1)|$$
$$=\left|\frac{x_0-h(0,x_0)}{f(0,x_0)}+\int_0^{t_2}(t_2-s)g\big(s,x_n(s)\big)\,ds-\left(\frac{x_0-h(0,x_0)}{f(0,x_0)}+\int_0^{t_1}g(t_1-s)\big(s,x_n(s)\big)\,ds\right)\right|$$
$$|\mathbb{B}x_n(t_2)-\mathbb{B}x_n(t_1)| = \left|\int_0^{t_2}(t-s)g\big(s,x_n(s)\big)\,ds-\int_0^{t_1}(t-s)g\big(s,x_n(s)\big)\,ds\right|$$
$$|\mathbb{B}x_n(t_2)-\mathbb{B}x_n(t_1)| = \left|\int_0^{t_2}(t_2-s)g\big(s,x_n(s)\big)\,ds-\int_0^{t_1}(t_1-s)g\big(s,x_n(s)\big)\,ds\right|$$
$$|\mathbb{B}x_n(t_2)-\mathbb{B}x_n(t_1)| \le \left|\int_0^{t_2}(t_2-s)\|p\|\,ds-\int_0^{t_1}(t_1-s)\|p\|\,ds\right|$$
$$|\mathbb{B}x_n(t_2)-\mathbb{B}x_n(t_1)| \le \|p\|\left|\int_0^{t_2}(t_2-s)\,ds-\|p\|\int_0^{t_1}(t_1-s)\,ds\right|$$
$$|\mathbb{B}x_n(t_2)-\mathbb{B}x_n(t_1)| \le \|p\|\left(\left|\int_0^{t_2}(t_2-s)\,ds-\int_0^{t_1}(t_1-s)\,ds\right|\right)$$
$$|\mathbb{B}x_n(t_2)-\mathbb{B}x_n(t_1)| \le \|p\|\left(\frac{{t_2}^2}{2}-\frac{{t_1}^2}{2}\right) \to 0 \text{ as } t_1 \to t_2,\ \forall n \in \mathcal{N}$$

This shows that the sequence converges uniformly, and uniform convergence imply continuity by using this property of uniform convergence we can conclude that $\mathbb{B}$ is continuous on S.

Step III: To show $\mathbb{B}$ is compact operator on S.

For proving this, it is enough to show that $\mathbb{B}$ is uniformly bounded and equicontinuous in S.

First, we will show that $\mathbb{B}$ is uniformly bounded. Let $x \in$ S be arbitrary then

$$|\mathbb{B}x(t)| = \left|\frac{x_0-h(0,x_0)}{f(0,x_0)}+\int_0^t (t-s)g\big(s,x(s)\big)\,ds\right|$$
$$\le \left|\frac{x_0-h(0,x_0)}{f(0,x_0)}\right|+\int_0^t \left|(t-s)g\big(s,x(s)\big)\right|ds$$
$$\le \left|\frac{x_0-h(0,x_0)}{f(0,x_0)}\right|+\rho_2(t)$$

Taking supremum over t, we obtain

$$\|\mathbb{B}x\| \leq \left|\frac{x_0 - h(0,x_0)}{f(0,x_0)}\right| + \sup_{t\geq 0}\rho_2(t) = \left|\frac{x_0 - h(0,x_0)}{f(0,x_0)}\right| + \mathcal{K}_2, \forall\, t \in \mathbb{R}_+$$

Hence $\mathbb{B}$ is uniformly bounded subset of S.

Now to show $\mathbb{B}$ is equicontinuous on S.

Let $t_1, t_2 \in \mathbb{R}_+$ then,

$$|\mathbb{B}x(t_1) - \mathbb{B}x(t_2)| = \left|\frac{x_0 - h(0,x_0)}{f(0,x_0)} + \int_0^{t_1}(t_1 - s)g(s,x(s))\,ds - \left(\frac{x_0 - h(0,x_0)}{f(0,x_0)} + \int_0^{t_2}(t_2 - s)g(s,x(s))\,ds\right)\right|$$

$$|\mathbb{B}x(t_1) - \mathbb{B}x(t_2)| \leq \left|\|p\|\int_0^{t_1}(t_1 - s)\,ds - \|p\|\int_0^{t_2}(t_2 - s)\,ds\right|$$

$$\leq \left|\|p\|\left(\frac{{t_1}^2}{2} - \frac{{t_2}^2}{2}\right)\right| \quad \to 0 \text{ as } t_1 \to t_2$$

Implies that $\mathbb{B}(S)$ is equicontinuous.

Hence $\mathbb{B}$ is compact subset of S. Hence the conclusion, that $\mathbb{B}$ is completely continuous on S.

Step IV: To show $x = \mathbb{A}x(t)\mathbb{B}y(t) + \mathbb{C}x(t) \Longrightarrow x \in S, \forall y \in S$.

Let $x \in \mathbb{X}$, and $y \in S$ such that $x = \mathbb{A}x(t)\mathbb{B}y(t) + \mathbb{C}x(t)$

By assumptions (**$\mathcal{H}_1$-$\mathcal{H}_4$**)

$$|x(t)| = |\mathbb{A}x(t)\mathbb{B}y(t) + \mathbb{C}x(t)|$$

$$\leq |\mathbb{A}x(t)||\mathbb{B}y(t)| + |\mathbb{C}x(t)|$$

$$\leq |f(t,x(t))|\left(\left|\frac{x_0 - h(0,x_0)}{f(0,x_0)}\right| + \int_0^t |(t-s)g(s,y(s))|ds\right) + |h(t,x(t))|$$

$$\leq \mathbb{F}\left(\left|\frac{x_0 - h(0,x_0)}{f(0,x_0)}\right| + \int_0^t |(t-s)g(s,y(s))|ds\right) + \mathbb{H}$$

$$\leq \mathbb{F}\left(\left|\frac{x_0 - h(0,x_0)}{f(0,x_0)}\right| + \rho_2(t)\right) + \mathbb{H}$$

Taking supremum over t on $\mathbb{R}_+$, we obtain $\|x\| \leq \|\xi\|\left(\left|\frac{x_0 - h(0,x_0)}{f(0,x_0)}\right| + \mathcal{K}_2\right) + \|\eta\|$, $\forall x \in S$.

That is, we have, $\|x\| = \|\mathbb{A}x(t)\mathbb{B}y(t) + \mathbb{C}x(t)\| \leq r$, $\forall x \in S$.

Hence assumption(c) of theorem (2.2.2) is proved.

Step V: Finally, we show that $\zeta\mathbb{M} + \eta < 1$ that is condition (d) of theorem (2.2.2) holds.

Since $\mathbb{M} = \|\mathbb{B}(\mathbb{S})\| = sup_{x\in\mathbb{S}}\{sup_{t\in\mathbb{R}_+}|\mathbb{B}x(t)|\}$

$$= sup_{x\in\mathbb{S}}\left\{sup_{t\in\mathbb{R}_+}\left|\frac{x_0 - h(0,x_0)}{f(0,x_0)} + \int_0^t (t-s)g(s,x(s))ds\right|\right\}$$

$$\leq sup_{x\in\mathbb{S}}\left\{sup_{t\in\mathbb{R}_+}\left|\frac{x_0 - h(0,x_0)}{f(0,x_0)}\right| + \left|\int_0^t (t-s)g(s,x(s))ds\right|\right\}$$

$$\leq sup_{x\in\mathbb{S}}\left\{\left|\frac{x_0 - h(0,x_0)}{f(0,x_0)}\right| + \rho_2(t)\right\}$$

$$\leq \left|\frac{x_0 - h(0,x_0)}{f(0,x_0)}\right| + \rho_2(t)$$

and therefore $\zeta\mathbb{M}+\eta = \left(\|\zeta\|\left(\left|\frac{x_0-h(0,x_0)}{f(0,x_0)}\right|+\mathcal{K}_2\right)+\|\eta\|\right)<1,$

Thus, all the conditions of theorem (2.2.2) are satisfied and hence the operator equation $x=\mathbb{A}x\mathbb{B}x+\mathbb{C}x$ has a solution in $\mathbb{S}$.

Step VI: Now to show the solution is locally attractive on $\mathbb{R}_+$. Then we have

$$|x(t)-y(t)| = \left|\begin{array}{l}\left\{[f(t,x(t))]\left(\frac{x_0-h(0,x_0)}{f(0,x_0)}+\int_0^t (t-s)g(s,x(s))\,ds\right)+h(t,x(t))\right\}- \\ \left\{[f(t,y(t))]\left(\frac{x_0-h(0,x_0)}{f(0,x_0)}+\int_0^t (t-s)g(s,y(s))ds\right)+h(t,y(t))\right\}\end{array}\right|$$

$$\leq |f(t,x(t))|\left(\frac{x_0-h(0,x_0)}{f(0,x_0)}+\int_0^t (t-s)g(s,x(s))\,ds\right)+|h(t,x(t))|$$

$$+|f(t,y(t))|\left(\frac{x_0-h(0,x_0)}{f(0,x_0)}+\int_0^t (t-s)g(s,y(s))\,ds\right)+|h(t,y(t))|$$

$$\leq 2\mathbb{F}\left(\frac{x_0-h(0,x_0)}{f(0,x_0)}+\rho_2(t)\right)+2\mathbb{H},\ \forall t\in\mathbb{R}_+$$

Since $\lim_{t\to\infty}\rho_2(t)=0$, $\lim_{t\to\infty}h(t,x(t))=0$

For $\epsilon>0$, there is real number $\mathbb{T}'>0,\mathbb{T}''>0$ such that $\rho_2(t)\leq\frac{\epsilon}{4F}-\frac{x_0-h(0,x_0)}{f(0,x_0)}$ $\forall$ $t\geq\mathbb{T}'$ and $|h(t,x(t))|<\frac{\epsilon}{4}$, $\forall\ t\geq\mathbb{T}''$ if we choose $\mathbb{T}^*=max\{\mathbb{T}',\mathbb{T}''\}$

Then from above inequality it follows that $|x(t)-y(t)|<\epsilon$ for all $t\geq\mathbb{T}^*$.

Hence SNQDE (1.1) has a locally attractive solution on $\mathbb{R}_+$.

This completes the proof.

Existence the extremal solutions for SNQDE (2.1.2)

Following definitions are useful in the forthcoming analysis for proving the existence of extremal solution.

Definition 2.5.8[15, 16, 18, 22, 23]: A function $a\in AC(\mathbb{R}_+,\mathbb{R})$ is called lower solution of (2.1.2) if

$$a(t)\leq h(t,a(t))+f(t,a(t))\left[\frac{x_0-h(0,x_0)}{f(0,x_0)}+\int_0^t g(s,a(s))\,ds\right] \tag{2.5.5}$$

$t\in\mathbb{R}_+$.

Definition 2.5.9[15, 16, 18,22,23,25]: A function $b\in AC(\mathbb{R}_+,\mathbb{R})$ is called upper solution of (2.1.2) if

$$b(t)\geq h(t,b(t))+f(t,b(t))\left[\frac{x_0-h(0,x_0)}{f(0,x_0)}+\int_0^t (t-s)g(s,x(s))\,ds\right] \tag{2.5.6}$$

Definition 2.5.10a[[15, 16, 18,22,23,25]: A solution x_M of the SNQDE (2.1.2) is said to be maximal if x is any other solution of SNQDE (2.1.2) on $\mathbb{R}_+$ then we have $x(t)\leq x_M(t)$ for all $t\in\mathbb{R}_+$.

Definition 2.5.10 b [[15, 16, 18,22,23,25]: A solution x_m of the SNQDE (2.1.2) is said to be minimal if x is any other solution of SNQDE (2.1.2) on $\mathbb{R}_+$ then we have $x(t)\geq x_m(t)$ for all $t\in\mathbb{R}_+$.

We consider the following assumptions.

($\mathfrak{B}_5$) The function $h(t,x(t))$, and function is monotone increasing in x almost every where for $t\in\mathbb{R}_+$.

($\mathfrak{B}_6$) The function $f(t,x(t))$ be monotone increasing in x a. e. for $t\in\mathbb{R}_+$

($\mathfrak{B}_7$) The function $g(t,x(t))$ is monotone increasing in x a.e. for $t\in\mathbb{R}_+$

($\mathfrak{B}_8$) The SNQDE (2.1.2) has a lower solution a and an upper solution b with $a\leq b$.

Remark (2.5.11): suppose that $\mathfrak{B}_5$- $\mathfrak{B}_8$ is satisfied then $\exists$ a function $h_q(t)=\left|\int_0^t (t-s)g(s,a(s))ds+\int_0^t (t-s)g(s,b(s))ds\right|$ is Lesbegue measurable.

Theorem 2.5.12: Assume that the hypothesis $\mathcal{H}_5 - \mathcal{H}_8$ and **($\mathfrak{B}_5$)- ($\mathfrak{B}_8$)** holds then SNQDE (2.1.2) has a minimal and a maximal solution.

Proof: Let $\mathbb{X} = AC(\mathbb{R}_+, \mathbb{R})$ and consider an ordered interval $[a, b]$ in X which is well defined in view of hypothesis (B_4). Let $\mathbb{A}, \mathbb{B}$: $[a, b]$: $\longrightarrow \mathcal{K}$ and $\mathbb{C}$: $[a, b] \longrightarrow \mathbb{X}$be three operators as defined in (2.4.3) and (2.4.4) and (2.4.5).

We will show that the operator , $\mathbb{B}$, and $\mathbb{C}$ are monotone increasing.

Let $x, y \in [a, b]$ be such that $x \leq y$,

$\mathbb{A}x = f(t, x(t))$ then by (B_1)

$f(t, x(t)) \leq f(t, y(t)).\, f(t, y(t))$

$$= \mathbb{A}y$$

$\mathbb{A}x \leq \mathbb{A}y$

$$\mathbb{B}x(t) = \frac{x_0 - h(0, x_0)}{f(0, x_0)} + \int_0^t (t - s) g(s, x(s)) ds$$

$$\mathbb{B}x(t) \leq \frac{x_0 - h(0, x_0)}{f(0, x_0)} + \int_0^t (t - s) g(s, y(s)) ds$$

$$\mathbb{B}x \leq \frac{x_0 - h(0, x_0)}{f(0, x_0)} + \int_0^t g(s, y(s)) ds$$

$$\mathbb{B}x \leq \mathbb{B}y$$

$$\mathbb{C}x(t) = h(t, x(t))$$

$$h(t, x(t)) \leq h(t, y(t))$$

$$h(t, y(t)) = \mathbb{C}y(t)$$

$$\mathbb{C}x(t) \leq \mathbb{C}y(t)$$

Hence the operators $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ are strictly monotone increasing.

It can be shown, as in the proof of theorem (2.4.7) that operator $\mathbb{A}$ and $\mathbb{C}$ are Lipschitz.

It can be shown, using remark (2.5.11) as in the poof of theorem (2.4.7) that $\mathbb{B}$ is completely continuous. And $\alpha M + \beta < 1$,

Let $x \in [a, b]$ be any element then by (B_4) we have,

$$a \leq \mathbb{A}a.\mathbb{B}a + \mathbb{C}a \leq \mathbb{A}x.\mathbb{B}x + \mathbb{C}x \leq \mathbb{A}b.\mathbb{B}b + \mathbb{C}b \leq b$$

which shows that $\mathbb{A}x\mathbb{B}x + \mathbb{C}x \subset [a, b]$.

Thus, the operators $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ satisfy all the conditions of theorem (2.5.2) which yields that the operator inclusion $x \in \mathbb{A}x\mathbb{B}x + \mathbb{C}x$ and consequently the SNQDE (2.1.2) has maximal and minimal solution. This completes the proof.

REFERENCES

[1] A. Bahkani, V.Dfatardar-Gejii, Existence of Positive Solutions for Nonlinear Fractional Differential Equation, Math., Appl., Vol. 278, pp. 443- 442, (2003).

[2] Ahmad B., Alsaedi. A., Existence and uniqueness of solutions for coupled systems of higher order nonlinear fractional differential equations, Fixed pointtheory Appl., Article ID 364560, (2010).

[3] Almeida R. and Torres D.F.M, Calculus of Variations with Fractional derivatives and integrals, Appl. Math. Lett. 22 (12), pp. 1816-1820, (2009).

[4] B.C. Dhage, A Nonlinear alternative in Banach algebras with application to Fractional Differential Equations, Nonlinear Funct. Anal. Appl., 8, pp. 563- 573, (2004).

[5] B. C. Dhage, Attractivity and positivity results for nonlinear functional integral equation, Diff. Eqn, & Applicn,, Vol. 2(3), pp. 298-318, (2010).

[6] B.C. Dhage, on a condensing mapping in Banach algebras, The Mathematical Student, Vol. 63(4), pp. 146-152, (1994).

[7] B.C. Dhage, on existence of extremal solutions of nonlinear functional integral equations in Banach Algebras, Journal of applied mathematics and stochastic Analysis, Vol. 3, pp. 272-282, (2004).

[8] B.C. Dhage, on existence theorems for Nonlinear integral equations in Banach algebras via fixed point technique, East Asian Math. J., 17, pp. 33-45, (2001).

[9] B.D. Karande and A.F. Siddiqui, Existence the Common Solution for a system of Fractional Order Integral Inclusions, GIS Science Journal, Vol.8 (6), pp: 340-351, Sept 2021.

[10] B.D. Karande, P.M. More, S.P. Shinde, Locally attractive of functional quadratic volterra type integral equation of fractional order, Doc., Reserachgatenet,http://dx.doi.org/10.13140/RG.2.1.1471.7205, (2016).

[11] B.D.Karande, Pravin M.More, Existence and extremal solution of boundary value problem for nonlinear hybrid fractional differential equations in Banach algebra, Nepal Journal of Mathematics (NJMS), Vol. 1, pp. 23-32, (2020).

[12] B.D. Karande, P.M. More and S.N. Kondekar, Existence of Locally Attractive Solution to Fractional order Quadratic Functional Integral Equation, International Journal Scientific and Innovative Mathematical Research, Vol. 3(1), PP. 328-336, (2015).

[13] B. D. Karande, P.M. More and Sachin Kondekar, Existence of Locally Attractive Solution to Nonlinear Quadratic Volterra Integral Equation, Indian Streams Research Journal, Vol. 6(6), PP. 1-12, (2017).

[14] B.D. Karande, Some Studies in Nonlinear Differential and Integral Equation In Banach Space, Ph.D. Thesis, Swami Ramanand Teerth Marathwada University, Nanded, April. (2007).

[15] C. Corduneau, Integral Equations and applications, Cambridge University press, Cambridge, (1991).

[16] Das S., Functional fractional calculus systems Identification and controls, Berlin, Heidelberg: Springer-Verlag (2008).

[17] H.M.Srivastava, R.K.Saxsena, Operators of fractional integration and applications, Application of Mathematics and Computers, Vol. 118, pp. 147- 156, (2006).

[18] Hiller R., Application of fractional Calculus in Physics, World Scientific Publishing, River Edge, N. J. USA, (2000).

[19] I. Podubny, Fractional Differential Equation, Academic Press, San Diego (1993).

[20] I. Podubny, Fractional Differential equations Mathematics in Science and Engineering, Vol. (198), Academic press, New York (1999).

[21] M.M.EL. Borai, W.G.Sayed and M.I. Abbas, Monotonic Solution of class of quadratic singular integral equations of Volterra type, Int. J. contemn. Math. Sci., Vol. 2(2), pp. 89-102, (2007).

[22] Mehdi Rahimy, Application of fractional differential equations, Applied mathematical sciences, Vol. 4(50), pp. 2453-2461, (2010).

[23] MeralaF.C.,RoystonaT.J.,Margin R., Fractional calculus in viscoelasticity an experimental study, Common Nonlinear Sci., Number Simul.,Vol.15, pp. 939-945, (2010).

[24] Mohamad I. Abbas, on the existence of local attractive solution of nonlinear quadratic volterra integral equation of fractional order, Advance in differentialequation, Article id. 127093, pp. 1-11, (2011).

[25] Ortigueira M.D., Fractional Calculus for scientists and engineering, Springer,New York (1993).

MATHEMATICAL MODELING IN REAL LIFE PROBLEMS

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ABSTRACT

Mathematical modeling is an essential tool in applied mathematics that enables the representation, analysis, and solution of real-life problems using mathematical concepts. In the modern scientific world, complex systems arising in engineering, medicine, economics, environmental science, and social sciences cannot be understood through direct observation alone. Mathematical models provide a structured and simplified framework to study such systems effectively.

A mathematical model uses variables, parameters, equations, and logical relationships to represent real-world phenomena. These models help in predicting system behavior, testing various scenarios, and supporting informed decision-making without conducting costly or time-consuming real experiments. This paper presents a comprehensive study of mathematical modeling, including its concept, modeling process, assumptions, validation techniques, types of models, and applications in various fields. The importance and limitations of mathematical modeling are also discussed to provide a balanced understanding of its role in solving real-life problems.

Keywords: *Mathematical Modeling, Applied Mathematics, Real-Life Problems, Differential Equations, Optimization*

1. INTRODUCTION

Mathematics has always played a central role in the development of science and technology. It provides a universal language for describing natural phenomena and solving practical problems. In earlier times, mathematics was mainly concerned with numbers, geometry, and abstract reasoning. However, with the advancement of science and technology, mathematics has evolved into a powerful problem-solving tool through its application in real-life situations.

Mathematical modeling is one of the most important applications of mathematics. It involves the formulation of mathematical representations of real-world problems in order to analyze, understand, and predict system behavior. Many real-life problems are too complex to be solved by direct experimentation. Mathematical models simplify such problems by focusing on essential variables and ignoring less significant factors, making analysis possible.

In today’s interdisciplinary research environment, mathematical modeling is widely used in engineering design, medical research, economics, environmental studies, industrial planning, and social sciences. Differential equations, probability theory, numerical methods, and optimization techniques form the foundation of most mathematical models. The purpose of this paper is to present a detailed study of mathematical modeling, its process, applications, advantages, and limitations, and to demonstrate its importance in solving real-life problems.

2. CONCEPT OF MATHEMATICAL MODELING

A mathematical model is a simplified representation of a real-world system expressed in mathematical form. It may consist of algebraic equations, differential equations, inequalities, graphs, or algorithms that describe the relationship between different variables of the system. The main objective of a mathematical model is to understand system behavior and to make predictions under various conditions.

Mathematical models are constructed by identifying key variables and parameters that influence the system. Assumptions are made to reduce complexity and to focus on dominant factors. Although models may not capture every detail of reality, they provide valuable insights into the working of complex systems.

Mathematical modeling acts as a bridge between theory and practice. It allows theoretical mathematical concepts to be applied to real-life situations, thereby enhancing their practical relevance and usefulness.

3. PROCESS OF MATHEMATICAL MODELING

The process of mathematical modeling involves several systematic steps. Each step plays a crucial role in developing an effective and reliable model.

3.1 Problem Identification
The first step is to clearly define the real-life problem to be studied. A well-defined problem ensures that the model focuses on relevant aspects of the system.

3.2 Formulation of Assumptions
Assumptions are made to simplify the system. These assumptions may include ideal conditions, constant parameters, or neglecting minor effects.

3.3 Identification of Variables and Parameters
Variables represent quantities that change, while parameters are constants that define system characteristics.

3.4 Model Formulation
Mathematical equations or relationships are developed based on assumptions and system behavior.

3.5 Solution of the Model
The model is solved using analytical or numerical methods depending on complexity.

3.6 Interpretation and Validation
Results are interpreted in real-world terms and validated using experimental or observed data.

4. ASSUMPTIONS AND MODEL VALIDATION
Assumptions are necessary for simplifying real-life problems; however, unrealistic assumptions can reduce model accuracy. Therefore, assumptions must be carefully chosen based on practical considerations.

Model validation is the process of checking the reliability of the model by comparing its predictions with real-world data. A validated model can be used confidently for prediction and decision-making. If discrepancies occur, the model must be revised and improved.

5. TYPES OF MATHEMATICAL MODELS

Mathematical models are classified based on their structure and application:

5.1 Deterministic Models
These models produce exact outcomes for given inputs without randomness.

5.2 Stochastic Models
These models involve probability and randomness and are useful when uncertainty is present.

5.3 Static Models
Static models describe systems at a particular point in time.

5.4 Dynamic Models
Dynamic models describe systems that change with time and often involve differential equations.

6. MATHEMATICAL MODELS USING DIFFERENTIAL EQUATIONS

Differential equations are widely used in modeling dynamic systems involving change. They describe the relationship between variables and their rates of change.

6.1 Population Growth Model
\frac{dP}{dt} = kP

This model explains exponential population growth under ideal conditions.

6.2 Newton’s Law of Cooling
\frac{dT}{dt} = -k(T - T_s)

This model describes temperature change over time.

Differential equation models are fundamental in physics, biology, engineering, and economics.

7. APPLICATIONS OF MATHEMATICAL MODELING
Mathematical modeling has extensive applications across various disciplines.

7.1 Engineering and Physical Sciences
Models are used in structural analysis, fluid mechanics, electrical circuits, and mechanical systems to improve safety and efficiency.

7.2 Biological and Medical Sciences

Epidemic models, population models, and physiological models help in disease control, medical planning, and healthcare management.

7.3 Economics and Finance

Models assist in analyzing supply-demand systems, economic growth, investment strategies, and risk management.

7.4 Environmental and Ecological Studies

Models are used to study pollution, climate change, and natural resource management.

7.5 Technology and Social Sciences

Mathematical modeling supports artificial intelligence, traffic flow analysis, urban planning, and decision-making processes.

8. IMPORTANCE OF MATHEMATICAL MODELING

Mathematical modeling simplifies complex problems, reduces cost and risk, improves planning, and supports innovation. It enhances understanding of systems and promotes interdisciplinary research.

9. LIMITATIONS OF MATHEMATICAL MODELING

Models depend on assumptions and data quality. Complex real-world systems may not be fully represented by simplified models. Therefore, careful interpretation of results is necessary.

10. CONCLUSION

Mathematical modeling is a powerful and indispensable tool in applied mathematics. It provides effective solutions to real-life problems by combining theoretical concepts with practical applications. With proper assumptions and validation, mathematical models significantly contribute to scientific and technological advancement.

REFERENCES

1. Giordano, F. R., Weir, M. D., & Fox, W. P., A First Course in Mathematical Modeling, Brooks/Cole.
2. Meyer, W. J., Concepts of Mathematical Modeling, McGraw-Hill.
3. Edwards, C. H., & Penney, D. E., Differential Equations and Boundary Value Problems, Pearson Education.
4. Kreyszig, E., Advanced Engineering Mathematics, Wiley.
5. Murray, J. D., Mathematical Biology, Springer-Verlag.
6. Winston, W. L., Operations Research: Applications and Algorithms, Cengage Learning.
7. Banks, H. T., Modeling and Control in the Biomedical Sciences, Springer.
8. Kapur, J. N., Mathematical Modeling, Wiley Eastern Limited.

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THE BACKBONE OF ARTIFICIAL INTELLIGENCE: MATHEMATICS

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ABSTRACT

Artificial Intelligence (AI) is growing very fast in today's world. We can see its use in many areas like online shopping, medical diagnosis, banking systems, education platforms, and even agriculture. Many students think AI is mainly about coding or using powerful computers. But actually, the real foundation of AI is mathematics. In this paper, I explain how mathematics supports AI in different ways. Topics like linear algebra, calculus, probability, statistics, and optimization are very important in AI. After studying these concepts, it becomes clear that AI is not only technology, but also applied mathematics in practical form.

Keywords: *Artificial Intelligence, Mathematics, Machine Learning, Linear Algebra, Calculus, Probability, Statistics, Optimization, Neural Networks, Data Science.*

INTRODUCTION

Artificial Intelligence (AI) has become an important part of modern life and is now used in many fields such as education, healthcare, banking, agriculture, and online services. Although many people think that AI mainly depends on programming and powerful computers, the real strength of AI lies in mathematics. Mathematical concepts provide the basic structure for understanding data, building models, training algorithms, and making predictions. Topics such as linear algebra, calculus, probability, statistics, optimization, and discrete mathematics play a key role in the development of machine learning and deep learning systems. This paper aims to explain, in simple and clear language, how different branches of mathematics support Artificial Intelligence. By connecting theoretical concepts with practical applications, the paper shows that AI is not only a technological field but also a strong example of applied mathematics in real life

1. Understanding What AI Really Is

Artificial Intelligence means creating machines that can perform tasks which usually require human thinking. For example, recognizing faces, understanding speech, recommending products, or predicting future results. At the beginning, I also believed that AI is mostly about programming languages like Python and using libraries. But while learning machine learning concepts, I understood that behind every algorithm there is mathematics working in the background. Actually, computer programs are just tools to apply mathematical formulas. If mathematics is removed, AI cannot function. So we can say mathematics is the backbone of Artificial Intelligence.

2. How AI Understands Data: The Role of Linear Algebra

In AI, everything is converted into numbers. A computer does not understand image, text, or sound directly. For example, an image is made up of pixels. Each pixel has a number representing color or brightness. These numbers form a matrix. In the same way, text data is also converted into numerical vectors. Neural networks use matrix multiplication to process this data. Weights and biases inside the network are also stored as matrices. When data passes from one layer to another, matrix operations are performed. When we study matrices and vectors in mathematics class, sometimes it feels theoretical and difficult. But in AI, these same concepts are used in real applications. That is why linear algebra is very important in machine learning and deep learning.

3. How Machines Learn Step by Step

Learning means improving performance step by step. In AI, when a model makes a prediction, it checks how much error is there between predicted output and actual output. This error is called loss. To reduce this loss, the model changes its internal parameters slightly. Calculus helps in this process. Derivatives tell how much the output changes when input changes slightly. Using this idea, gradient descent method adjusts the parameters to reduce error slowly. Backpropagation is another important technique used in deep learning. It uses chain rule from calculus to update weights in multiple layers. Before studying AI, calculus felt very abstract. But after seeing its application in neural networks, it becomes more practical and meaningful.

4. Making Decisions When Things Are Uncertain

In real life, data is not always perfect. Sometimes information is incomplete or noisy. AI systems must still give output. Probability helps AI measure how likely something is. For example, in spam detection, the system gives probability that an email is spam instead of giving 100% sure answer. Statistics helps in checking model performance. Accuracy, precision, recall, and mean squared error are statistical measures used to evaluate models. In India, we see AI being used in areas like weather prediction and crop analysis. These systems also depend on probability because future events cannot be predicted with full certainty. So probability and statistics help AI make reasonable decisions in uncertain situations.

5. Improving the Model Again and Again

Most AI models try to find the best possible solution. This process is called optimization. A loss function measures how wrong the prediction is. The main aim of training is to minimize this loss. Optimization methods like gradient descent help in reaching better parameter values. In deep learning, there may be millions of parameters, so optimization becomes slightly complex. But mathematical techniques make it manageable. Without optimization, AI models would not improve even after seeing large data.

6. Understanding Connections and Relationships

Some AI problems involve relationships between different objects. For example, social media networks connect people. Online shopping websites recommend products based on user connections. Graph theory represents these relationships using nodes and edges. Search algorithms are also based on discrete mathematics.

Logical reasoning in AI systems is based on Boolean algebra and set theory. These mathematical concepts help in building rule-based systems.

7. What I Learned as a Student

As a student from Maharashtra studying engineering, I used to think mathematics is only for passing exams. Sometimes formulas looked difficult and not useful in real life. But while learning AI and machine learning, I slowly understood the importance of mathematics. Concepts like matrices, derivatives, and probability which I studied earlier are directly used in AI models. Now I feel mathematics gives confidence in understanding how AI actually works. Without knowing mathematics, we can run programs, but we may not understand what is happening internally. This realization changed my thinking about mathematics. It is not just theory; it is practical and powerful.

8. Final Thoughts

Artificial Intelligence looks modern and advanced, but its base is traditional mathematics. Linear algebra helps in representing data, calculus helps in learning, probability handles uncertainty, and optimization improves performance. From academic learning and personal experience, it is clear that mathematics is the backbone of AI. As AI continues to grow in India and across the world, students must build strong mathematical foundations to truly understand and innovate in this field.

REFERENCES

1. Goodfellow, I., Bengio, Y., & Courville, A. (2016). Deep Learning. MIT Press.
2. Bishop, C. M. (2006). Pattern Recognition and Machine Learning. Springer.
3. Murphy, K. P. (2012). Machine Learning: A Probabilistic Perspective. MIT Press.
4. Boyd, S., & Vandenberghe, L. (2004). Convex Optimization. Cambridge University Press.
5. Cover, T. M., & Thomas, J. A. (2006). Elements of Information Theory. Wiley.
6. Hastie, T., Tibshirani, R., & Friedman, J. (2009). The Elements of Statistical Learning: Data Mining, Inference, and Prediction. Springer.
7. Mitchell, T. M. (1997). Machine Learning. McGraw-Hill.
8. Russell, S., & Norvig, N. (2010). Artificial Intelligence: A Modern Approach (3rd ed.). Pearson.
9. Strang, G. (2016). Introduction to Linear Algebra (5th ed.). Wellesley-Cambridge Press.
10. Burden, R. L., & Faires, J. D. (2011). Numerical Analysis. Brooks/Cole.
11. James, G., Witten, D., Hastie, T., & Tibshirani, R. (2013). An Introduction to Statistical Learning. Springer.
12. Koller, D., & Friedman, N. (2009). Probabilistic Graphical Models: Principles and Techniques. MIT Press.

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THE STIELTJES MOMENT PROBLEMS

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ABSTRACT

The Stieltjes moment problem is a classical topic in mathematical analysis that concerns determining whether a sequence of real numbers can be represented as moments of a non-negative measure supported on the interval $[0, \infty)$*, and if so, whether this representation is unique. Formally, given a sequence* $\{\mu_n\}_{n=0}^{\infty}$*, the problem asks whether there exists a positive measure* μ*such that*

$$\mu_n = \int_0^{\infty} t^n \, d\alpha(t), n = 0,1,2, \ldots$$

This problem plays a significant role in functional analysis, probability theory, orthogonal polynomials, and spectral theory. Central issues include conditions for existence, determinacy versus indeterminacy of solutions, and characterization through positivity of Hankel matrices and Carleman-type criteria. The study of the Stieltjes moment problem also connects to continued fractions, analytic function theory, and applications such as signal processing and statistical reconstruction of distributions from finite data. Its theoretical framework provides powerful tools for understanding how sequences encode structural information about measures, making it an important bridge between analysis, algebra, and applied mathematics.

Keywords: *Stieltjes moment problem; moment sequence; positive measure; Hankel matrix; determinacy and indeterminacy; Carleman condition; orthogonal polynomials; functional analysis; measure theory; spectral theory; continued fractions; moment representation; non-negative support; analytic functions; inverse problems.*

INTRODUCTION

The Stieltjes moment problem is a fundamental topic in classical analysis that investigates the relationship between sequences of numbers and measures defined on the non-negative real axis. Given a sequence $\{\mu_n\}_{n=0}^{\infty}$, the central question is whether there exists a non-negative measure μsupported on $[0, \infty)$such that each term of the sequence can be expressed as a moment,

$$\mu_n = \int_0^{\infty} t^n \, d\alpha(t), n = 0,1,2, \ldots$$

This problem is one of the three classical moment problems—the others being the Hamburger moment problem (on $(-\infty, \infty)$) and the Hausdorff moment problem (on a finite interval). The Stieltjes case is distinguished by its domain restriction to non-negative values, which makes it particularly relevant in applications where quantities cannot be negative, such as probability distributions, physical measurements, and spectral densities.

The importance of the Stieltjes moment problem lies not only in determining whether such a measure exists but also in understanding whether it is unique. These two aspects are known as the **existence problem** and the **determinacy problem**. Mathematicians have developed several criteria to address these questions, including positivity conditions on Hankel matrices formed from the moment sequence and growth conditions such as Carleman's criterion. These tools provide deep insight into how numerical sequences encode structural information about functions or distributions.

Historically, the theory originated in the late nineteenth century through the work of Thomas Joannes Stieltjes, who studied continued fractions and their connections with integrals. Since then, the theory has expanded significantly and now plays an important role in functional analysis, operator theory, probability theory, and numerical analysis. In modern applications, moment problems arise in signal reconstruction, statistical inference, quantum mechanics, and approximation theory, where one often seeks to recover a function or distribution from partial information.

Thus, the Stieltjes moment problem serves as a bridge between abstract mathematical theory and practical applications. By linking sequences, measures, and analytic structures, it provides a powerful framework for understanding how discrete numerical data can represent continuous phenomena.

1.1 The Stieltjes Moment Problem Related to the Humberger Problem

The Stieltjes problem can also be applied as a special case of the Hamburger problem.

Axiom 1.1(a) A significant and required condition that there should exist an increasing function $\alpha(t)$ such as-

$$\mu_n = \int_0^\infty t^n d\alpha(t) \qquad (n = 0, 1, 2, ...), \qquad (1.1,1)$$

The integrals are all converging, thus; the sequences $\{\mu_n\}_0^\infty$ and $\{\mu_n\}_1^\infty$ should be non-negative, or that the quadratic forms

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\mu_{i+j}\xi_i\xi_j \qquad (n = 0, 1, 2, ...) \qquad (1.1,2)$$

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\mu_{i+j+1}\xi_i\xi_j \qquad (n = 0, 1, 2, ...) \qquad (1.1,3)$$

should be non-negative (definite or semi-definite).

According to Axiom- The equivalence of the two forms of the condition is apparent. We prove the result in the latter form involving quadratic forms. For the requirement, Let the sequence $\{\mu_n\}_0^\infty$ have the form (1.1,1). since we may regard $\alpha(t)$ as constant in the interval $(-\infty,0)$. Proceeding further,

$$\mu_{n+1} = \int_0^\infty t^n d\beta(t) \qquad (n = 0, 1, ...)$$

$$\beta(t) = \int_0^t u d\alpha(u) \qquad (t \geqq 0)\cdot$$

As $\beta(t)$ is also an increasing one, we know that there is a necessary and required condition that there should be an existence of at least one increasing $\alpha(t)$ in a way as

$$\mu_n = \int_{-\infty}^\infty t^n d\alpha(t) \qquad (n = 0, 1, 2, ...),$$

All the integrals converging is that the sequence $\{\mu_n\}_0^\infty$ should be non-negative. and a significantly required and necessary condition that the sequence $\{\mu_n\}_0^\infty$ should be non-negative definite (semi-definite) is that the quadratic forms

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\mu_{i+j}\xi_i\xi_j \qquad (n = 0, 1, 2, ...)$$

could be non-negative definite (semi-definite). show that the forms (1.1,3) are also non-negative.

Contradictory, suppose the forms (1.1,2) and (1.1,3) be non-negative and then considering the new sequence $\{\nu_n\}_0^\infty$ where,

$$\nu_{2n} = \mu_n \qquad (n = 0, 1, ...)$$

$$\nu_{2n+1} = 0 \qquad (n = 0, 1, ...).$$

If $n = \text{odd}$,

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\nu_{i+j}\xi_i\xi_j = \sum_{i=0}^{\frac{n-1}{2}}\sum_{j=0}^{\frac{n-1}{2}}\mu_{i+j}\xi_{2i}\xi_{2j}\sum_{i=0}^{\frac{n-1}{2}}\sum_{j=0}^{\frac{n-1}{2}}\mu_{i+j+1}\xi_{2i+1}\xi_{2j+1},$$

and now if n = even;

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\nu_{i+j}\xi_i\xi_j = \sum_{i=0}^{\frac{n}{2}}\sum_{j=0}^{\frac{n}{2}}\mu_{i+j}\xi_{2i}\xi_{2j}\sum_{i=0}^{\frac{n}{2}-1}\sum_{j=0}^{\frac{n}{2}-1}\mu_{i+j+1}\xi_{2i+1}\xi_{2j+1},$$

This proves that the sequence $\{\nu_n\}_0^\infty$ is non-negative, we know that there is a necessary and required condition that there should be an existence of at least one increasing $\alpha(t)$ in a way as

$$\mu_n = \int_{-\infty}^{\infty} t^n d\alpha(t) \qquad (n = 0, 1, 2, ...),$$

All the integrals converging is that the sequence $\{\mu_n\}_0^\infty$ should be non-negative. it stated that there lies an existence of an increasing function $\beta(t)$ in such a way-

$$\nu_n = \int_{-\infty}^{\infty} t^n d\beta(t) \qquad (n = 0, 1, 2, ...),$$

Otherwise,

$$\mu_n = \int_{-\infty}^{\infty} t^{2n} d\beta(t) \qquad (n = 0, 1, 2, ...)$$

$$0 = \int_{-\infty}^{\infty} t^{2n+1} d\beta(t) \qquad (n = 0, 1, 2, ...). \qquad (1.1,4)$$

Set

$$\gamma(t) = \frac{\beta(t) - \beta(-t)}{2} \qquad (-\infty < t < \infty).$$

Here, the function is odd which also satisfy the equations (1.1,4). It is increasing.

Set $\alpha(t) = 2\gamma(t^{1/2})(t \geqq 0)$. So, Then-

$$\mu_n = \int_{-\infty}^{\infty} t^{2n} d\gamma(t) = \int_0^{\infty} t^{2n} d\gamma(t) - \int_0^{\infty} t^{2n} d\gamma(-t)$$

by an obvious change of variables. But since $\gamma(t)$ is odd, this gives

$$\mu_n = 2\int_0^{\infty} t^{2n} d\gamma(t) \qquad (n = 0, 1, 2, ...)$$

$$= \int_0^{\infty} t^n d[2\gamma(t^{1/2})]$$

$$= \int_0^{\infty} t^n d\alpha(t).$$

Since, $\alpha(t)$ is increasing in the interval $0 \leqq t < \infty$, we reached the expected outcome.

Clearly stated that we also have the following results.

Axiom 1.1(b) A significant and required condition that equations (1.1,1) should have an increasing solution $\alpha(t)$ with infinitely many points of non-decrease is that the forms (1.1,2) and (1.1,3) should all be non-negative definite or that the zero must be

$$\mu_0, \quad \begin{vmatrix} \mu_0 & \mu_1 \\ \mu_1 & \mu_2 \end{vmatrix}, \quad \begin{vmatrix} \mu_0 & \mu_1 & \mu_2 \\ \mu_1 & \mu_2 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \end{vmatrix}, \ldots \qquad (1.1,5)$$

$$\mu_1, \quad \begin{vmatrix} \mu_1 & \mu_2 \\ \mu_2 & \mu_3 \end{vmatrix}, \quad \begin{vmatrix} \mu_1 & \mu_2 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \\ \mu_3 & \mu_4 & \mu_5 \end{vmatrix}, \ldots$$

lesser than all the determinants.

Axiom 1.1(c) A required and significant condition that equations (1.1,1) should have an increasing solution $\alpha(t)$ with a finite number of points of non-decrease is that the forms (1.1,2) and (1.1,3) should all be non-negative & at least one of them being non-negative semi-definite.

We observe that it is not enough that the determinants (1.1,4) should be all positive.

1.2 The Humburger Problem-

According to the Hausdorff problem one may expect that it could be expectable to cope with the Hamburger and Stieltjes problems for the case in which $\alpha(t)$ is of closely related variation on the appropriate infinite interval. R.P. Boas [1939] has observed that in this case there is hardly any problem, as every sequence leads to a soluble Stieltjes or Hamburger problem if we consider any function of closely related variation as a solution.

We provide the proof of Boas. which will of course be enough to initialize the Stieltjes case.

Axiom 1.2(a) The equations

$$\mu_n = \int_0^\infty t^n d\alpha(t) \qquad (n = 0, 1, 2, ...)$$

forever have a solution $\alpha(t)$ of closely related variation for which

$$\int_0^\infty |d\alpha(t)| < \infty .$$

We formed the other two sequences $\{\lambda_n\}_0^\infty, \{v_n\}_0^\infty$, such as;

$$\mu_n = \lambda_n - v_n \tag{1.2,16}$$

$$\lambda_n = \int_0^\infty t^n d\beta(t) \tag{1.2,17}$$

$$v_n = \int_0^\infty t^n d\gamma(t) \qquad (n = 0, 1, 2, ...), \tag{1.2,18}$$

Now, here $\beta(t)$ and $\gamma(t)$ are closely related increasing functions. First choose $\lambda_0, \lambda_1, v_0, v_1$ as any non-negative numeric satisfying (1.2,16). Proceeding by induction, Let, λ_k, v_k for k = 0, 1, 2, ...,2n – 1 so that (1.2,16) holds and so that the determinants

$$[\lambda_0, \lambda_1, ..., \lambda_{2k}] = \begin{vmatrix} \lambda_0 & \lambda_1 & \cdots & \lambda_k \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{k+1} \\ . & . & \cdots & . \\ \lambda_k & \lambda_{k+1} & \cdots & \lambda_{2k} \end{vmatrix} \tag{1.2,19}$$

$$[\lambda_1, \lambda_2, ..., \lambda_{2k+1}] = \begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k+1} \\ \lambda_2 & \lambda_3 & \cdots & \lambda_{k+2} \\ . & . & \cdots & . \\ \lambda_{k+1} & \lambda_{k+2} & \cdots & \lambda_{2k+1} \end{vmatrix},$$

$$[v_0, v_1, ..., v_{2k}], \quad [v_1, v_2, ..., v_{2k+1}] \tag{1.2,20}$$

are non-negative for k = 0, 1, ...n – 1. Thus, we explain &define $\lambda_{2n}, v_{2n}, \lambda_{2n+1}, v_{2n+1}$. We have undetermined λ_{2n}

$$[\lambda_0, \lambda_1, ..., \lambda_{2n}] = \lambda_{2n} [\lambda_0, \lambda_1, ..., \lambda_{2n-2}] + P, \tag{1.2,21}$$

Here, P is a polynomial in $\lambda_0,\lambda_1,\ldots,\lambda_{2n-1}$; and similarly for $\left[\nu_0,\nu_1,\ldots,\nu_{2n}\right]$. Since zero is smaller than $\left[\lambda_0,\lambda_1,\ldots,\lambda_{2n-2}\right]$ and $\left[\nu_0,\nu_1,\ldots,\nu_{2n-2}\right]$ so, we can choose λ_{2n} and ν_{2n} non-negative and so large that $\lambda_{2n}-\nu_{2n}=\mu_{2n}$ as-

$$\left[\lambda_0,\lambda_1,\ldots,\lambda_{2n}\right]>0, \quad \left[\nu_0,\nu_1,\ldots,\nu_{2n}\right]>0$$

It has been observed that (1.2,21) holds with all subscripts increased by unity, P now being a polynomial in $\lambda_1,\lambda_2,\ldots,\lambda_{2n}$, a similar equation holding for the ν_k. With λ_{2n} and ν_{2n} now determined we proceed exactly as above to determine λ_{2n+1} and ν_{2n+1}. Thus, the induction is completed. Following Axiom 1.1(b) if the determinants (1.2,19) and (1.2,20) are non-negative for k = 0, 1, 2, ... equations (1.2,17) and (1.2,18) have closely related solutions $\beta(t)$ and $\gamma(t)$ respectively, so that when $\alpha(t)$ is explained & defined as $\beta(t)-\gamma(t)$ our proof is proved.

Stieltjes is mentioned to prove there exists a function, not a constant, all the moments of which are zero. It too follows from Axiom 1.2(a). For, by this result, there is an existence of a non-constant function $\alpha(t)$ such as-

$$\int_0^\infty t^n d\alpha(t)=1 \qquad (n = 1)$$

(n = 0, 2, 3, 4, ...)

$$\int_0^\infty \left|d\alpha(t)\right|<\infty .$$

Putting;

$$\beta(t)=\alpha\left(t^{1/2}\right),$$

we form-

$$\int_0^\infty t^n d\beta(t)=\int_0^\infty t^{2n} d\alpha(t)=0 \qquad (n = 0, 2, 3, 4, ...).$$

The function $\beta(t)$ is the required example .

1.3 The condition for solving moment problem

Boas showed that any sequence which increases sufficiently rapidly leads to a soluble Stieltjes problem [with increasing $\alpha(t)$] by a slight modification of the method employed in Section 14. More precisely, the outcome is

Axiom 1.3(a) If

$$\mu_0 \geqq 1, \quad \mu_n \geqq \left(n\mu_{n-1}\right)^n \qquad (n = 1, 2, \ ...) \qquad (1.3,22)$$

So, the equations

$$\mu_n=\int_0^\infty t^n d\alpha(t) \qquad (n = 0, 1, 2, ...)$$

have an increasing solution $\alpha(t)$

For example- Given sequence satisfying (1.2,16) is $\mu_0=1,\mu_n=n^{n^n}$ for n = 1, 2, ... as considered in the last section,

$$\left[\mu_0,\mu_1,\ldots,\mu_{2n}\right]=\mu_{2n}\left[\mu_0,\mu_2,\ldots,\mu_{2n-2}\right]+\sum_{k=n}^{2n-1}\pm\mu_k D_k, \qquad (1.2,23)$$

Where, the D_k are n-rowed minors of $\left[\mu_0,\mu_1,\ldots,\mu_{2n}\right]$ not containing μ_{2n}. Similarly

$$[\mu_1,\mu_2,\ldots,\mu_{2n+1}] = \mu_{2n+1}[\mu_1,\mu_2,\ldots,\mu_{2n-1}] + \sum_{k=n+1}^{2n} \pm \mu_k D'_k \qquad (1.3,24)$$

the D'_k are again n-rowed minors of $[\mu_1,\mu_2,\ldots,\mu_{2n+1}]$ not containing μ_{2n+1}.

Let, $k \leqq m-1$ we have given that-

$$[\mu_0,\mu_1,\ldots,\mu_{2k}] \geqq 1, \qquad [\mu_1,\mu_2,\ldots,\mu_{2k+1}] \geqq 1 \qquad (1.3,25)$$

To prove the same inequalities for k = m, using (1.3,22) we observe that the sequence $\{\mu_n\}_0^\infty$ is an increasing function and thus, unity is greater than then no element of the sequence. So,

$$\mu_n > 2\left(\frac{n}{2}\right)^{\frac{n+4}{4}} (\mu_{n-1})^{\frac{n+2}{2}} \qquad (n = 2, 3, \ldots).$$

Particularly-

$$\mu_{2n} > 2n^{\frac{n+4}{4}} (\mu_{2n-1})^{n+1} \geqq 1 + n^{\frac{n+2}{2}} (\mu_{2n-1})^{n+1} \qquad (n = 2, 3, \ldots)$$

$$\mu_{2n+1} > 2n^{\frac{n+4}{4}} (\mu_{2n})^{n+1} \geqq 1 + n^{\frac{n+2}{2}} (\mu_{2n})^{n+1} \qquad (n = 2, 3, \ldots) \qquad (1.3,26)$$

The elements of D_k are smaller than μ_{2m-1} and those of D'_k are also smaller than μ_{2m} where k ranges over the integer mentioned in the summations (1.3,23) and (1.3,24). Thus, by Hadamard's upper bound for a determinant- we get,

$$|D_k| \leqq m^{\frac{m}{2}} (\mu_{2m-1})^m \qquad (k = m, m+1, \ldots, 2m-1)$$

$$|D'_k| \leqq m^{\frac{m}{2}} (\mu_{2m})^m \qquad (k = m+1, m+2, \ldots, 2m).$$

Thus;

$$\left|\sum_{k=m}^{2m-1} \pm \mu_k D_k\right| \leqq m(\mu_{2m-1}) m^{\frac{m}{2}} (\mu_{2m-1})^m$$

$$\left|\sum_{k=m}^{2m} \pm \mu_k D'_k\right| \leqq m(\mu_{2m}) m^{\frac{m}{2}} (\mu_{2m})^m,$$

so that by (1.23) and (1.24)

$$[\mu_0,\mu_1,\ldots,\mu_{2m}] \geqq \mu_{2m} - m^{\frac{m+2}{2}} (\mu_{2m-1})^{m+1}$$

$$[\mu_1,\mu_2,\ldots,\mu_{2m+1}] \geqq \mu_{2m+1} - m^{\frac{m+2}{2}} (\mu_{2m})^{m+1}$$

As per our observation by (1.3,26) we concluded that (1.3,25) is established for k = m. Through induction (1.3,25) now holds for all k, and by Axiom 1.1(b) the Stieltjes moment problem similar to the moments (1.3,22) has an increasing solution $\alpha(t)$.

Hence, proved!.

1.4 Determining and non-determining solution

Boas showed that any sequence of sufficiently continuous strong growth leads to a Stieltjes problem which has more than one increasing solution by making use of the last outcome.

Axiom 1.4(a) If

$$\lambda_0 \geqq 1$$

$$\lambda_2 \geqq (2\lambda_1 + 2)^2$$

$$\lambda_n \geqq (n\lambda_{n-1})^n \qquad (n = 1, 3, 4, 5, ...)$$

$$\mu_n = \lambda_{2n} \qquad (n = 0, 1, 2, ...),$$

So, then there are at least two necessary and significant different increasing functions $\alpha(t)$ such as-

$$\mu_n = \int_0^\infty t^n d\alpha(t) \qquad (n = 0, 1, 2, ...). \qquad (1.4,27)$$

There exists a function $\beta(t)$ which is non-negative, increasing, such as given below;

$$\lambda_n = \int_0^\infty t^n d\beta(t^{1/2}) \qquad (n = 0, 1, ...),$$

Thus,

$$\mu_n = \int_0^\infty t^n d\beta(t^{1/2}) \qquad (n = 0, 1, 2, ...).$$

Next, we suppose $\{v_n\}_0^\infty$ be a sequence which is corresponding to the sequence $\{\lambda_n\}_0^\infty$ with an exception $v_1 = \lambda_1 + 1$.

$$v_n \geqq (nv_{n-1})^n \qquad (n = 1, 2, 3, ...).$$

Obviously, if n is neither 1 nor 2. Then;

$$\lambda_1 + 1 = v_1 \geqq v_0 = \lambda_0$$

$$v_2 = \lambda_2 \geqq (2v_1)^2 = (2\lambda_1 + 2)^2.$$

According to Axiom 1.3(a) there is a non-negative increasing function $\gamma(t)$ such as-

$$v_n = \int_0^\infty t^n d\gamma(t) \qquad (n = 0, 1, 2, ...)$$

$$\mu_n = \int_0^\infty t^n d\gamma(t^{1/2}) \qquad (n = 0, 1, 2, ...)$$

Clearly $\beta(t^{1/2})$ is significantly different from $\gamma(t^{1/2})$, for otherwise we should have

$$\int_0^\infty t d\beta(t) = \int_0^\infty t d\gamma(t),$$

which is not possible since, v_1 and λ_1 are unequal.

CONCLUSION

The Stieltjes moment problem provides a fundamental framework for understanding how numerical sequences can represent continuous measures on the non-negative real axis. Through its formulation, it establishes precise mathematical conditions under which a sequence can be realized as a system of moments, thereby connecting discrete data with integral representations. The theory demonstrates that solvability depends primarily on positivity conditions such as the non-negativity of quadratic forms and Hankel determinants, which ensure the existence of a corresponding non-negative measure.

An important aspect of the problem is the distinction between determinate and indeterminate cases. Some moment sequences uniquely determine a measure, while others admit infinitely many solutions, depending on growth properties and analytic constraints. These results highlight the delicate balance between algebraic structure and analytic behavior in moment theory.

Overall, the Stieltjes moment problem stands as a central topic in classical analysis with wide-ranging applications in probability theory, spectral analysis, orthogonal polynomials, and inverse problems. Its study not only deepens theoretical understanding but also provides essential tools for reconstructing functions or distributions from limited information, demonstrating its lasting significance in both pure and applied mathematics.

REFERENCES

1. **Thomas Joannes Stieltjes** (1894). Recherches sur les fractions continues. Annales de la Faculté des Sciences de Toulouse.
2. — Original papers introducing moment problems and continued fraction methods.
3. **Hans Ludwig Hamburger** (1920–1921). Über eine Erweiterung des Stieltjesschen Momentenproblems. Mathematische Annalen.
   — Classical work formulating the general Hamburger moment problem.
4. **The Problem of Moments — James Alexander Shohat & Jacob David Tamarkin** (1943).
   — A foundational monograph covering Stieltjes, Hamburger, and Hausdorff moment problems.
5. **The Classical Moment Problem — Naum Il′ich Akhiezer** (1965).
   — One of the most authoritative texts on determinacy, positivity conditions, and applications.
6. **Ralph Philip Boas Jr.** (1939). Papers on moment sequences and growth conditions.
   — Source of several results about solvability and indeterminacy conditions cited in your section.
7. **Orthogonal Polynomials — Gábor Szegő** (1939, later editions).
   — Contains moment problem connections via Hankel determinants and orthogonal polynomial theory.
8. **An Introduction to the Theory of Reproducing Kernel Hilbert Spaces** (for advanced perspective).
   — Useful for functional analytic formulations of moment problems.
9. **Barry Simon** (1998). The Classical Moment Problem as a Self-Adjoint Finite Difference Operator. Advances in Mathematics.
   — Modern operator-theoretic approach to determinacy and spectral interpretation.
10. [If you need a short reference list, items **3, 4, and 5** are usually sufficient because they directly treat theorems about positivity of quadratic forms, determinant conditions, and solvability criteria you cited in Sections 1.1–1.4.]

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ETHICAL ISSUES AND RISK OF ARTIFICIAL INTELLIGENCE

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ABSTRACT

The rapid maturation of Artificial Intelligence (AI) has transitioned the technology from a specialized academic pursuit into a ubiquitous infrastructure that is fundamentally redefining the trajectory of modern civilization. Its integration across a vast spectrum of domains—ranging from the high-stakes precision of autonomous vehicle navigation and diagnostic medical imaging to the complex predictive modeling of global finance and industrial robotics—has catalyzed a profound shift in productivity and economic logic. By automating cognitive tasks and optimizing systemic efficiencies, AI has unlocked unprecedented benefits for both the global economy and the daily convenience of individuals. However, this deep penetration into the socio-economic fabric is not without significant friction, as the speed of technological adoption frequently outpaces the development of regulatory and moral frameworks.

Keyword: -*Scope and Methodology, Ethical Issues and Risks of AI, Categorization Based on Algorithm, Data, Application, and Long-Term and Indirect Ethical Risks,) Categorization Based on the Deployment of AI*

INTRODUCTION

Artificial intelligence (AI) has achieved rapid and remarkable development during the last decade. AI technologies such as machine learning (ML), natural language processing, and computer vision are increasingly permeating and spreading to various disciplines and aspects of our society. AI is increasingly taking over human tasks and replacing human decision-making. It has been widely used in a variety of sectors, such as business, logistics, manufacturing, transportation, health care, education, state governance, etc.

The application of AI has brought about efficiency improvement and cost reduction, which are beneficial for economic growth, social development, and human well-being.

1) Scope and Methodology

In this section, we first clarify the aspects and topics covered in this review and the links between these topics. Then, we describe the methodology followed in conducting this survey, including the literature search strategy and selection criteria.

A. Scope

The scope and topics of this article is described as follows. Investigation of ethical issues and risks of AI is the starting point of this review, since it is because of the existence of ethical issues in AI that the research field of AI ethics exists. Thus, it is necessary and important to clarify and understand the ethical problems existed in AI.

B. Methodology

This review covers a wide variety of documents, including academic, organizational, government grey literature sources, and news report. The search of relevant literature was conducted in two phases.

2) Ethical Issues and Risks of AI

To address the ethical problems of AI, we must first recognize and understand the potential ethical issues or risks that AI may bring. Then, the necessary AI ethical guidelines, policies, principles, rules can be formulated appropriately. With the adequate ethics of AI, we can design and build AI that behaves ethically. The ethical issue of AI generally refers to the morally bad things or problematic outcomes relevant to AI that need to be addressed. Many ethical issues, such as lack of transparency, privacy and accountability, bias and discrimination, safety and security problems, the potential for criminal and malicious use, and the main goal of this section is to discuss and clarify the ethical issues of AI so that practitioners can recognize and understand these issues, and then help them to further study how to address AI ethical issues.

a) Ethical issues caused by features of AI

Transparency: ML is the core technology of current AI, especially (deep) neural networks

Data Security and Privacy: The performance of current AI strongly depends on the training data. Usually, a huge amount of data, which probably includes personal data and private data, is required to train an AI model, particularly the deep learning model

Autonomy, Intentionality, and Responsibility: With the advancement of AI, current AI systems or agents, such as healthcare robots, have a certain degree of autonomy, intentionality, and responsibility.

b) Ethical issues caused by human factors

Accountability: When an AI system or agent fails in a specified task and results in bad consequences, who should be responsible. Thus, accountability is an ethical issue that concerns the human factors involved in the designing, implementation, deployment, and usage of AI.

Ethical Standards: As the ultimate goal of AI ethics is to create ethical AI that can follow ethical principles and behave ethically it is crucial to form comprehensive and unbiased ethical standards for training or regulating AI to be ethical.

Human Rights Laws: The designer, software engineers, and other participants in AI system design and application should be taught human rights laws.

c) Ethical issues from the vulnerabilities of AI

ML is data hungry: Usually, ML requires a large amount of data to work well. Therefore, this motivates companies and organizations to collect or purchase data, including sensitive personal data, even if doing so may violate the individual's right to privacy.

Garbage in/garbage out: The performance of a ML algorithm heavily depends on the data from which it learns. If one ML algorithm is trained on insufficient or inaccurate data, it will provide undesirable results even it is well designed.

Faulty algorithms: Even if a ML algorithm is input with enough and accurate data, if the algorithm itself is bad, it will also make bad predictions.

Deep learning is a black box: Deep learning is a black box, which raises issues such as explain ability, interpretability, and trust

d) Ethical issues from the vulnerabilities of human

Abuse of AI: AI technologies, such as facial recognition and image generation, can work better than humans

Job replacement: Since intelligent robots can perform certain tasks faster and better than humans, many people worry that robots and other AI technologies will replace a large part of current human labor in the near future.

Issues about robotic companions: As AI robots become more and more sophisticated, they have begun to be regarded as companions of humans.

3) Categorization Based on Algorithm, Data, Application, and Long-Term and Indirect Ethical Risks

In the analysis report of AI ethical risks released by the Chinese National AI Standardization General Working Group, AI ethical issues are

Categorized into the following four aspects:

- ethical issues related to AI algorithms;
- ethical issues related to data;
- ethical issues related to the application of AI;
- long-term and indirect ethical risks.

a) Ethical issues related to algorithms

Algorithm security: The AI algorithms pose several security issues. First, there is a risk of algorithm or model leakage.

Algorithm explain ability: Due to the black-box characteristic of many ML algorithms [33], especially the popular deep learning or neural networks, the decision process of AI algorithms is hard to understand.

Algorithmic decision dilemma: After obtaining the AI model, the result of the algorithm is usually unpredictable for us.

b) Ethical issues related to data

Privacy protection: With the development of big data and AI, the tension between AI technology and user privacy protection has become more and more serious.

Recognizing and processing personal and sensitive information: Traditional laws and regulations only focus on the protection of personal and sensitive information.

c) Ethical issues related to application

Algorithm discrimination: The execution results of algorithms directly affect the decision-making of AI systems.

Algorithm abuse: Algorithm abuse refers to the situation where people use algorithms for analysis, decision-making, coordination, and other activities, but their use purpose, use method, use range, etc., have deviations and cause adverse effects.

d) Long-term and indirect ethical risks

Employment: With the fast advancement and widespread application of AI, more and more work can be completed by some AI products. This will have a significant influence on the employment problem.

Ownership: As AI continues to improve, the intellectual differences between AI agents and humans will gradually shrink.

Competition: Unfair competition, malicious competition, and monopolistic behaviors with technological advantages will all have an impact on social stability and market freedom, fairness, and equal value, and will seriously damage the interests of consumers and hinder the improvement of social welfare. When companies, organizations or individuals use AI algorithms, they should follow competitive ethics and not go beyond legal boundaries.

Responsibility: With the widespread application of AI, many cases in which AI products violate the laws or ethics, such as personal injury and algorithmic bias, have been observed. A fundamental problem that arises in these cases is who is responsible for these bad consequences.

4) Categorization Based on the Deployment of AI

In European Parliamentary Research Service's latest study on the ethical implications and moral questions brought by AI the ethical issues are mapped into different categories according to the ethical impacts of AI on human society, human psychology, financial system, legal system, environment and the planet, and trust.

a) Impact on society

The labor market: AI has already been applied in finance, advanced manufacturing, transportation, energy development, healthcare, and many other sectors.

Inequality: AI technologies are expected to enable companies to streamline their business operations and make them more efficient and productive.

Privacy, human rights, and dignity: AI is already affecting privacy, human rights, and dignity in many ways.

Bias: Human bias, such as gender prejudice and racism bias, may be inherited by AI.

Democracy. The implementation and adoption of AI can threaten democracy in several ways.

b) Impact on human psychology

Relationships: AI is getting better and better at imitating human thought, experience, action, dialogue, and relationships. In the future, we will frequently interact with machines or AI products as if they are humans.

Personhood: AI systems are increasingly taking on tasks and decisions that are traditionally performed by humans.

c) Impact on the financial system

The application of AI in financial markets has significantly improved transaction efficiency and trading volume. Markets are very suitable for automation, because they now operate almost entirely electronically and a huge amount of data is generated at a high rate, which requires the employment of algorithms to digest and analyze it

d) Impact on the legal system

Criminal law: According to current criminal law, a crime consists of two elements, that is, a voluntary act (or omission) and an intention to commit a crime.

Tort law: Tort law covers situations such as one person's behavior case injury, suffering, unfair loss, or harm to another person.

e) Impact on the environment and the planet

Use of natural resources: The development and application of AI will increase the demand of many natural resources, such as rare earth metals like nickel, cobalt, graphite, and so on.

Pollution and waste: The increase in production and consumption of AI technological devices such as robots will exacerbate pollution and waste, such as the accumulation of heavy metals and toxic materials in the environment.

Energy concerns: Employing AI technology, particularly deep learning, generally involves training ML models on a huge amount of data, which usually consumes large amounts of energy.

f) Impact on trust

AI promises numerous changes and benefits to individual's lives and the society. It is changing our daily lives in many domains, such as transportation, service industry, healthcare, education, public safety and security, and entertainment.

Fairness: In order to trust AI, it must be fair and impartial. As more and more decisions are delegated to AI, we must ensure that these decisions are free from bias and discrimination.

Transparency: Transparency is important for building trust in AI since it should be a must to know why an AI system made a particular decision, especially if that decision caused undesirable consequences or harm.

Accountability: Accountability ensures that if an AI system makes a mistake or hurts someone, then someone can be held responsible, whether it is the designer, developer, or company selling the AI.

Control: Another issue that affects the public trust in AI is the controllability of AI

CONCLUSION

Based on our review of AI ethics and the many complexities and challenges described in this article, it is clear that attempting to address ethical issues in AI and to design ethical AI systems that are able to behave ethically is a tricky and complex task. However, whether AI can play an increasingly important role in our future society largely depends on the success of ethical AI systems. The discipline of AI ethics requires a joint effort of AI scientists, engineers, philosophers, users, and government policymakers.

REFERENCE

1) Noble, Safiya Umoja. Algorithms of Oppression: How Search Engines Reinforce Racism. NYU Press, 2018.

2) Binns, Reuben. Fairness in Machine Learning: Lessons from Political Philosophy. 2018. (available as a preprint)

3) Rejeb, A.; Rejeb, K.; Appolloni, A.; Treiblmaier, H.; Iranmanesh, M. Exploring the impact of ChatGPT on education: A web mining and machine learning approach. Int. J. Manag. Educ. **2024.**

4) Yılmaz Virlan, A.; Tomak, B. AI tools for writing: A Q-method study with Turkish instructors of English. Educ. Inf. Technol. **2025**

5) Shahsavar, Z.; Kafipour, R.; Khojasteh, L.; Pakdel, F. Is artificial intelligence for everyone? Analyzing the role of ChatGPT as a writing assistant for medical students. Front. Educ. **2024**.

6) Bai, Y.; Kosonocky, C.W.; Wang, J.Z. How our authors are using AI tools in manuscript writing. Patterns **2024**.

7) Tang, A.; Li, K.K.; Kwok, K.O.; Cao, L.; Luong, S.; Tam, W. The importance of transparency: Declaring the use of generative artificial intelligence (AI) in academic writing. J. Nurs. Scholarsh. **2024**.

8) Barrot, J.S. Leveraging Google Gemini as a Research Writing Tool in Higher Education. Technol. Knowl. Learn. **2025**.

9) Sajid, M.; Sanaullah, M.; Fuzail, M.; Malik, T.S.; Shuhidan, S.M. Comparative analysis of text-based plagiarism detection techniques. PLoS ONE **2025**.

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APPLICATION OF LAPLACE TRANSFORM ON THE SPRING FOR FREE DAMPED MOTION

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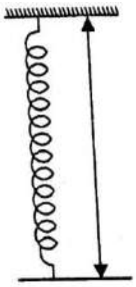
ABSTRACT

The Laplace transform is a critical tool in analyzing and solving the motion of mass spring damper systems because it transforms complex, second order linear differential equations into simpler algebraic equations. It is extensively used in engineering to model, analyze and design particularly for understanding transient behaviors in mechanical systems.

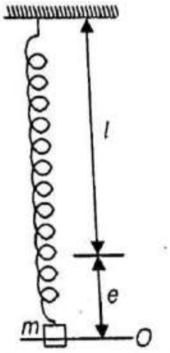
Keywords: *Differential equation, Laplace transform, Inverse Laplace transform, free damped motion*

1. INTRODUCTION

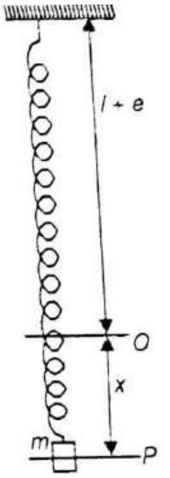
The study of spring–mass systems is a fundamental topic in engineering, physics, and applied mathematics. When a mass is attached to a spring and allowed to move freely, it performs oscillatory motion. In practical situations, resistive forces such as friction or air resistance oppose the motion. This resistance causes the amplitude of oscillation to gradually decrease with time, and such motion is called free damped motion.



(a) **Natural length *l***



(b) **Mass in equilibrium position; spring has stretched length *l* + *e***



(c) **Mass distance x below equilibrium position; spring stretched to length *l* + *e* + *x***

According to Hooke's law the magnitude of the force needed to produce a certain elongation of a spring is directly proportional to the amount of this elongation. The positive constant of proportionality is k .

Thus $|F| = ks$ where F is the magnitude of the force and s is the amount of elongation.

Let the coil spring have natural length l.The mass m is attached by an amount e so that its stretched length is $l + e$. In the position of equilibrium O,the mass m is acted upon by two forces (i) weight mg acting vertically downwards (ii) the spring force ke acting vertically upwards thus we have $mg = ke$

We choose the axis along the line of the spring with the origin at equilibrium position O and the position direction downward. Let P be the position of the mass at any time t such that $OP = x$. Then x is positive, zero or negative according to whether the mass is below at or above its equilibrium position.

When the mass is situated at Pit is acted upon by the following forces. The forces tending to pull the mass downward are positive while those pulling it vertically upward are negative.

i) $F_1 = mg$ acting in the vertically downward direction

ii) Let F_2 be the restoring force of the spring. When the mass is at P, F_2 is acting is the upward direction and so it is negative. By Hooks law we have $F_2 = -k(x + e) = -kx - ke = -kx - mg$

iii) Let F_3 be the resistance force of the medium called the damping force. It is known that for small velocities F_3 is approximately proportional to the magnitude of the velocity. When the mass is moving downward F_3 is negative and is given by $F_3 = -a\left(\frac{dx}{dt}\right)$ where $a > 0$ is called damping constant.

iv) External impressed force $F(t)$ acting in downward direction. By Newton's second law

$F = ma$ where $F = F_1 + F_2 + F_3 + F_4$ and $a = \left(\frac{d^2x}{dt^2}\right)$

$$\therefore m\left(\frac{d^2x}{dt^2}\right) = mg - kx - mg - a\left(\frac{dx}{dt}\right) + F(t)$$

$$\Rightarrow m\left(\frac{d^2x}{dt^2}\right) + a\left(\frac{dx}{dt}\right) + kx = F(t)$$

which is the differential equation for the motion of the mass on the spring. If $a = 0$ the motion is called undamped otherwise damped. If there are no external impressed forces. $F(t) = 0$ for all t and the motion is called free otherwise it is called forced.

2. APPLICATION OF LAPLACE TRANSFORMS

By applying the Laplace transform to the equation of free damped motion, the problem becomes easier to handle. After solving the algebraic equation in the Laplace domain, the inverse Laplace transform is used to obtain the displacement function$x(t)$. This approach not only simplifies calculations but also clearly incorporates initial conditions into the solution.The application of the Laplace transform to free damped spring motion is widely used in mechanical engineering, structural analysis, vibration studies, and control systems. It helps engineers predict system behavior under different damping conditions such as under damping, critical damping, and over damping. Thus, the Laplace transform plays a crucial role in analyzing and understanding the behavior of damped spring–mass systems, making it an essential mathematical tool in applied sciences.

We now wish to examine the effect of the resistance of the medium upon the mass on the spring. We assume that no external force acts on the mass. Thus we have the so called free damped motion. Hence setting $F(t) = 0$, the basic differential equation of the vibrations of a mass on the spring for free damped motion is

$$m\left(\frac{d^2x}{dt^2}\right) + a\left(\frac{dx}{dt}\right) + kx = 0\,,\quad a > 0, k > 0$$

Here we using Laplace transform to find solution of three case; I) Motion of an over damped system, II) Motion of a critically damped system III) Motion of an underdamped system

Apply Laplace transform both side of equation (1)

$$\mathcal{L}\{\, m\left(\frac{d^2x}{dt^2}\right) + a\left(\frac{dx}{dt}\right) + kx\} = 0$$

$$m\,\mathcal{L}\left(\frac{d^2x}{dt^2}\right) + a\,\mathcal{L}\left(\frac{dx}{dt}\right) + k\,\mathcal{L}(x) = 0 \quad (1)$$

Using the differentiation property of Laplace transform

$$\mathcal{L}\left(\frac{dx}{dt}\right) = \mathcal{L}(x') = sX(s) - x(0) \quad (2)$$

$$\mathcal{L}\left(\frac{d^2x}{dt^2}\right) = \mathcal{L}(x'') = s^2X(s) - sx(0) - x'(0) \quad (3)$$

Using (2) and (3) in (1)

$$m\big(s^2X(s) - sx(0) - x'(0)\big) + a\,\big(sX(s) - x(0)\big) + k\,X(s) = 0$$

$$X(s)(ms^2 + as + k) - msx(0) - mx'(0) - ax(0) = 0$$

$$X(s) = \frac{msx(0) + mx'(0) + ax(0)}{(ms^2 + as + k)} \quad (4)$$

The general solution for the displacement $x(t)$ depends on the initial conditions $x(0) = x_0$ and $x'(0) = v_0$

$\therefore$ Equation (4) become

$$X(s) = \frac{msx_0 + mv_0 + ax_0}{ms^2 + as + k}$$

Take Inverse Laplace transform

$$x(t) = \mathcal{L}^{-1}\left\{\frac{msx_0 + mv_0 + ax_0}{ms^2 + as + k}\right\}$$

To find $x(t)$ apply the Inverse Laplace Transform. The specific form of the solution depends on the discriminant $\Delta = a^2 - 4mk$

Case 1) Over damped$(a^2 > 4mk)$: Two distinct real roots the solution involves sum of exponentials

Case 2) critically damped $(a^2 = 4mk)$: One repeated real root the solution involves te^{rt}

Case 3) Under damped $(a^2 < 4mk)$: Complex conjugate roots the solution involves decaying sines and cosines

3. HERE SOME EXAMPLE SOLVE BY LAPLACE TRANSFORM FOR ABOVE CASE

a) An 8 pound weight stretches a spring 2 feet. Assuming that a damping force numerically equal to 2 times the instantaneous velocity acts on the system, determine the equation of motion if the weight is released from the equilibrium position with an upward velocity of 3 ft/sec.

Solution:- Using Hooke's law we have $8 = k \times 2$ so that $k = 4\ lb/ft$ .Again $w = mg \Rightarrow 8 = m \times 32$ so that $m = 1/4$ sulg. Also here damping factor = 2. Using the above facts the basic differential equation of the vibrations of the given mass on the spring for free damped motion

$$m\left(\frac{d^2x}{dt^2}\right) + a\left(\frac{dx}{dt}\right) + kx = 0$$

$$a^2 = 4, and\ 4mk = 4$$

Here motion is critically damped system, the initial condition are $x_0 = 0\ and\ v_0 = -3$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{msx_0 + mv_0 + ax_0}{ms^2 + as + k}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{\frac{-3}{4}}{\frac{s^2}{4} + 2s + 4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{-3}{(s+4)^2}\right\}$$

$$x(t) = -3te^{-4t}$$

$$x'(t) = -3e^{-4t}(1 - 4t)$$

Showing that $x'(t) = 0$ when $t = 1/4$. the corresponding extreme displacement is give by

$$x(1/4) = -3 \times \left(\frac{1}{4}\right) \times e^{-1} = -0.276$$

The weight reaches the maximum height of 0.276 foot above the equilibrium position.

b) A $32-lb$ weight is attached to the lower end of a coil spring suspended from the ceiling. The weight comes to rest in its equilibrium position. Thereby stretching the spring $2\ ft$. The weight is then pulled down 6 inches below its equilibrium position and released at $t = 0$. No external forces are present, but the resistance of the medium is numerically equal to $4\left(\frac{dx}{dt}\right)$ where $\frac{dx}{dt}$ is the instantaneous velocity in $feet\ per\ second$. Determine the resulting motion of the weight on the spring.

Solution:- Using Hooke's law we have $32 = k \times 2$ so that $k = 16\ lb/ft$ .Again $w = mg \Rightarrow 32 = m \times 32$ so that $m = 1$ sulg. Also here damping factor = 4. Using the above facts the basic differential equation of the vibrations of the given mass on the spring for free damped motion

$$m\left(\frac{d^2x}{dt^2}\right) + a\left(\frac{dx}{dt}\right) + kx = 0$$

$a^2 = 16, and\ 4mk = 64$ there fore $a^2 < 4mk)$

Here motion is under damped system, the initial condition is $x_0 = 1/2\ and\ v_0 = 0$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{msx_0 + mv_0 + ax_0}{ms^2 + as + k}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{\frac{s}{2} + 2}{s^2 + 4s + 16}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{\frac{1}{2}(s + 2) + 1}{(s + 1)^2 + (2\sqrt{3})^2}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{(s + 2)}{(s + 1)^2 + (2\sqrt{3})^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s + 1)^2 + (2\sqrt{3})^2}\right\}$$

$$x(t) = \frac{1}{2}e^{-2t}\, cos\left(2\sqrt{3}\right)t + \frac{\sqrt{3}}{6}e^{-2t}\, sin\left(2\sqrt{3}\right)t$$

4. CONCLUSION

Above differential equation can be difficult to solve directly using classical methods, especially when initial conditions are involved. The Laplace transform provides a powerful and systematic technique to solve such equations. It converts linear differential equations in the time domain (t) into algebraic equations in the frequency domain (s). Significantly reducing the mathematical burden and accelerating the solution process. Unlike traditional methods that require solving a general solution first, the Laplace transform incorporates initial conditions $x(0) = x_0$ and $x'(0) = v_0$ directly into the transformation streamlining the calculation of specific solutions. The Laplace transform is exceptionally effective at handling discontinuous, external or impulsive forces often applied to spring

5. REFERENCE

[1] Ravish R Singh, Mukul Bhatt, Engineering Mathematics A Tutorial Approach, McGraw Hill Education (India) Private Limited, 2010

[2] M. D. Raisinghania, Ordinary and Partial Differential Equations, S. Chand & Co. Ltd, 2011

[3] H. K. Dass, Advanced Engineering Mathematics, S. Chand & Co. Ltd, 2007

[4] Ahsan, Z., Differential Equation and Their Application, PHI, 2006

[5] S. Arumugam, A. ThangapandiIsaac, A. Somasundaram, Differential Equations, Yes Dee Publishing, 2020

[6] George Finlay Simmons, Differential Equation with Applications and Historical Notes, McGraw Hill Education (India) Private Limited, 2017

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THE ADOMIAN DECOMPOSITION METHOD FOR SOLVING DIFFERENT EQUATION

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ABSTRACT

This paper presents the application of Adomian decomposition method for various types of problems. This method is applied to find successive approximate solutions of first order differential equation with single condition, second order differential equations with two conditions, an isoperimetric problem and Volterra integral equation of second kind. It was shown that they are converging to their exact solutions. The successive approximation and exact solution are shown graphically.

Keywords: *Adomian decomposition method; Differential equation; Integral equation; Initial value problem.*

1. INTRODUTION

The Adomian decomposition method (ADM) was firstly introduced by George Adomian in 1981 and developed in [1]. This method has been applied to solve differential and integral equations of linear and non-linear problems in mathematics, physics, biology and chemistry and up to now a large number of research papers have been published to show the feasibility of the decomposition method.

The main advantage of this method is that it can be applied directly to all types of differential and integral equations, linear or non-linear, homogeneous or inhomogeneous, with constant or variable coefficients.

2. THE ADOMIAN DECOMPOSITION METHOD

Consider the differential equation

$$L_y + R_y + N_y = g(x) \tag{1}$$

where N is a non-linear operator, L is the highest order derivative which is assumed to be invertible and R is a linear differential operator of order less than L. Making L_y subject of the formula, we get

$$L_y = g(x) - R_y - N_y. \tag{2}$$

By solving (2) for L_y, since L is invertible, we can write

$$L^{-1}L_y = L^{-1}g(x) - L^{-1}R_y - L^{-1}N_y \tag{3}$$

For initial value problems we conveniently define L^{-1} for L $=\frac{d^n}{dx^n}$ as the n-fold definite integration from 0 to x. If L is a second-order operator, L^{-1} is a two fold integral and so by solving (3) for y, we get

$$y = A + Bx + L^{-1}g(x) - L^{-1}R_y - L^{-1}N_y, \tag{4}$$

where A and B are constants of integration and can be found from the initial or boundary conditions.

The Adomian method consists of approximating the solution of (1) as an infinite series

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \tag{5}$$

and decomposing the non-linear operator N as

$$N(y) = \sum_{n=0}^{\infty} A_n, \tag{6}$$

where $A_n$are Adomian polynomials [2] of $y_0, y_1, y_2, \ldots . y_n$n given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\sum_{i=0}^{\infty} \lambda^i y_i)]_{\lambda=0}, n = 0,1,2, \ldots ..$$

Substituting (5) and (6) into (4) yields

$$\sum_{n=0}^{\infty} y_n(x) = A + Bx + L^{-1}g(x) - L^{-1}R\sum_{n=0}^{\infty} y_n(x) - L^{-1}N\sum_{n=0}^{\infty} A_n,$$

The recursive relationship is found to be

$$y_0 = g(x),$$
$$y_{n+1} = -L^{-1}R_y - L^{-1}A_n.$$

Using the above recursive relationship, we can construct the solution y as

$$y = \lim_{n\to\infty} \Phi_n(y), \qquad (7)$$

were

$$\Phi_n(y) = \sum_{n=0}^{\infty} y_i \qquad (8)$$

3. APPLICATION TO FIRST ORDER DIFFERENTIAL EQUATION WITH SINGLE GIVEN CONDITION

Example: Suppose $y'(x) + y = 2e^x$ and the given condition is y (0)=1.

The given equation is

$$y'(x) + y = 2e^x \qquad y\,(0)=1 \qquad (1)$$

Rewrite the given equation in operator form

$$y'(x) = -y + 2e^x \qquad (2)$$

Applying the inverse operator $L^{-1}$and using the initial condition gives

$$y(x) = y(0) + L^{-1}[-y + 2e^x]$$
$$y(x) = 1 + L^{-1}[-y + 2\mathrm{e}^x]$$
$$y(x) = 1 + L^{-1}[-y] + L^{-1}[2e^x]$$

The recursive relation

$$y_0 = 1 + L^{-1}[2e^x]$$
$$y_{n+1} = L^{-1}[-y_n]$$

Now, the first few terms

$$y_0 = 1 + L^{-1}[2e^x]$$
$$= 1 + \int_0^x 2e^t dt$$
$$y_0 = 2e^x - 1$$
$$y_1 = L^{-1}[-y_0]$$
$$= -\int_0^x (2e^t - 1)dt$$
$$y_1 = -2e^x + 2 + x$$
$$y_2 = L^{-1}[-y_1]$$
$$= -\int_0^x (-2e^t + 2 + t)$$
$$y_2 = 2e^x - 2 - 2x - \frac{x^2}{2}$$

And so on.

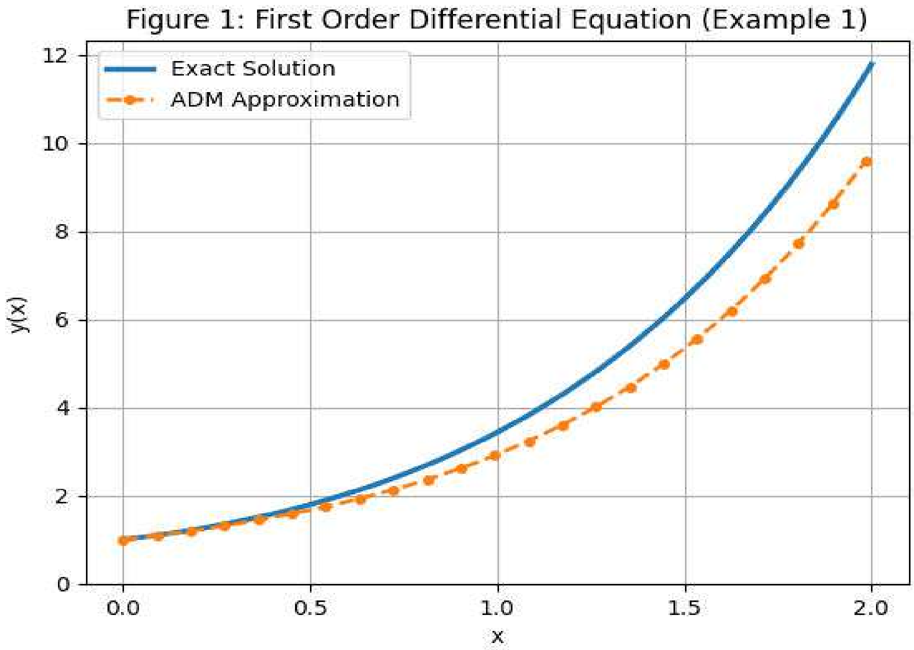
The solution in a series form is given by

$$y(x) = y_0(x) + y_1(x) + y_2(x) + \cdots$$

$$= (2e^{x} - 1) + (-2e^{x} + 2 + x) + \left(2e^{x} - 2 - 2x - \frac{x^2}{2}\right)$$

$$= 2e^{x} - x - \frac{x^2}{2} - 1 + \cdots$$

$$y(x) = 2e^{x} - (1 + x + \frac{x^2}{2} + \cdots)$$



Example2. Suppose $y'(x) - 2x = cosx$ and given condition is $y(0) = 0$.

The given equation is

$y'(x) - 2x = cosx \qquad y(0) = 0 \qquad (1)$

Rewrite the given equation in operator form

$$y'(x) = 2x + cosx$$

Applying the inverse operator $L^{-1}$and using the initial condition gives

$$y(x) = y(0) + L^{-1}[2x + cosx\,]$$

$$y(x) = L^{-1}[2x + cosx\,]$$

The recursive relation

$$y_0(x) = y(0) = 0$$

$$y_{n+1}(x) = L^{-1}[2x + cosx\,]$$

Now, the first few terms

$$y_0(x) = 0$$

$$y_1(x) = L^{-1}[2x + cosx\,]$$

$$= \int_0^x [2\mathrm{t} + cost\,]dt$$

$$y_1(x) = x^2 + sinx$$

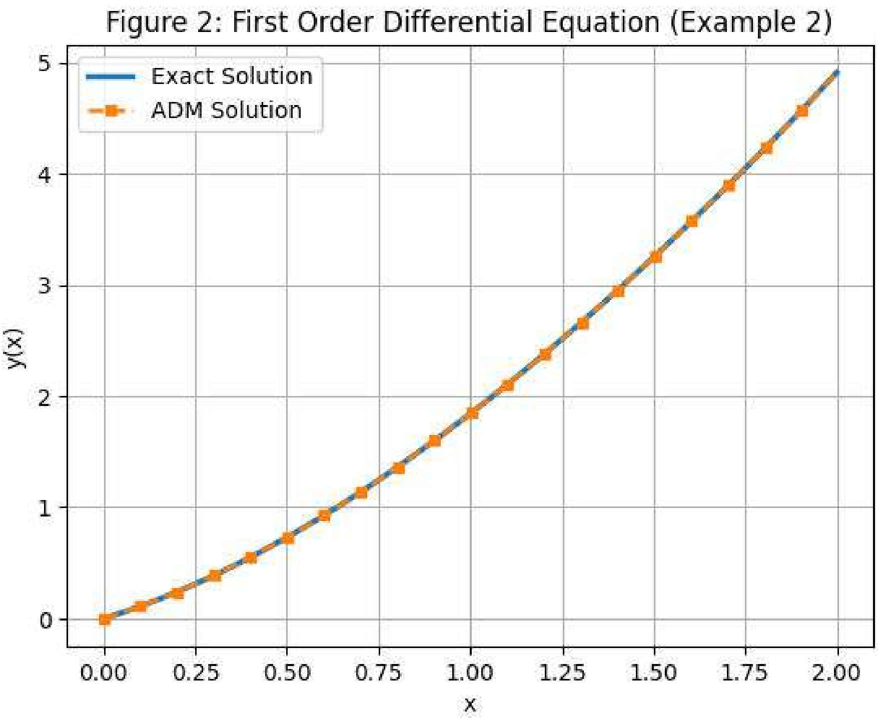
Since the equation has no recursion

$$\therefore y_2(x) = 0, y_3(x) = 0, \ldots$$

The solution in a series form is given by

$$y(x) = y_0(x) + y_1(x) + y_2(x) + \cdots$$

$$y(x) = x^2 + sinx.$$



4. APPLICATION TO SECOND ORDER DIFFERENTIAL EQUATION WITH TWO CONDITIONS

Example1. Suppose the second order differential equation is $y'' - y + x = 0$and the given two condition are$y(0) = 0 \; and \; y'(0) = 0$.

The given equation is

$$y'' - y + x = 0 \qquad y(0) = 0\,, y'(0) = 0. \tag{1}$$

Rewrite the given equation in operator form

$$y'' = y - x$$

Applying the inverse operator $L^{-1}$and using the initial condition gives

$$y(x) = y(0) + xy'(0) + L^{-1}(y) - L^{-1}(x)$$

$$y(x) = L^{-1}(y) - L^{-1}(x).$$

The recursive relation

$$y_0(x) = -L^{-1}(x).$$

$$y_{n+1}(x) = L^{-1}(y_n).$$

Now, the first few terms

$$y_0(x) = -L^{-1}(x).$$

$$= -\frac{x^3}{3!}$$

$$y_1(x) = L^{-1}(y_0).$$

$$= L^{-1}[-\frac{x^3}{3!}]$$

$$= -\frac{x^5}{5!}$$

$$y_2(x) = L^{-1}(y_1).$$

$$= L^{-1}[-\frac{x^5}{5!}]$$

$$= -\frac{x^7}{7!}$$

And so on.

The solution in a series form is given by

$$y(x) = y_0(x) + y_1(x) + y_2(x) + \cdots$$

$$y(x) = -\frac{x^3}{3!} - \frac{x^5}{5!} - \frac{x^7}{7!} - \frac{x^9}{9!} - \cdots$$

$$y(x) = -(sinhx - x).$$

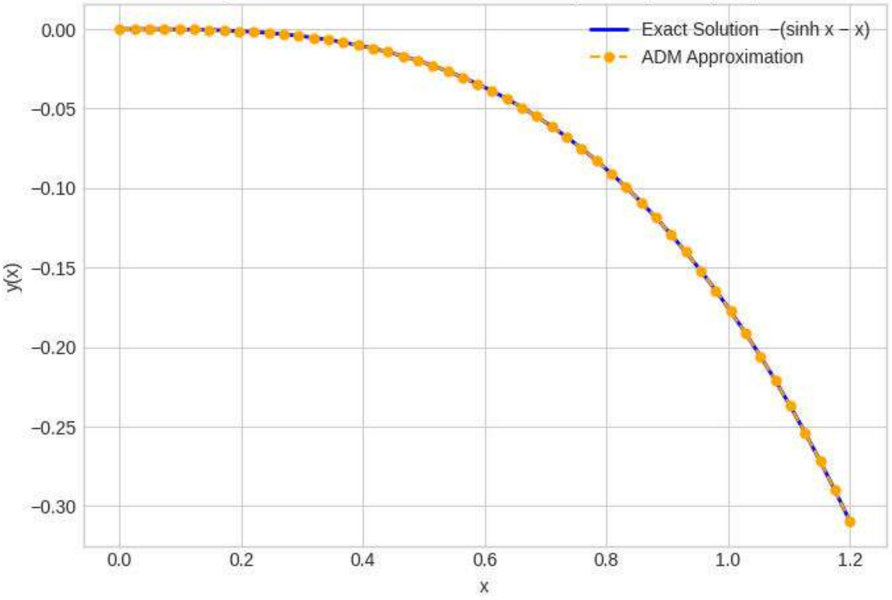


Figure 3: Second Order Differential Equation (Example 1)

Example2. we know that $y''(x) + 16y(x) = 0$ with the condition $y(0) = 1$ *and* $y'(0) = 0$. represent simple harmonic motion.

The given equation is

$$y''(x) + 16y(x) = 0, \quad y(0) = 1 \text{ and } y'(0) = 0. \qquad (1)$$

Rewrite the given equation in operator form

$$y''(x) = -16y(x)$$

Applying the inverse operator $L^{-1}$and using the initial condition gives

$$y(x) = y(0) + y'(0)x + L^{-1}[-16y(x)]$$

$$y(x) = 1 - 16L^{-1}[y(x)]$$

The recursive relation

$$y_0(x) = 1.$$

$$y_{n+1}(x) = -16L^{-1}(y_n).$$

Now, the first few terms

$$y_0(x) = 1.$$

$$y_1(x) = -16L^{-1}(y_0).$$

$$= -16L^{-1}(1)$$

$$y_1(x) = -8x^2.$$

$$y_2(x) = -16L^{-1}(y_1).$$

$$= -16L^{-1}(-8x^2)$$

$$y_2(x) = \frac{(4x)^4}{4!}.$$

$$y_3(x) = -16L^{-1}(y_2).$$

$$= -16L^{-1}\left(\frac{(4x)^4}{4!}\right).$$

$$y_3(x) = -\frac{(4x)^6}{6!}.$$

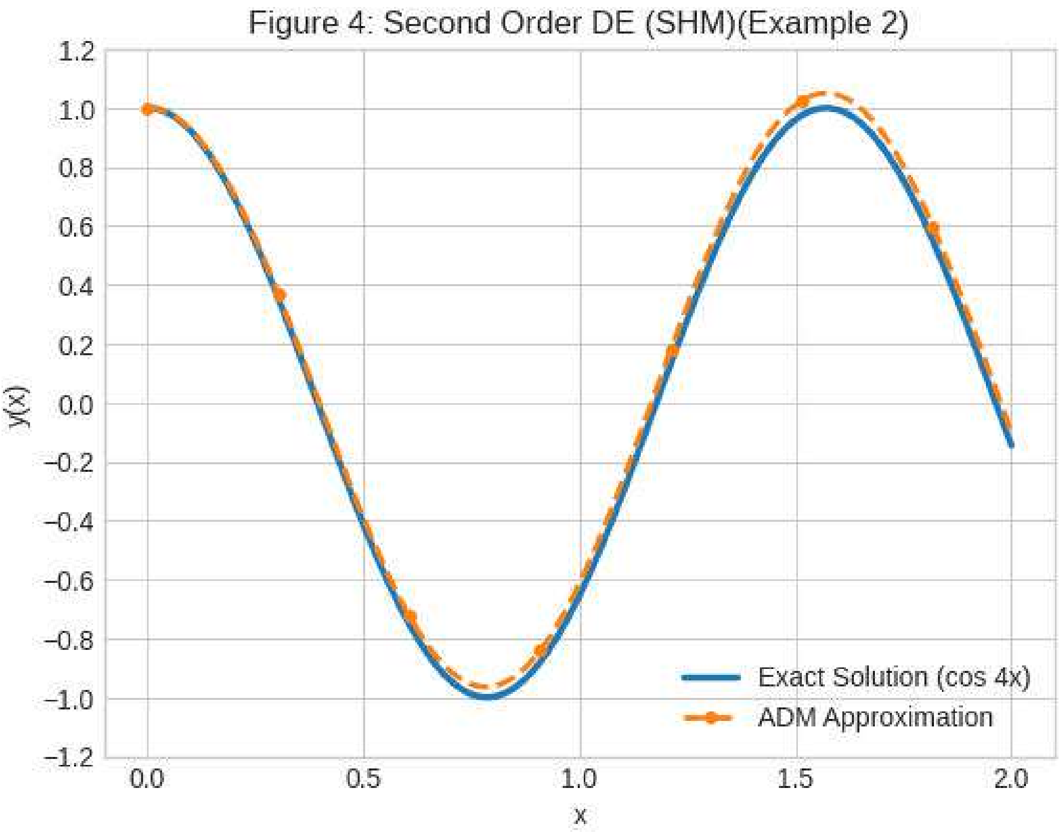
And so on.

The solution in a series form is given by

$$y(x) = y_0(x) + y_1(x) + y_2(x) + \cdots$$

$$y(x) = 1 - \frac{(4x)^2}{2!} + \frac{(4x)^4}{4!} - \frac{(4x)^6}{6!}$$

$$y(x) = cos(4x)$$



5. APPLICATION TO VOLTERRA INTEGRAL EQUATION

Example1, Let the Volterra integral equation be

$$y(x) = x - \int_0^x (x-t)y(t)dt$$

The given equation is

$$y(x) = x - \int_0^x (x-t)y(t)dt$$

Assume the solution as an infinite series

$$y(x) = \sum_{n=0}^{\infty} y_n(x)$$

The recursive relation is

$$y_0(x) = x$$

$$y_{n+1}(x) = -\int_0^x (x-t)y_n(t)dt$$

Now, the first few terms

$$y_0(x) = x$$

$$y_1(x) = -\int_0^x (x-t)y_0(t)dt$$

$$= -\int_0^x (x-t)tdt$$

$$= -\int_0^x (xt - t^2)dt$$

$$= -\left|\frac{xt^2}{2} - \frac{t^3}{3}\right|_0^x$$

$$= -\left(\frac{x^3}{2} - \frac{x^3}{3}\right)$$

$$= -\frac{x^3}{3!}$$

$$y_2(x) = -\int_0^x (x-t)y_1(t)dt$$

$$= \int_0^x (x-t)(-\frac{t^3}{6})dt$$

$$= \frac{1}{6}\int_0^x (x-t)t^3dt$$

$$= \frac{1}{6}\left[\left|\frac{xt^4}{4}\frac{t^5}{5}\right|_0^x\right]$$

$$= \frac{1}{6}\left[\frac{x^5}{4} - \frac{x^5}{5}\right]$$

$$= \frac{1}{6}\frac{x^5}{20}$$

$$y_2(x) = \frac{x^5}{5!}$$

And so on

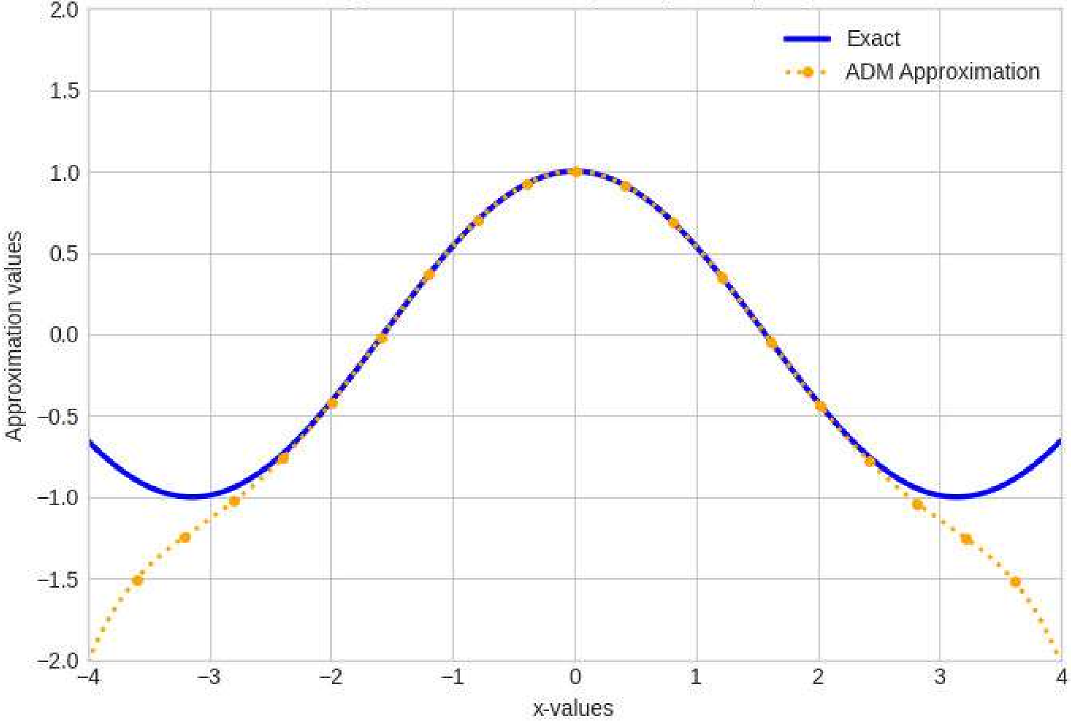
The solution in a series form is given by

$$y(x) = y_0(x) + y_1(x) + y_2(x) + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$= sin\ (x)$$

Figure 5: Volterra Integral Equation (sin x)



Example2. The equation given below is a Volterra integral equation

$$y(x) = 1 + x + \int_0^x (x - t)y(t)dt$$

The given equation is

$$y(x) = 1 + x + \int_0^x (x - t)y(t)dt$$

Assume the solution as an infinite series

$$y(x) = \sum_{n=0}^{\infty} y_n(x)$$

The recursive relation is

$$y_0(x) = 1 + x$$

$$y_{n+1}(x) = \int_0^x (x - t)y_n(t)dt$$

Now, the first few terms

$$y_0(x) = 1 + x$$

$$y_1(x) = \int_0^x (x - t)y_0(t)dt$$

$$= \int_0^x (x - t)(1 + t)dt$$

$$= \int_0^x (x - t + xt - t^2)dt$$

$$= \frac{x^2}{2} + \frac{x^3}{2} - \frac{x^3}{3}$$

$$y_1(x) = \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$y_2(x) = \int_0^x (x-t) y_1(t) dt$$

$$= \int_0^x (x-t)(\frac{t^2}{2} + \frac{t^3}{6}) dt$$

$$= \int_0^x (\frac{xt^2}{2} + \frac{xt^3}{6} - \frac{t^3}{2} - \frac{t^4}{6}) dt$$

$$= \frac{x^4}{6} + \frac{x^5}{24} - \frac{x^4}{8} - \frac{x^5}{30}$$

$$y_2(x) = \frac{x^4}{4!} + \frac{x^5}{5!}$$

And so on

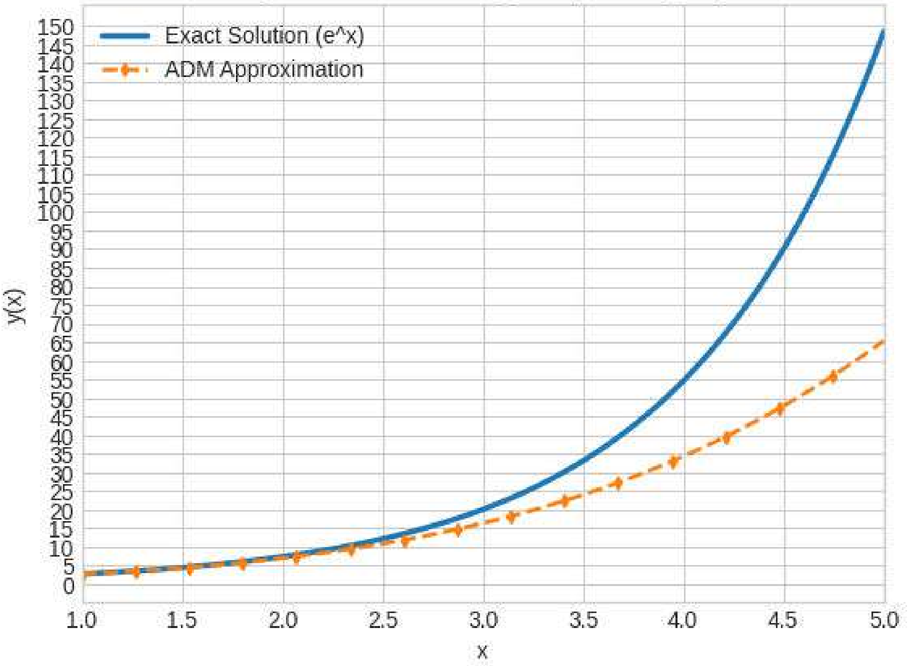
The solution in a series form is given by

$$y(x) = y_0(x) + y_1(x) + y_2(x) + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$y(x) = e^x.$$

Figure 6: Volterra Integral Equation (e^x)



6. CONCLUSIONS

This paper is aimed to obtain the approximate solution of initial value problems of first and second orders, Volterra integral equation. By observing the solutions of problems that we considered one can say that Adomian decomposition Method developed by adomian can be applied to various types of problems. By observing the graphs of exact and approximate solutions we can say that solution is obtained.

7. REFERENCES

1. G. Adomian, A Review of the Decomposition Method in Applied Mathematics, J. Math. Anal. Appl., 135(1988), 501-544.
2. G. Adomian, Solving frontier problems of physics, the Decomposition Method, Kluwer Academic Publishers, Boston, 1994.

3. G.B. Whitham, Linear and Nonlinear Waves, John Wiley, New York, (1976).
4. A.M. Wazwaz, A new technique for calculating Adomian polynomial for nonlinear polynomials, Appl. Math. Comput. 111(1), 2000, 33-51.
5. G. Nhawu, P. Mafuta, J. Mushanyu, The Adomian decomposition method for numerical solution of first order differential equations, Journal of Mathematics and Computer Science, 6 (2016), 307-314.
6. J. Biazar, E. Babolian, R. Islam, Solution of the system of ordinary differential equations by Adomian decomposition method, Applied Mathematics and Computation, 147 (2004) 713-719.
7. J. Biazar and S.M. Shafiof, A Simple Algorithm for Calculating Adomian Polynomials, Int. J. Contemp. Math. Sciences, Vol. 2, no. 20, 975-982, 2007.
8. Y. Cherruault and G. Adomian Decomposition Methods, A new proof of convergence, Math. Comput. Modelling 18 (12) (1993),103-106.
9. M. Sezer, A method for the approximate solution of the second order linear differential equations in terms of Taylor polynomials, Int. J. Math. Educ. Sci. Technol., 27 (6), pp. 821-834, 1996.
10. Fadugba Sunday Emmanuel, Solution of Integral Equations of Volterra Type Using The Adomian Decomposition Method (ADM), MathLAB Journal Vol 7 (2020) ISSN: 2582-0389

INTEGRAL TRANSFORM METHODS : AN OVERVIEW

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ABSTRACT

Integral transform methods play a fundamental role in solving a wide range of problems in applied mathematics, physics, and engineering. These methods convert complex differential equations into simpler algebraic forms, making them easier to analyze and solve. This paper presents a comprehensive review of major integral transforms, including the Laplace transform, Sumudu transform, Elzaki transform etc. It discusses their definitions and basic properties. The objective of this review is to provide a unified understanding of integral transform methods and to emphasize their significance as powerful analytical tools in both classical and modern scientific research.

Keywords: *Integral transforms, Laplace transform, Sumudu transform, Elzaki transform, Shehu transform etc.*

INTRODUCTION

Integral transform methods are among the most powerful analytical techniques used in mathematics and engineering to solve differential equations and boundary value problems. These methods work by transforming functions from one domain, typically the time or spatial domain, into another domain such as the frequency or complex domain. This transformation simplifies the mathematical structure of problems, often reducing differential equations to algebraic equations that are easier to handle.

Over the years, several integral transforms have been developed, each tailored to specific types of problems. The Laplace transform is particularly effective in solving initial value problems in control systems and circuit analysis. The importance of integral transform methods lies not only in their ability to simplify complex problems but also in their versatility across various disciplines. This paper aims to review the fundamental concepts and properties of major integral transform methods.

Transform Methods and Their Standard Properties :

1. Laplace Transform:

The origin of integral transforms can be linked to the pioneering work of Pierre-Simon Laplace in the late 18th century, particularly in his studies on probability theory. His contributions laid the foundation for what is now known as the Laplace transform, one of the earliest and most widely applied integral transforms in mathematics. Today, it remains a fundamental tool in solving differential equations and modelling physical systems.

The transformation of a function $f(t)$ using the Laplace transform is defined as:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt\ , \quad Re(s) > 0$$

The Laplace transform for some fundamental functions is given below:

1. $L\{t^n\} = \frac{n!}{s^{n+1}}, n \in N$

2. $L\{e^{at}\} = \frac{1}{s-a}, a \in R$

3. $L\{sinat\} = \frac{a}{s^2+a^2}$

4. $L\{cosat\} = \frac{s}{s^2+a^2}$

5. $L\{f'(t)\} = s\,F(s) - f(0)$

6. $L\{f''(t)\} = s^2\,F(s) - s\,f(0) - f'(0)$

2. Sumudu Transform:

The Sumudu transform, introduced in 1993 by G. K. Watugala, is defined for a function $f(t)$ as:

$$S\{f(t)\} = \frac{1}{v}\int_0^\infty e^{\frac{-t}{v}} f(t)\,dt = G(v)\,, v \in (-k_1, k_2), k_1, k_2 > 0, v \neq 0$$

The Sumudu transform for some fundamental functions is given below:

1. $S\{t^n\} = \frac{v^n}{n!}\,, \quad n \in N$
2. $S\{e^{at}\} = \frac{1}{1-av}\,, \quad a \in R$
3. $S\{sinat\} = \frac{av}{1+a^2v^2}$
4. $S\{cosat\} = \frac{1}{1+a^2v^2}$
5. $S\{f'(t)\} = \frac{G(v)-f(0)}{v}$
6. $S\{f''(t)\} = \frac{G(v)-f(0)}{v^2} - \frac{f'(0)}{v}$

3. Tarig Transform:

A new integral transform, called the Tarig transform, was introduced in 2011 by Tarig M. E. and Salih M. E., and is defined for $f(t)$ as:

$$T\{f(t)\} = \frac{1}{u}\int_0^\infty e^{\frac{-t}{u^2}} f(t)\,dt = F(u), u \neq 0$$

The Tarig transform for some fundamental functions is given below:

1. $T\{t^n\} = n!\,u^{2n+1}\,, \quad n \in N$
2. $T\{e^{at}\} = \frac{u}{1-au^2}\,, \quad a \in R$
3. $T\{f'(t)\} = \frac{1}{u^2}\,F(u) - \frac{1}{u}f(0)$
4. $T\{f''(t)\} = \frac{1}{u^4}F(u) - \frac{1}{u^3}f(0) - \frac{1}{u}f'(0)$

4. Elzaki Transform:

In 2011, Tarig M. Elzaki introduced the Elzaki transform as a new integral transform technique. It is closely related to classical transforms and is widely used for solving differential equations and engineering problems. The transform is defined for a function $f(t)$ as:

$$E\{f(t)\} = v\int_0^\infty e^{\frac{-t}{v}} f(t)\,dt = T(v)\,, v \in (k_1, k_2), k_1, k_2 > 0,$$

The Elzaki transform for some fundamental functions is given below:

1. $E\{t^n\} = n!\,v^{n+2}\,, \quad \text{n} \in N$
2. $E\{e^{at}\} = \frac{v^2}{1-av}\,, \quad a \in R$
3. $E\{sinat\} = \frac{av^3}{1+a^2v^2}$
4. $E\{cosat\} = \frac{v^2}{1+a^2v^2}$
5. $E\{f'(t)\} = \frac{T(v)}{v} - vf(0)$

6. $E\{f''(t)\} = \frac{T(v)}{v^2} - f(0) - vf'(0)$

5. Aboodh Transform:

The Aboodh transform was proposed in 2013 by Khalid Suliman Aboodh. It provides an effective technique for handling linear differential equations and related problems. The transform is defined for the function $f(t)$ as:

$$A\{f(t)\} = \frac{1}{v}\int_0^{\infty} e^{-vt} f(t)dt = K(v),\ t \geq 0.$$

The Aboodh transform for some fundamental functions is given below:

1. $A\{t^n\} = \frac{n!}{v^{n+2}}$, $n \geq 0$

2. $A\{e^{at}\} = \frac{1}{v^2 - av}$, $a \in R$

3. $A\{sinat\} = \frac{a}{v(a^2+v^2)}$

4. $A\{cosat\} = \frac{1}{(a^2+v^2)}$

5. $A\{f'(t)\} = v\,K(v) - \frac{f(0)}{v}$

6. $A\{f''(t)\} = v^2K(v) - \frac{f'(0)}{v} - f(0)$

6. Mahgoub Transform:

A new integral transform, known as the Mahgoub transform, was proposed by Mohand M. et al. This transform is defined for a function $f(t)$ as:

$$M\{f(t)\} = v\int_0^{\infty} e^{-vt}\, f(t)\, dt = H(v), \qquad v \in [k_1, k_2], t \geq 0$$

The Mahgoub transform for some fundamental functions is given below:

1. $M\{t^n\} = \frac{n!}{v^n}$, $n \in N$

2. $M\{e^{at}\} = \frac{v}{v - a}$, $a \in R$

3. $M\{sinat\} = \frac{av}{(a^2+v^2)}$

4. $M\{cosat\} = \frac{v^2}{(a^2+v^2)}$

5. $M\{f'(t)\} = v\,H(v) - v\,f(0)$

6. $M\{f''(t)\} = v^2H(v) - v^2f(0) - vf'(0)$

7. Kamal Transform:

The Kamal transform was proposed by Abdelilah Kamal et al. as a novel integral transform. It is defined for a function $f(t)$ as:

$K\{f(t)\} = \int_0^{\infty} e^{\frac{-t}{v}}\, f(t)\, dt = G(v), v \in (k_1, k_2), k_1, k_2 > 0,$

The Kamal transform for some fundamental functions is given below:

1. $K\{t^n\} = n!\, v^{\,n+1}$, $n \in N$

2. $K\{e^{at}\} = \frac{v}{1 - av}$, $a \in R$

3. $K\{sinat\} = \frac{av^2}{(1+a^2v^2)}$

4. $K\{cosat\} = \frac{v}{(1+a^2v^2)}$

5. $K\{f'(t)\} = \frac{1}{v}\ G(v) -\ f(0)$

6. $K\{f''(t)\} = \frac{1}{v^2}G(v) - \frac{1}{v}f(0) - f'(0)$

8. Mohand Transform :

Mohand M. et al. introduced an integral transform referred to as the Mohand transform. The transform is defined for a function $f(t)$ as:

$$M\{f(t)\} = v^2 \int_0^\infty e^{-vt}\, f(t)\, dt = R(v), v \in [k_1, k_2],$$

The Mohand transform for some fundamental functions is given below:

1. $M\{t^n\} = \frac{n!}{v^{n-1}}\ ,\quad n \in N$

2. $M\{e^{at}\} = \dfrac{v^2}{v-a}\ ,\ a \in R$

3. $M\{sinat\} = \frac{av^2}{(a^2+v^2)}$

4. $M\{cosat\} = \frac{v^3}{(a^2+v^2)}$

5. $M\{f'(t)\} = v\, R(v) - v^2 f(0)$

6. $M\{f''(t)\} = v^2 R(v) - v^3 f(0) - v^2 f'(0)$

9. Shehu Transform:

A generalized form of integral transform, known as the Shehu transform, was presented by Shehu Maitama and Weidong Zhao. It extends the concepts of the Laplace transform and Sumudu transform. This transform is defined for the function $f(t)$ as:

$S\{f(t)\} = \int_0^\infty e^{\frac{-st}{v}}\, f(t)\, dt = F(s,v), \quad s, v > 0$

The Shehu Transform for some fundamental functions is given below:

1. $S\{t^n\} = \frac{n!v^{n+1}}{s^{n+1}}\ ,\quad n \in N$

2. $S\{e^{at}\} = \dfrac{v}{s-av}\ ,\ a \in R$

3. $S\{sinat\} = \frac{av^2}{(s^2a^2+v^2)}$

4. $S\{cosat\} = \frac{vs}{(s^2a^2+v^2)}$

5. $S\{f'(t)\} = \frac{s}{v}F(s,v) - f(0)$

6. $S\{f''(t)\} = \frac{s^2}{v^2}F(s,v) - \frac{s}{v}F(s,v) - f'(0)$

10. Sawi Transform :

The Sawi transform was introduced by Mahgoub et al. as an integral transform technique. It is defined for a function $f(t)$ as:

$$S\{f(t)\} = \frac{1}{v^2}\int_0^\infty e^{\frac{-t}{v}}\, f(t)\, dt = T(v), v > 0$$

The Sawi transform for some fundamental functions is given below:

1. $S\{t^n\} = n!\, v^{n-1}\ ,\quad n \in N$

2. $S\{e^{at}\} = \dfrac{1}{v(1-av)}\ ,\ a \in R$

3. $S\{sinat\} = \frac{a}{(1+ a^2v^2)}$

4. $S\{cosat\} = \frac{1}{v(1+ a^2v^2)}$

5. $S\{f'(t)\} = \frac{1}{v}T(v) - \frac{1}{v^2}f(0)$

6. $S\{f''(t)\} = \frac{1}{v^2}T(v) - \frac{1}{v^2}f'(0) - \frac{1}{v^3}f(0)$

11. Rohit Transform :

The Rohit transform was proposed by S. L. Rohit Gupta as an integral transform technique. It is defined for a function $f(t)$ as:

$R\{f(t)\} = r^2 \int_0^\infty e^{-rt} f(t)\, dt = F(r),$ r is real or complex parameter

The Rohit transform for some fundamental functions is given below:

1. $R\{t^n\} = \frac{n!}{r^{n-2}}, \quad n \in N$

2. $R\{e^{at}\} = \dfrac{r^3}{r-a}\ , \ r > a$

3. $R\{sinat\} = \frac{ar^3}{(a^2+r^2)}$

4. $R\{cosat\} = \frac{r^4}{(a^2+r^2)}$

5. $R\{f'(t)\} = rF(r) - r^3f(0)$

6. $R\{f''(t)\} = r^2F(r) - r^4f(0) - r^3f'(0)$

12. Gupta Transform :

An integral transform known as the Gupta transform was introduced by Rahul Gupta et al. This transform is convergent and defined for the function $f(t)$ as:

$$G\{f(t)\} = \frac{1}{p^3}\int_0^\infty e^{-pt} f(t)\, dt = F(p), p\ is\ real\ or\ complex\ parameter$$

The Gupta transform for some fundamental functions is given below:

1. $G\{t^n\} = \frac{n!}{p^{n+4}}, \quad n \in N$

2. $G\{e^{at}\} = \dfrac{1}{p^3(p-a)}\ , \ p > a$

3. $G\{sinat\} = \frac{a}{p^3(a^2+p^2)}$

4. $G\{cosat\} = \frac{1}{p^2(a^2+p^2)}$

5. $G\{f'(t)\} = pF(p) - \frac{1}{p^3}f(0)$

6. $G\{f''(t)\} = p^2F(p) - \frac{1}{p^2}f(0) - \frac{1}{p^3}f'(0)$

13. Soham Transform :

The Soham transform was developed by D. P. Patil et al. as an integral transform. It is defined for a function $f(t)$ as:

$$S\{f(t)\} = \frac{1}{v}\int_0^\infty e^{-v^{\beta}t} f(t)dt = P(v), \beta \neq 0, v \in [\ k_1, k_2]$$

The Soham transform for some fundamental functions is given below:

1. $S\{t^n\} = \frac{n!}{v^{\beta(n+1)+1}}\ , \quad n \in N$

2. $S\{e^{at}\} = \dfrac{1}{v(v^{\beta}+a)}$, $a \in R$

3. $S\{sinat\} = \frac{a}{v(v^{2\beta}+a^2)}$

4. $S\{cosat\} = \frac{v^{\beta}}{v(v^{2\beta}+a^2)}$

5. $S\{f'(t)\} = v^{\beta}P(v) - \frac{1}{v}f(0)$

6. $S\{f''(t)\} = \frac{1}{v^{2\beta}}P(v) - v^{\beta-1}f(0) - \frac{1}{v}f'(0)$

14. Sadik Transform :

An integral transform known as the Sadik transform was proposed by S. L. Shaikh. This transform is useful in mathematical analysis and is defined for the function $f(t)$ as:

$$S\{f(t)\} = F(v^{\alpha},\beta) = \frac{1}{v^{\beta}}\int_0^{\infty} e^{-tv^{\alpha}} f(t)dt$$

Where v is a complex variable, α is any nonzero real numbers, and β is any real number.

The Sadik transform for some fundamental functions is given below:

1. $S\{t^n\} = \frac{n!}{v^{n\alpha+(\alpha+\beta)}}$, $n \geq 0$

2. $S\{e^{at}\} = \dfrac{v^{-\beta}}{v^{\alpha}-a}$, $a\ is\ constant$

3. $S\{sinat\} = \frac{av^{-\beta}}{v^{2\alpha}+a^2}$

4. $S\{cosat\} = \frac{v^{\alpha-\beta}}{v^{2\alpha}+a^2}$

5. $S\{f'(t)\} = \mathrm{v}^{\alpha}\mathrm{F}(\mathrm{v}^{\alpha},\beta) - \mathrm{v}^{-\beta}\mathrm{f}(0)$

6. $\mathrm{S}\{\mathrm{f}''(\mathrm{t})\} = \mathrm{v}^{2\alpha}\mathrm{F}(\mathrm{v}^{\alpha},\beta) - \mathrm{v}^{-\beta}\mathrm{f}'(0) - \mathrm{v}^{\alpha-\beta}\mathrm{f}(0)$

15. The Dinesh Verma Transform :

The Dinesh Verma transform (DVT) was introduced by Dinesh Verma as a convergent integral transform. It is applied in solving differential equations and is defined for a function $\mathrm{f(t)}$ as:

$\mathrm{D}\{\mathrm{f(t)}\} = \mathrm{F(p)} = \mathrm{p}^5\int_0^{\infty} \mathrm{e}^{-\mathrm{pt}}\,\mathrm{f(t)}\,\mathrm{dt}$, p is real or complex parameter

The Dinesh Verma transform for some fundamental functions is given below:

1. $\mathrm{D}\{\mathrm{t}^{\mathrm{n}}\} = \frac{\mathrm{n}!}{\mathrm{p}^{\mathrm{n}-4}}$, $\mathrm{n} \in \mathrm{N}$

2. $\mathrm{D}\{\mathrm{e}^{\mathrm{at}}\} = \dfrac{\mathrm{p}^5}{\mathrm{p}-\mathrm{a}}$

3. $\mathrm{D}\{\mathrm{sinat}\} = \frac{\mathrm{ap}^5}{(\mathrm{a}^2+\mathrm{p}^2)}$

4. $\mathrm{D}\{\mathrm{cosat}\} = \frac{\mathrm{p}^6}{(\mathrm{a}^2+\mathrm{p}^2)}$

5. $\mathrm{D}\{\mathrm{f}'(\mathrm{t})\} = \mathrm{pF(p)} - \mathrm{p}^5\mathrm{f}(0)$

6. $\mathrm{D}\{\mathrm{f}''(\mathrm{t})\} = \mathrm{p}^2\mathrm{F(p)} - \mathrm{p}^6\mathrm{f}(0) - \mathrm{p}^5\mathrm{f}'(0)$

Comparison of Integral Transforms :

Sr. No	Name of Transforms	Applications	Limitations
01	Laplace transform	Solving differential equations, control systems, circuit analysis	Requires convergence conditions, not suitable for all nonlinear problems
02	Sumudu transform	Engineering models, heat transfer, signal processing	Limited awareness, fewer standard tables
03	Tarig transform	Differential equations, applied mathematics	Limited literature and applications
04	Elzaki transform	Mathematical physics, engineering problems	Not widely adopted, fewer references
05	Aboodh transform	Linear differential equations, system modelling	Limited theoretical development
06	Mahgoub transform	Applied mathematics, integral equations	Lack of extensive research work
07	Kamal transform	Engineering and physics problems	Limited practical applications
08	Mohand transform	Differential equations, modelling problems	Less known and fewer applications
09	Shehu transform	Generalization of Laplace & Sumudu, engineering problems	Complex formulation in some cases
10	Sawi transform	Mathematical analysis, applied problems	Limited published work
11	Rohit transform	Differential equations, system analysis	Very limited usage
12	Gupta transform	Convergent transform applications, engineering problems	Less theoretical background
13	Soham transform	Applied mathematics, differential equations	Limited recognition
14	Sadik transform	Mathematical modelling, engineering systems	Fewer applications available
15	Dinesh Verma transform	Emerging applications in applied mathematics	Still under development, limited studies

Key Observations:

1. Integral transforms are widely used to obtain exact or simplified solutions of various scientific and engineering problems.
2. The Laplace transform provides a foundational basis for many integral transforms, and several modern transforms are developed from or inspired by it.
3. All the transforms discussed in this study are continuous in nature and are defined over continuous domains.
4. Each transform has its own advantages and is suitable for specific types of problems; therefore, no single transform can be considered universally superior to others.
5. Recently developed transforms such as Elzaki transform, Shehu transform, and Aboodh transform provide alternative approaches that simplify computations and offer efficient solutions for differential equations.

CONCLUSION

In this study, various integral transforms have been reviewed and compared based on their applications and limitations. Classical transforms such as the Laplace transform and Sumudu transform remain widely used due to their strong theoretical foundation and extensive applicability. On the other hand, recently developed transforms provide alternative approaches for solving differential equations and modelling complex systems, although many of them are still in the early stages of development. Overall, each transform offers specific advantages depending on the nature of the problem, and further research is required to enhance their applicability and establish broader acceptance in mathematical and engineering fields.

REFERENCES

[1] Debnath L., and Bhatta D. , "Integral Transforms and Their Applications", 2nd ed. Chapman & Hall/CRC, 2007.

[2] F. Bin Muhammed Belgacem, "Sumudu transform fundamental properties investigations and applications," Journal of Applied Mathematics and Stochastic Analysis, vol. 2006, pp. 1-23, 2006,

[3] Shaheed N. H., Muhammed Y., "Adomian Decomposition Tarig Transform Method", Journal of Physics: Conference Series, 2022,

[4] Tarig. M. Elzaki, The new integral transform "Elzaki transform", Global Journal of pure and Applied Mathematics, Vol.7, No.1, 2011, pp. 57–64.

[5] Khalid Suliman Aboodh, The New Integral Transform “Aboodh Transform” Global Journal of pure and Applied Mathematics, Vol.9, No.1,2013,pp. 35-43.

[6] Mohand M., Abdelrahim Mahgoub, The New Integral Transform "Mahgoub Transform", Advances in Theoretical and Applied Mathematics,Vol.11, No.4, 2016, pp. 391- 398.

[7] Abdelilah Kamal. H. Sedeeg, The new integral transform "Kamal Transform", Advances in Theoretical and Applied Mathematics, Vol. 11, No.4, 2016, pp. 451- 458.

[8] Mohand M. Abdelrahim Mahgoub, The New Integral Transform "Mohand Transform", Advances in Theoretical and Applied Mathematics, Vol. 12,No.2, 2017, pp.113- 120.

[9] Maitama S. and Zhao W., New integral transform: Shehu transform a generalization of Sumudu and Laplace transform for solving differential equations, International Journal of Analysis and Applications, Vol.17, No.2, 2019, pp.167- 190,

[10] Higazy M. and Aggarwal S., Sawi transformation for system of ordinary differential equations with application, Ain Shams Engineering Journal, Vol.12,2021, pp.3173- 3182,

[11] Rohit Gupta, On Novel Integral Transform: Rohit Transform and Its Application to Boundary Volue Problems, ASIO Journal of Chemistry, Physics, Mathematics and Applied Sciences, Vol.4, No.1, 2020, pp.8-13.

[12] Rahul Gupta, Rohit Gupta, Dinesh Verma, Application of Novel Integral Transform: Gupta Transform to Mechanical and Electrical - Oscillators, ASIO Journal of Chemistry, Physics, mathematics, and Applied sciences, Vol.4, No.1, 2020, pp. 4-7.

[13] D.P. Patil, Savita Santu Khakale, The New Integral Transform “Soham Transform”, International Journal of Advances in Engineering and Management, Vol.3. No.10, 2021, pp.126-132.

[14] Sadikali Latif Shaikh, Introducing a new integral transform: Sadiq transform. American International Journal of Research in Science, Technology, Engineering and Mathematics, Vol.22, No.1, 2018, pp.100-102.

[15] Dinesh Verma, Putting Forward a Novel Integral Transform Dinesh Verma Transform (DVT) and Its Applications, International Journal of Scientific Research in Mathematical and Statistical Sciences, Vol.7, No.2, 2020, pp. 139-145.

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A DIVERGENCE THEOREM APPROACH TO THE CLASSICAL HEAT WAVE EQUATION

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ABSTRACT

In this paper we focus our study to create a mathematical model of how heat spreads, or diffuses through an object, such as a metal rod, or a body of water. To do this we take advantage of our knowledge of vector calculus and the divergence theorem to set up a partial differential equation those models such a situation. The heat equation and the wave equation are two fundamental partial differential equations in the mathematical physics. We study of the heat equations and the wave equation, including their derivation, mathematical formulation, solution techniques, and applications in science and engineering.

INTRODUCTION

The heat equation models the distribution of the temperature in a given region over the time. It describes how heat diffuses through a given medium. The wave equation models oscillatory phenomena such as vibrations in strings, sound waves, and electromagnetic waves. These equations have influenced mathematics, physics, and engineering since the work of Joseph Fourier and Jean le Rond d'Alembert in the eighteenth century.[2] The key to this approach will be the observation that heat tends to flow in the direction of decreasing temperature. Bigger the difference in the temperature, faster the heat flow, or heat loss as per Newton's heating and cooling differential equation.

The Heat Equation

If the function $u(x, y, z, t)$ gives the temperature at time t at any point (x, y, z) in an object, then in mathematical terms the direction of fastest decreasing temperature away from a specific point (x, y, z), is just the gradient of u which is calculated at that point (x, y, z) and at particular time t. Here note that we are considering the gradient of u as just being with respect to the spatial coordinates x, y and z, so that we write,

$$grad(u) = \nabla u = \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k}$$

Thus the rate at which heat flows away or toward the point is proportional to this gradient, so that if F is the vector field that gives the velocity of the heat flow, then

$$\mathbf{F} = -k(grad(u))$$

The negative sign shows the flow is in the direction of fastest decreasing temperature.[5] The constant, k, is called the thermal conductivity of the object, and it determines the rate at which heat is passed through the material that the object is made of Some metals, for instance, conduct heat quite rapidly, and so have high values for the constant k, while other materials act more like insulators, with a much lower value of k as a result.[2]

Now suppose we know the temperature function, $u(x, y, z, t)$, for an object, but just at an initial time, when t = 0, i.e. we just know $u(x, y, z, 0)$. Suppose we also know the thermal conductivity of the material. What we would like to do is to figure out how the temperature of the object, $u(x, y, z, t)$, changes over the time t.[5-6] The goal is to use the observation about the rate of heat flow to set up a partial differential equation involving the function $u(x, y, z, t)$ which is the Heat Equation, and then solve this partial differential equation to find $u(x, y, z, t)$.

Deriving the Heat wave Equation using the Divergence Theorem

To get to a partial differential equation, the easiest route to take is to invoke something called the Divergence Theorem. It's such a neat application of the use of the Divergence Theorem, however, that at this point you should just skip to the end of this short section and take it on faith that we will get a partial differential equation in this situation (i.e. skip to equation (10) below. Then be sure to come back and read through this section once you’ve learned about the divergence theorem.

First notice if E is the region in the body of interest like the metal bar, the pool of water, etc. then the amount of heat that leaves E per unit time is simply a surface integral. More exactly, it is the flux integral over the surface of E of the heat flow vector field F.[2] Recall that F is the vector field that gives the velocity of the heat flow -

it's the one we wrote down as $\mathbf{F}=-k\nabla u$. Thus the amount of heat leaving E per unit time is just as $\iint_{\mathbf{S}} \mathbf{F}\cdot d\mathbf{S}$ where S is the surface of E. But, we have the highly convenient divergence theorem that tells us that

$$\iint_{\mathbf{S}} \mathbf{F}\cdot d\mathbf{S} = -k\iiint_{E} div(grad(u))dV \qquad (1)$$

Okay, now what is div(grad(u))? Given that

$$grad(u) = \nabla u = \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k} \qquad (2)$$

then div(grad(u)) is just equal to

$$div(grad(u)) = \nabla\cdot(\nabla u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \qquad (3)$$

Incidentally, this combination of divergence and gradient is used so often that it's given a name, the Laplacian.[1] The notation $div(grad(u) = \nabla\cdot(\nabla u)$ is usually shortened up to simply $\nabla^2 u$. So we could rewrite (1), the heat leaving region E per unit time as

$$\iint_{\mathbf{S}} \mathbf{F}\cdot d\mathbf{S} = -k\iiint_{E} (\nabla^2 u)dV \qquad (4)$$

On the other hand, we can calculate the total amount of heat, H, in the given region, E, at the particular time, t, by computing the triple integral over E:

$$H = \iiint_{E} (\sigma\delta)u(x,y,z,t)dV \qquad (5)$$

Where the δ is the density of the material and the constant σ is the specific heat of that material, don't worry about all these extra constants for now - we will lump them all together in one place in the end. How does this relate to the earlier integral? On one hand (4) gives the rate of heat leaving E per unit time. This is just the same as $-\frac{\partial H}{\partial t}$, where H gives the total amount of heat in E.[3-4] This means we actually have two ways to calculate this same thing, because we can calculate $\frac{\partial H}{\partial t}$ by differentiating equation (5) giving H, that is

$$-\frac{\partial H}{\partial t} = -\iiint_{E} (\sigma\delta)\frac{\partial u}{\partial t}dV \qquad (6)$$

Now since both (4) and (6) give the rate of heat leaving E per unit time, then these two equations must equal each other, so…

$$-\frac{\partial H}{\partial t} = -\iiint_{E} (\sigma\delta)\frac{\partial u}{\partial t}dV = -k\iiint_{E} (\nabla^2 u)dV \qquad (7)$$

For these two integrals to be equal means that their two integrands must equal each other. Since these integral holds over any arbitrary region E in the object being studied, so…

$$(\sigma\delta)\frac{\partial u}{\partial t} = k(\nabla^2 u) \qquad (8)$$

or, if we let $c^2 = \frac{k}{\sigma\delta}$, and write out the Laplacian, $\nabla^2 u$, then this works out simply as

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \qquad (9)$$

This, then, is the partial differential equation that models the diffusion of heat in an object, i.e. the Heat Equation! This particular version (9) is the three-dimensional heat equation.

Solution of Heat Wave Equation

We simplify the heat diffusion modeling by considering the specific case of heat flowing in a long thin bar or wire, where the cross-section is very small and constant, and insulated in such a way that the heat flow is just along the length of that bar or wire. In this slightly contrived situation, we can model the heat flow by keeping track of the temperature at any point along the bar using just one spatial dimension, measuring the position along the bar.

This means that the function, u, that keeps track of the temperature, just depends on x, the position along the bar, and at the time t, and so this heat equation becomes the so called one-dimensional heat equation:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

One of the interesting things to note at this point is how similar this partial differential equation appears to the wave equation partial differential equation. [8] However, the resulting solution functions are remarkably different in the nature. Remember that the solutions to the wave equation has to do with oscillations, dealing with vibrating strings and all that.

Here the solutions to the heat equation deal with temperature flow, not oscillation, so that means the solution functions will likely look quite different.[10] If you are familiar with the solution to Newton's heating and cooling differential equations, then you might expect to see some type of exponential decay function as the part of the solution function.

Before we start to solve this equation, let's mention a few more conditions that we will need to know to nail down a specific solution. If the metal bar that we are studying has a specific length, l, then we need to know the temperatures at the both ends of the bars.

These temperatures will give us boundary conditions similar to the ones we worked with for the wave equation.[9] To make life a bit simpler for us as we solve the heat equation, let's start with the case when the ends of the bar, at $x = 0$ and $x = l$ both have temperature equal to 0 for all time. One can picture this situation as a metal bar with the ends stuck against blocks of ice, or some other cooling apparatus keeping the ends exactly at zero degrees. Thus we will be working with the same boundary conditions as before, namely

$u(0,t) = 0$ and $u(l,t) = 0$ for all values of t

Finally, to pick out a particular solution, we also need to know the initial starting temperature of the entire bar, namely we need to know the function $u(x,0)$. Interestingly, that is all we would need for an initial condition this time around, recall that to specify a particular solution in the wave equation we needed to know the two initial conditions i.e. $u(x,0)$ and $u_t(x,0)$.

The nice thing now is that since we have already solved the partial differential equation, then we can try following the same basic approach as the one we used to solve the last partial differential equation, namely separation of variables.[4-5] With any luck, we will end up solving this new partial differential equation. So, remembering back to what we did in that case, let's start by writing

$$u(x,t) = F(x)G(t) \quad (1)$$

where F, and G, are single variable functions.[7] Differentiating this equation for $u(x,t)$ with respect to each variable yields

$$\frac{\partial^2 u}{\partial x^2} = F''(x)G(t) \text{ and } \frac{\partial u}{\partial t} = F(x)G'(t) \quad (2)$$

When we substitute these two equations back into the original heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (3)$$

we get

$$\frac{\partial u}{\partial t}=F(x)G'(t)=c^2\,\frac{\partial^2 u}{\partial x^2}=c^2F''(x)G(t) \qquad (4)$$

If we now separate the two functions F and G by dividing through both sides, then we get

$$\frac{G'(t)}{c^2G(t)}=\frac{F''(x)}{F(x)} \qquad (5)$$

Just as before, the left-hand side only depends on the variable t, and the right-hand side just depends on x. As a result, to have these two be equal can only mean one thing, that they are both equal to the same constant, k:

$$\frac{G'(t)}{c^2G(t)}=\frac{F''(x)}{F(x)}=k \qquad (6)$$

As before, let's first take a look at the implications for $F(x)$ as the boundary conditions will again limit the possible solution functions.[11] From (8) we get that $F(x)$ has to satisfy

$$F''(x)-kF(x)=0 \qquad (7)$$

Just as before, one can consider the various cases with the constant k being positive, zero, or negative. Just as before, to meet the boundary conditions, it turns out that k must in fact be negative, otherwise $F(x)$ ends up being identically equal to zero, and we end up with the trivial solution $u(x,t)=0$.

So skipping ahead a bit, let's assume we have figured out that constant k must be negative. One can check the other two cases just as before to see that what we have just written is true!.[9] To indicate this, we write, as before, that $k=-\omega^2$, so that we now need to look for solutions to $F''(x)+\omega^2F(x)=0$.

Then solutions are just the same as before, namely the general solution is:

$$F(x)=A\cos(\omega x)+B\sin(\omega x) \qquad (8)$$

where again A and B are constants and now we have $\omega=\sqrt{-k}$. Next, let's consider the boundary conditions $u(0,t)=0$ and $u(l,t)=0$.[11] These are equivalent to stating that $F(0)=F(l)=0$. Substituting 0 for x in equation (8) leads to

$$F(0)=A\cos(0)+B\sin(0)=A=0 \qquad (9)$$

Which gives that $F(x)=B\sin(\omega x)$.

Now, consider $F(l)=B\sin(\omega l)=0$. As before, we check that the constant B can't equal 0, otherwise $F(x)=0$ which would then mean that $u(x,t)=F(x)G(t)=0\cdot G(t)=0$, the trivial solution, again. With $B\neq 0$, then it must be the case that $\sin(\omega l)=0$ in order to have $B\sin(\omega l)=0$. Again, the only way that this can happen is for ωl to be a multiple of π.

$$\omega l=n\pi \text{ or } \omega=\frac{n\pi}{l}, \qquad (10)$$

Where n is constant and so get that

$$F(x)=\sin\left(\frac{n\pi}{l}x\right) \qquad (11)$$

where n is an integer. Now we solve for $G(t)$, using equation (6) again. So, rewriting (6), we see that this time

$$G'(t)+\lambda_n{}^2G(t)=0 \qquad (12)$$

here $\lambda_n=\frac{cn\pi}{l}$, since we had already consider the contant $k=-\omega^2$, and we just determined that $\omega=\frac{n\pi}{l}$ during the solution for $F(x)$.[12] The general solution to this first order differential equation is just

$$G(t) = Ce^{-\lambda_n^2 t}$$

So, now we can put it all together to find out that

$$u(x,t) = F(x)G(t) = C\sin\left(\frac{n\pi}{l}x\right)e^{-\lambda_n^2 t} \qquad (13)$$

where n is an integer, C is an arbitrary constant, and $\lambda_n = \frac{cn\pi}{l}$. As is always the case, given a supposed solution to a differential equation, which is a solution to the original heat equation, and that it satisfies the two considered boundary conditions.

CONCLUSION

Using conservation of energy and the Divergence Theorem, we derived the classical heat equation in a rigorous and systematic manner. The knowledge of the particular partial differential equation can be used to model situations involving many sorts of diffusion processes, not just heat. For instance the partial differential equation that we have derived can be used to model the spread of a drug in an organism, of the diffusion of pollutants in a water supply. The approach emphasizes the geometric and physical foundations underlying parabolic partial differential equations. Their study has significantly influenced both theoretical mathematics and practical engineering applications.

REFERENCES

[1] M. Reed and B. Simon, Methods of Modern Mathematical Physics, Volume I, Reprint Elsevier (Singapore, 2013).

[2] Mathew J Hancock, The 1-D Wave Equation 18.303 Linear Partial Differential Equation (2006)

[3] R.W Ziolkowski , I.M Besieris, A.M. Shaarawi, Aperture realizations of exact solutions to homogeneous wave equations. J OPT.SOC Am .A .10 ,75-87(1993).

[4] A.M. Shaarawi, I.M Besieris, On The superluminal propagation of X-shaped localized wave. J. Phys. A 33 , 7227-7254(2000).

[5] A. Robinson and J. Holland, Quantum wave theory of unity, Nature,2011.

[6] Vijay P. Sangale, Govardhan K. Sanap, Solution of the Wave Equation using a special Transformation of variables, IJIRSET, Vol 9 issue 9, Sept 2020 , 8806-8812

[7] L. Evans , Partial Differential Equations. American mathematical society providence 1998.

[8] R. Courant, D. Hilbert, Methods of Mathematical Physics, Vol II. Inter science (Wiley) New York, 1962.

[9] V. P. Sangale, Innovare Journal of Sciences, vol 8, Special issue I,2020,108-110

[10] Hochstadt, H. Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable. SIAM Review. Vol. 22.No. 3. Jul. 1980, 378.

[11] N. Gisinetal, “Quantum cryptography,” Rev. Mod. Phys., Vol. 74, No. 1, January 2002.

[12] K. karunagaran, Applications of D’alemberts Solution, IJISET, Vol 4, Issue 7, July 2015, 5743-5745

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ANALYSIS OF FRACTAL DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper, our study includes an exploration of method analogues of the separable method and integrating factor technique for solving $\propto -order$ differential equations. This paper provides a summary of the fractal calculus framework. It presents higher -order homogeneous and non-homogeneous linear fractal differential equation with $\propto -order$. Solutions for these equation with constant coefficients are obtained through the method of variation of parameters and the method of undetermined coefficients. We will also look on the basic concept regarding fractal derivative.

Keywords: *Fractal derivative, $\propto -order$ fractal differential equations, Homogeneous, Non-homogenous, Higher order fractal differential equation.*

1 INTRODUCTION

Fractal geometry mathematically describes complex shapes that are not described by Euclidean geometry. These shapes are found in nature such as clouds, mountains, lightning etc. which are called Fractals. The most important properties of fractals are self-similarities and have non-integer dimensions. Fractals are non-differentiable in the sense of ordinary calculus since they have a rough structure rather smooth. Their fractal dimensions exceed their topological dimensions and appear similar at various scales.

Fractals have different measures like Hausdorff measure - In this context, ordinary calculus which is based on length, area and volume fails to define derivatives and integrals on them. Many researches have tried to formulate analysis on fractals in order to explain their physical properties i.e. harmonic analysis, measure theory, fractional Brownian motion and probability theoretical approaches fractional space, fractional calculus. In seminal papers, ordinary calculus was adopted to define their derivatives and integrals of functions with fractal support like cantor sets and Koch curves. This new frame work which is a generalization of ordinary calculus is called Fractal calculus or $F\propto$-calculus. Fractal calculus is simple constructive and algorithmic and applied in physics.

Fractal derivative is a new concept of differentiation that extends the standard derivative for discontinuous fractal media. In 2006, Chen introduced the concept of Hausdorff derivative of a function with respect to a fractal measure tn, where n is the order of the fractal derivative.

Fractal calculus was developed in different branches such as stability of solutions of fractal differential equations, nonlocal reverse Minkowski's fractal integral inequalities and properties of staircase function. Fractal calculus was generalized on fractal cubes and tartan canto spaces and laplace equations on fractal cubes were solved. Fractal derivatives and integrals were worked out fractal interpolation functions and Weierstrass functions. Fractal calculus is a way to extend calculus and deal with equations that have solutions, in form of functions with fractal properties like fractal sets and curves.

The fractal derivative or Hausdorff derivative is a non-Newtonian generalization of derivative dealing with measurement of fractals, defined in fractal geometry. This generalization is not merely a purely mathematical curiously, but it has demonstrated its application in various disciplines such as physics, biology, engineering and economics.

Fractal calculus is a way to extend calculus and deal with equations that have solutions, in the form of functions with fractal properties like fractal sets and curves. The beauty of fractal calculus lies in its simplicity and algorithmic approaches when compared to methods. The generalization of $F\propto$-calculus (FC) has been achieved through the utilization of the gauge integral method. Fractal calculus has been extended to include Cantor cubes and Cantor tartan and the Laplace equation has been defined within this framework. This paper introduces the utilization of an integrating factor to solve fractal α-order differential equations. In this paper we outline the application of the method of separation to fractal differential equations and introduces higher α-order linear fractal differential equations. Also, the homogeneous higher α-order linear fractal differential equation is introduced and solved. We will discuss nonhomogeneous higher α-order linear fractal differential equations & Homogeneous higher α-order linear fractal differential equations with constant coefficients are solved. The specific solutions for nonhomogeneous higher α-order linear fractal differential equations are obtained using the variation of parameters and the method of undetermined coefficients.

2.1 Overview of Fractal Calculus on Fractal Sets

In this section, we present a concise overview of fractal calculus applied to fractal sets.

Definition 1. The flag function of a set F and a closed interval I is defined as:

$$\rho(F,I)=\begin{cases}1, & if\ F\ \cap I=\emptyset;\\ 0, & otherwise.\end{cases}$$

Definition 2. For a fractal set $F\subset[a,b]$, a subdivision $P_{[a,b]}$ of $[a,b]$ and a given $\delta>0$, the coarse-grained mass of $F\cap[a,b]$ is defined by

$$\gamma_\delta^\alpha(F,a,b)=\underset{|P|\le\delta}{inf}\sum_{i=0}^{n-1}\Gamma(\alpha+1)(t_{i+1}-t_i)^\alpha\rho(F,[t_i,t_{i+1}])$$

where, $|P|=max_{0\le i\le n-1}(t_{i+1}-t_i), and\ 0\ <\ \alpha\ \le\ 1.$

Definition 3. The mass function of a fractal set F is defined as the limit of the coarse-grained mass as δ approaches zero:

$$\gamma^\alpha(F,a,b)=\lim_{\delta\to0}\gamma_\delta^\alpha(F,a,b)$$

Definition 4. The γ-dimension of $F\cap[a,b]$ is defined as:

$$\begin{aligned}dim_\gamma(F\cap[a,b])&=\inf\{\alpha:\gamma^\alpha(F,a,b)=0\}\\&=\sup\{\alpha:\gamma^\alpha(F,a,b)=\infty\}\end{aligned}$$

Definition5.The integral staircase function of order α for a fractal set F is given by:

$$S_F^\alpha(x)=\begin{cases}\gamma^\alpha(F,a_0,x), & if\ x\ge a_0\\ -\gamma^\alpha(F,x,a_0), & otherwise\end{cases}$$

Where a_0 is an arbitrary fixed real number.

Definition 6. Let F be an $\alpha-perfect$ fractal set, let f be a function defined on F and let $x\in F$ The $F^\alpha-derivative$ of f at the point x is defined as follows:

$$D_F^\alpha f(x)=\begin{cases}F-\lim\limits_{y\to x}\dfrac{f(y)-f(x)}{S_F^\alpha(y)-S_F^\alpha(x)}, & if x\in F\\ 0, & otherwise\end{cases}$$

if the fractal limit $F-lim$ exists.

Definition 7. Let $I=[a,b]$. Let F be a $\alpha-perfect$ fractal set such that S_F^α is fi
Let f be a bounded function defined on F and let $x\in F$. The $F^\alpha-integral$ of
defined as:

$$\int_a^b f(x)\, d_F^\alpha x = \underset{P_{[a,b]}}{sup} \sum_{i=0}^{n-1} \underset{x\,\in\, F\,\cap\, I}{inf} f(x)(\,S_F^\alpha(x_{i+1}) - S_F^\alpha(x_i))$$

$$= \underset{P_{[a,b]}}{inf} \sum_{i=0}^{n-1} \underset{x\,\in\, F\,\cap\, I}{sup} f(x)(\,S_F^\alpha(x_{i+1}) - S_F^\alpha(x_i)$$

2.2 Fractal Calculus on Fractal Curves

We begin with defining the key concepts in fractal calculus on fractal curves. By the way, we recall that a fractal curve $F\subset[a,\,b]$ is parametrizable if there exists a bijective and continuous function $\boldsymbol{w}:[a0,\,b0]\to R$. Moreover, we recall also that by $(a,\,b)$ we denote the segment of the curve lying between the points (a) and (b) on the fractal curve F.

Definition 8. For a fractal curve denoted as F and a subdivision denoted as $P_{[a,b]}$, where $[a,b]\subset$ R, the mass function is given by

$$\gamma^\alpha(F,a,b) = \lim_{\delta\to 0} \underset{|P|\,\le\,\delta}{inf} \sum_{i=0}^{n-1} \frac{|\boldsymbol{w}(t_{i+1}) - \boldsymbol{w}(t_i)|^\alpha}{\Gamma(\alpha+1)}$$

Where $|\cdot|$ represents the Euclidean norm in $R^n, 1\le\alpha\le n$, $P_{[a,b]}=\{a=t_0,\ldots\ldots t_n=b\}$ and $|P|=max_{0\le i\le n-1}(t_{i+1}-t_i)$ for a subdivision $P_{[a,b]}$

Definition 9. The γ-dimension of fractal curve F is defined as:

$$dim_\gamma(F) = \inf\{\alpha:\gamma^\alpha(F,a,b)=0\}$$

$$= \sup\{\alpha:\gamma^\alpha(F,a,b)=\infty\}$$

Definition 10. Let $P_0\in[a_0,b_0]$ be arbitrary but fixed. The mass of the fractal of a fractal curve F is defined as:

$$S_F^\alpha(u) = \begin{cases} \gamma^\alpha(F,p_0,u), & u\ge p_0 \\ -\gamma^\alpha(F,u,p_0), & u<p_0 \end{cases}$$

The mass of the fractal curve F up to point u is provided by $S_F^\alpha(u)$, where $u\in[a_0,b_0]$.

Definition 11. Let $S_F^\alpha(u)=J(\theta)$. The fractal $F^\alpha-derivative$ of a function f at a point $\theta\in F$ is defined as:

$$D_F^\alpha f(x) = F-\lim_{\theta'\to\theta} \frac{f(\theta')-f(\theta)}{J(\theta')-J(\theta)},$$

if the fractal limit $F-lim$ exists.

Remark 1: It is worth noting that the Euclidean distance from the origin to a point θ = **w**(u) is given by L(θ) = L(**w**(u)) = |**w**(u)|

Definition 12. The fractal integral or F^α-integral is defined as

$$\int_{C(a,b)} f(\theta)\, d_F^\alpha\theta = \underset{P_{[a,b]}}{sup} \sum_{i=0}^{n-1} \underset{\theta \in C(t_i,\, i_{i+1})}{inf} f(\theta)(J(\theta_{i+1}) - J(\theta_i))$$
$$= \underset{P_{[a,b]}}{inf} \sum_{i=0}^{n-1} \underset{\theta \in C(t_i,\, i_{i+1})}{sup} f(\theta)(J(\theta_{i+1}) - J(\theta_i))$$

Where $t_i = w^{-1}(\theta_i)$ and f is a bounded function on a fractal curve F

3.1 Solving Fractal Differential Equations by Method of Integrating Factor

In this section, we delve into the concept of differential equations on fractal curves and fractal sets. We start by considering an α-order linear differential equation on a fractal curve $F \subset R^n$:

$$D_F^\alpha y(\theta) + p(\theta)y(\theta) = g(\theta), \qquad \theta \in F, \tag{1}$$

where p and g are F -continuous functions defined on the fractal curve F, with $\varphi_1 < \theta < \varphi_2$ and $\varphi_1, \varphi_2 \in F$.

Definition 1 Let $\psi: F \subset R^n \to R$ be a function. If ψ has fractal $F^\alpha - derivative$ at each point $\theta \in F$, therefore ψ is called the solution of the α-order differential equation if substituted in the Eq. (1) satisfies it.

Theorem 1 (Method of the integration factor) Let $F \subset R^n$ be a fractal curve, therefore there exists a fractal $F^\alpha - derivative$ function defined on a fractal curve F called integration factor, such that all the solutions of the Eq. (1) are expressed by:

$$y(\theta) = \frac{\int \mu(\theta)g(\theta)d_F^\alpha\theta + J(c)}{\mu(\theta)} \tag{2}$$

Here $\mu(\theta)$ is the integration factor and $J(c)$ is an arbitrary constant.

Proof 1 To solve Eq. (1), we introduce an integrating factor $\mu(\theta)$ and multiply both sides of the equation by it:

$$\mu(\theta)D_F^\alpha y(\theta) + \mu(\theta)p(\theta)y(\theta) = \mu(\theta)g(\theta) \tag{3}$$

For this modified equation to hold, we require the following relationship:

$$D_F^\alpha \mu(\theta) = p(\theta)\mu(\theta) \tag{4}$$

Assuming $\mu(\theta) > 0$ we can express Eq. (4) as:

$$\frac{D_F^\alpha \mu(\theta)}{\mu(\theta)} = p(\theta) \tag{5}$$

By applying fractal integration, we arrive at the integral equation:

$$\ln(\mu(\theta)) = \int p(\theta)\, d_F^\alpha\theta + J(k) \tag{6}$$

where $J(k)$ is an arbitrary constant of integration. Setting $J(k) = 0$ we obtain the expression for the integrating factor:

$$\mu(\theta) = \exp\,(\int p(\theta)\, d_F^\alpha\theta) \tag{7}$$

After determining $\mu(\theta)$, we substitute it back into Eq. (3), yielding:

$$D_F^\alpha\big(\mu(\theta)y(\theta)\big) = \mu(\theta)g(\theta) \tag{8}$$

Integrating both sides of the equation using fractal integration, we arrive at the solution for the original differential equation (1):

$$y(\theta) = \frac{\int \mu(\theta)g(\theta)d_F^\alpha\theta + J(c)}{\mu(\theta)} \tag{9}$$

which completes the proof.

Remark 2 By the previous theorem it follows that there are infinitely many functions $y(\theta)$ that satisfy the given α-order fractal differential equation on the fractal curve F.

3.2 Solving Fractal Differential Equations by Method of Separation

The equation representing a separable α-order fractal differential equation is given as

$$D_F^\alpha y(\theta) = \frac{d_F^\alpha y}{d_F^\alpha \theta} = f(\theta, y), \qquad \theta \in F \tag{10}$$

with the initial condition

$$y(\theta_0) = y_0 \tag{11}$$

Here, $f(\theta, y)$ takes a linear form with respect to y. The equation (10) can be rearranged as:

$$M(\theta, y) + N(\theta, y)\frac{d_F^\alpha y}{d_F^\alpha \theta} = 0 \tag{12}$$

where $M(\theta, y) = -f(\theta, y)$ and $N(\theta, y) = 1$. By considering $M(\theta, y) = M(\theta)$and $N(\theta, y) = N(\theta)$, we arrive at:

$$M(\theta)d_F^\alpha\theta + N(y)d_F^\alpha y = 0 \tag{13}$$

This equation is referred to as a separable fractal differentiable equation. To solve (13), we introduce functions $D_F^\alpha H_1(\theta) = M(\theta)\ and\ D_F^\alpha H_2(y) = N(y)$.

Therefore Eq. (13) has the following expression:

$$D_F^\alpha H_1(\theta) + D_F^\alpha H_2(y)\frac{d_F^\alpha y}{d_F^\alpha \theta} = 0 \tag{14}$$

Now by fractal chain rule, we have:

$$D_F^\alpha H_2(y)\frac{d_F^\alpha y}{d_F^\alpha \theta} = \frac{d_F^\alpha}{d_F^\alpha \theta}H_2(y) \tag{15}$$

Consequently, by Eq. (14) and Eq. (15) we have:

$$\frac{d_F^{\alpha}}{d_F^{\alpha}\theta}[H_1(\theta)+H_2(y)]=0 \tag{16}$$

and applying fractal integration, we obtain:

$$H_1(\theta)+H_2(y)=c \tag{17}$$

The Eq. (17) is the implicit solution of the Eq. (13). Now by substituting the initial condition into Eq. (17) we get:

$$H_1(\theta_0)+H_2(y_0)=c \tag{18}$$

Finally, by replacing (18) into (17), we arrive at:

$$H_2(y)-H_2(y_0)=\int_{y_0}^{y} N(s)d_F^{\alpha}s,\ \ H_1(\theta)-H_1(\theta_0)=\int_{C(\theta_0,\theta)}^{\cdot} M(s)d_F^{\alpha}s \tag{19}$$

This leads to:

$$\int_{y_0}^{y} N(s)d_F^{\alpha}s+\int_{C(\theta_0,\theta)}^{\cdot} M(s)d_F^{\alpha}s=0 \tag{20}$$

which is the implicit solution of (13), satisfying the initial condition.

Example Let's consider the fractal differential equation on a fractal curve given by

$$D_F^{\alpha}y(\theta)=\frac{3J(\theta)^2+4J(\theta)+2}{2(y-1)},\qquad y(0)=-1 \tag{21}$$

We can rewrite Eq. (21) as follows:

$$2(y-1)d_F^{\alpha}y=(3J(\theta)^2+4J(\theta)+2)d_F^{\alpha}\theta \tag{22}$$

Moreover, by fractal integration with respect to y on the left side and with respect to θ on the right side, we obtain:

$$y^2-2y=J(\theta)^3+2J(\theta)^2+2J(\theta)+c \tag{23}$$

Finally, by using the initial condition $y(0)=-1$ in Eq. (23) we get:

$$y^2-2y=J(\theta)^3+2J(\theta)^2+2J(\theta)+3 \tag{24}$$

It can be further simplified to:

$$y(\theta)=1-\sqrt{J(\theta)^3+2J(\theta)^2+2J(\theta)+4}$$

4.1 Higher α-Order Linear Fractal Differential Equations

Within this section, we present higher-order linear fractal differential equations and their corresponding solutions. A $n^{th}\ \alpha - order$ linear fractal differential equation can be expressed in the following form:

$$P_0(x)D_F^{n\alpha}f(x) + P_1(x)D_F^{(n-1)\alpha}f(x) + \cdots + P_{n-1}(x)D_F^{\alpha}f(x) + P_n(x)f(x) = G(x) \qquad (25)$$

In cases where the functions $P_0, \dots P_n$ and $G(x)$ exhibit $F - continuity$ over the interval $I: a < x < b$, with the additional condition that P_0 is non-zero everywhere in this interval , dividing Eq.(25) by $P_0(x)$ results in:

$$L[f] = D_F^{n\alpha}f(x) + p_1(x)D_F^{(n-1)\alpha}f(x) + \cdots + p_{n-1}(x)D_F^{\alpha}f(x) + p_n(x)f(x) = g(x) \qquad (26)$$

To derive a unique solution for Eq. (26), it is necessary to have n initial conditions, as outlined below:

$$f(x_0) = f_0,\ \ D_F^{\alpha}f(x)|_{x=x_0} = D_F^{\alpha}f(x_0), \dots, D_F^{\alpha(n-1)}f(x)|_{x=x_0} = D_F^{\alpha(n-1)}f(x_0) \qquad (27)$$

Here, x_0 represents any point within the interval (a, b) and

$$f_0, D_F^{\alpha}f(x_0), \dots, D_F^{\alpha(n-1)}f(x_0)$$

can take any real constant numbers.

Definition. The solution space for higher α -order linear fractal differential equations has a dimensionality of $n\alpha$.

4.2 The Homogeneous Higher α –order Linear Fractal Differential Equation

The $n^{th}\ \alpha - order$ fractal differential equation in its homogeneous form can be expressed as:

$$\begin{aligned} L[f] &= D_F^{n\alpha}f(x) + p_1D_F^{(n-1)\alpha}f(x) + \cdots + p_{n-1}D_F^{\alpha}f(x) + p_n(t)f(x) = 0 \\ &= f^{n\alpha}(x) + p_1f^{(n-1)\alpha}(x) + \cdots + p_{n-1}f^{\alpha}(x) + p_n(t)f(x) = 0 \end{aligned} \qquad (28)$$

If the functions $f_1, f_2, \dots, f_n$ constitute solutions to Eq. (28), it consequently follows that:

$$f(x) = c_1f_1(x) + c_2f_2(x) + \cdots + c_nf_n(x) \qquad (29)$$

Where $c_1, c_2, \dots, c_n$represent arbitrary constants, any solution of Eq. (29) can be expressed as:

$$c_1f_1(x_0) + c_2f_2(x_0) + \cdots + c_nf_n(x_0) = f(x_0) = f_0$$

$$c_1f_1^{\alpha}(x_0) + c_2f_2^{\alpha}(x_0) + \cdots + c_nf_n^{\alpha}(x_0) = f^{\alpha}(x_0) = f_0^{\alpha}$$

.

$$c_1 f_1^{(n-1)\alpha}(x_0) + c_2 f_2^{(n-1)\alpha}(x_0) + \cdots + c_n f_n^{(n-1)\alpha}(x_0) = f^{(n-1)\alpha}(x_0) = f_0^{(n-1)\alpha} \quad (30)$$

$here\ x \in (a,b)$ and $D_F^{n\alpha} f(x) = f^{n\alpha}(x)$.

For arbitrary values of $f_0, f^{\alpha}0, \ldots., f^{(n-1)\alpha}0$, a necessary and sufficient condition for the existence of a solution to the system of algebraic equations in (30) is that the Wronskian be:

$$W[f_0, f_1, \ldots, f_n] = \begin{vmatrix} f_1 & f_2 & \ldots\ldots f_n \\ f_1^{\alpha} & f_2^{\alpha} \quad \ldots\ldots & f_n^{\alpha} \\ f_1^{(n-1)\alpha} & f_2^{(n-1)\alpha} & f_n^{(n-1)\alpha} \end{vmatrix} \quad (31)$$

To ensure that the Wronskian $W[f_0, f_1, \ldots, f_n]$ is non-zero at $x = x_0$, where $x_0 \in (a,b)$, it is necessary and sufficient that $W[f_0, f_1, \ldots, f_n]$ is non-zero at every point within the interval (a,b).

Theorem. If the functions $p_1, p_2, \ldots, p_n$ are $F - continuouse$ on the open interval (a,b) and $f_1, f_2 \ldots, f_n$ are solutions to equation (28), with $W[f_1, f_2 \ldots, f_n] \neq 0$ for at least one point in (a,b), then every solution of equation (28) can be represented as a linear combination of the solutions $f_1, f_2 \ldots, f_n$.

Proof. Consider the equation

$$k_1 f_1(x) + k_2 f_2(x) + \cdots + k_n f_n(x) = 0 \quad (32)$$

By repeatedly differentiating (32), we obtain an additional $n - 1$ equations:

$$\begin{gathered} k_1 f_1^{\alpha}(x) + k_2 f_2^{\alpha}(x) + \cdots + k_n f_n^{\alpha}(x) = 0 \\ \vdots \\ k_1 f_1^{(n-1)\alpha}(x) + k_2 f_2^{(n-1)\alpha}(x) + \cdots + k_n f_n^{(n-1)\alpha}(x) = 0 \end{gathered} \quad (33)$$

These equations form a system of algebraic equations for n unknowns $k_1, k_{2,} \ldots, k_n$. The determinant of the coefficients for this system is the Wronskian $W[f_1, f_2 \ldots, f_n]$ of $f_1, f_2 \ldots, f_n$.

Theorem. If $f_1(x), f_2(x) \ldots, f_n(x)$ constitute a fundamental set of solutions for the homogeneous n^{th} order linear fractal differential equation:

$$L[f] = f^{n\alpha}(x) + p_1(x) f^{(n-1)\alpha}(x) + \cdots + p_{n-1}(x) f^{\alpha}(x) + p_n(x) f(x) = 0 \quad (34)$$

defined on an interval (a,b), then $f_1(x), f_2(x) \ldots, f_n(x)$ are linearly independent on (a,b). Conversely, if $f_1(x), f_2(x) \ldots, f_n(x)$ are linearly independent solutions of equation (34) on (a,b), then they form a fundamental set of solutions on (a,b).

Proof. To establish this theorem, let's initially assume that $f_1(x), f_2(x) \ldots, f_n(x)$ constitute a fundamental set of solutions for the homogeneous fractal differential equation (34) on (a,b). Consequently, the Wronskian $W[f_1, f_2 \ldots, f_n] \neq 0$ for every $x \in (a,b)$. Therefore, the system (32) and (33) has only the solution $k_1 = k_2 = \cdots = k_n = 0$ for every $x \in (a,b)$

Consequently, $f_1(x), f_2(x) \dots, f_n(x)$ must be linearly independent on (a, b). To demonstrate the converse, let $f_1(x), f_2(x) \dots, f_n(x)$ be linearly independent on (a, b). In order to show that they form a fundamental set of solutions, it is necessary to demonstrate that their Wronskian is never zero in (a, b). Suppose, for the sake of contradiction, that this is not true, indicating that there exists at least one-point x_0 where the Wronskian is zero. At this point, the system (15) and (16) has a non-zero solution denoted by $k_1, k_2, \dots, k_n$. Forming the linear combination:

$$\psi(t) = k_1 f_1(x) + k_2 f_2(x) + \cdots + k_n f_n(x) \qquad (35)$$

Then $f = \psi(t)$ satisfies the initial value problem:

$$L[f] = 0, \quad f(x_0) = 0, \quad f^{\alpha}(x_0) = 0, \quad f^{(n-1)\alpha}(x_0) = 0 \qquad (36)$$

The function ψ satisfies the fractal differential equation because it is a linear combination of solutions, and it satisfies the initial conditions because these are just the equations in the system (32) and (33) evaluated at x_0. However, the function $f(x) = 0$ for all $x \in (a, b)$ is also a solution of this initial value problem, and by Theorem 1, the solution to the initial value problem (36) is unique. Thus, $\psi(t) = 0$ for all $x \in (a, b)$. Consequently, $f_1(x), f_2(x) \dots, f_n(x)$ are linearly dependent on (a, b), which is a contradiction. Hence, the assumption that there is a point where the Wronskian is zero is untenable. Hence, the Wronskian is never zero on (a, b), as was to be proved.

5 The Non-homogeneous Higher α-Order Linear Fractal Differential Equation

In this section, we introduce non-homogeneous n^{th} $\alpha - order$ fractal differential equations along with their solutions. Consider the nonhomogeneous n^{th} $\alpha - order$ fractal differential equation:

$$L[f] = f^{n\alpha}(x) + p_1(x) f^{(n-1)\alpha}(x) + \cdots + p_{n-1}(x) f^{\alpha}(x) + p_n(x) f(x) = g(x) \qquad (37)$$

If $F_1 and\ F_2$ are any two solutions, it immediately follows from the linearity of the fractal operator L that:

$$L[F_1 - F_2](x) = L[F_1](x) - L[F_2](x) = g(x) - g(x) = 0 \qquad (38)$$

Hence, the difference of any two solutions of the non-homogeneous fractal differential Eq.(37) is a solution of the homogeneous fractal differential equation Eq.(28). Since any solution of the nonhomogeneous equation can be expressed as a linear combination of a fundamental set of solutions$f_1, f_2 \dots, f_n$, it follows that any solution of the nonhomogeneous fractal differential equation Eq.(37) can be written as:

$$f(x) = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) + F(x) \qquad (39)$$

where F(x) is some particular solution of Eq.(37). The linear combination Eq.(39) is called the general solution of the nonhomogeneous Eq.(37).

6. CONCLUSION

In conclusion, the study delved into a comprehensive exploration of various methods in the realm of fractal calculus. By investigating the method analogues of the separable method and integrating factor technique, we addressed α-order differential equations. Also, this paper has delved into the realm of nth α-order fractal homogeneous and non-homogeneous differential equations, providing solutions for both.

7. REFERENCES

[1] B. B. Mandelbrot, The Fractal Geometry of Nature, WH freeman New York, 1982.

[2] K. Falconer, Fractal Geometry: Mathematical Foundations and Applica tions, John Wiley & Sons, 2004.

[3] P. E. Jorgensen, Analysis and probability: wavelets, signals, fractals, Vol. 234, Springer Science & Business Media, 2006.

[4] P. R. Massopust, Fractal Functions, Fractal Surfaces, and Wavelets, Aca demic Press, 2017.

[5] M. L. Lapidus, G. Radunovi´c, D. ˇ Zubrini´c, Fractal Zeta Functions and Fractal Drums, Springer International Publishing, 2017.

[6] C. A. Rogers, Hausdorff Measures, Cambridge University Press, 1998.

[7] N. Lesmoir-Gordon, B. Rood, Introducing Fractal Geometry, Icon Books, 2000.

[8] M. F. Barnsley, Fractals Everywhere, Academic Press, 2014.

[9] T. G. Dewey, Fractals in Molecular Biophysics, Oxford University Press, 1998.

[10] E. Rosenberg, Fractal dimensions of networks, Vol. 1, Springer, 2020.

[11] L. Pietronero, E. Tosatti (Eds.), Fractals in Physics, Elsevier, 1986.

[12] C. J. Bishop, Y. Peres, Fractals in probability and analysis, Vol. 162, Cam bridge University Press, 2017.

[13] A. Bunde, S. Havlin, Fractals in Science, Springer, 2013.

[14] J. Kigami, Analysis on fractals, no. 143, Cambridge University Press, 2001.

[15] R. S. Strichartz, Differential Equations on Fractals, Princeton University Press, 2018.

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Sillick, T. J., & Schutte, N. S. (2006). Emotional intelligence and self-esteem mediate between perceived early parental love and adult happiness. *E-Journal of Applied Psychology*, *2*(2), 38-48. Retrieved from http://ojs.lib.swin.edu.au/index.php/ejap

- **Unpublished dissertation/ paper:**

Uddin, K. (2000). A Study of Corporate Governance in a Developing Country: A Case of Bangladesh (Unpublished Dissertation). Lingnan University, Hong Kong.

- **Article in newspaper:**

Yunus, M. (2005, March 23). Micro Credit and Poverty Alleviation in Bangladesh. *The Bangladesh Observer*, p. 9.

- **Article in magazine:**

Holloway, M. (2005, August 6). When extinct isn't. *Scientific American, 293*, 22-23.

- **Website of any institution:**

Central Bank of India (2005). *Income Recognition Norms Definition of NPA*. Retrieved August 10, 2005, from http://www.centralbankofindia.co.in/ home/index1.htm, viewed on

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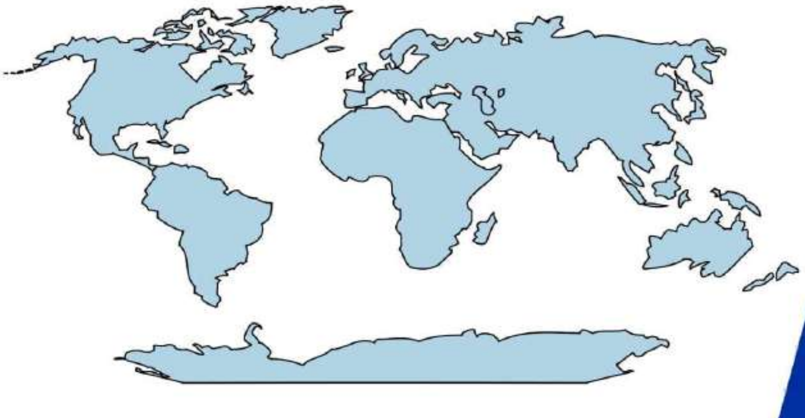

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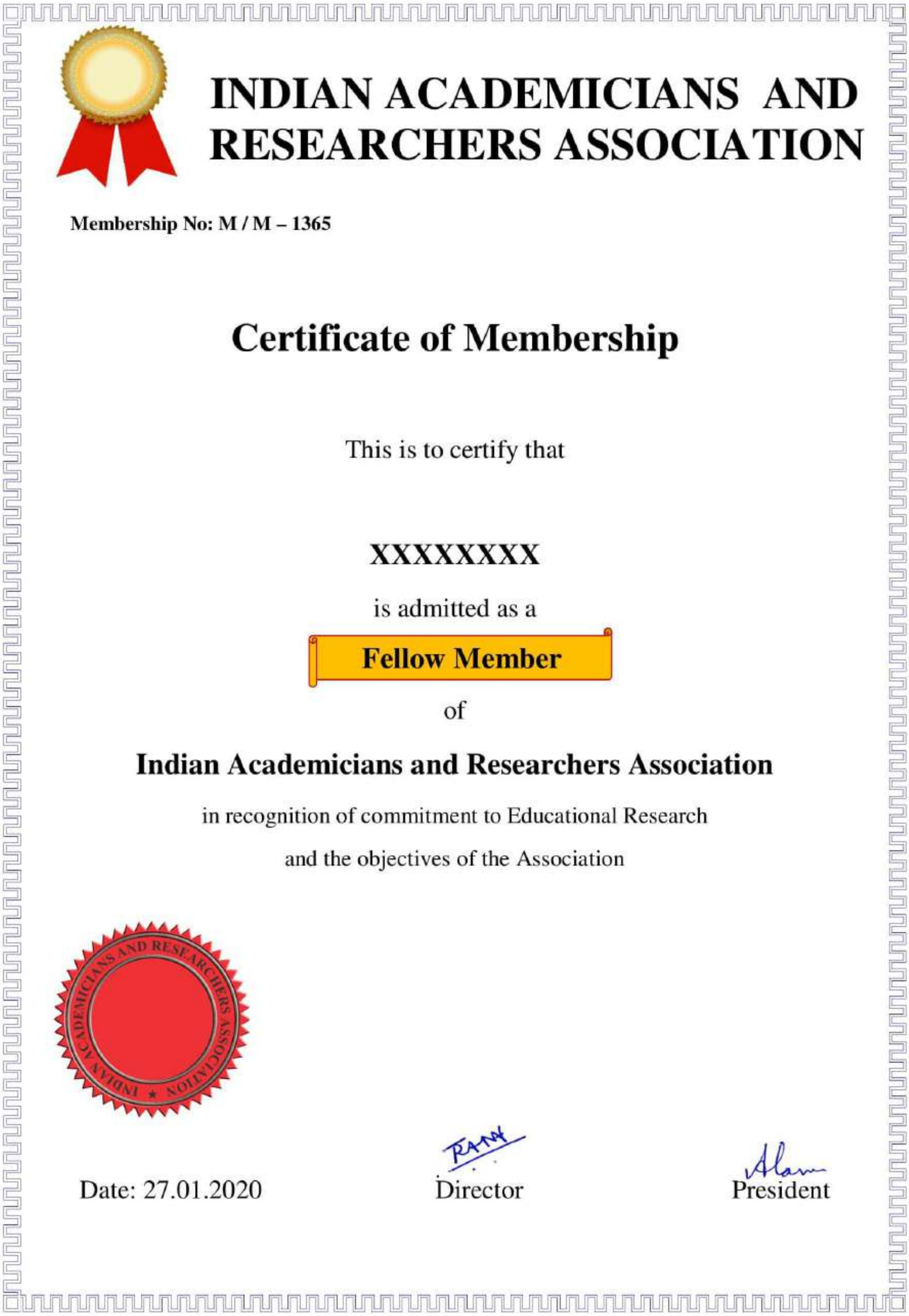

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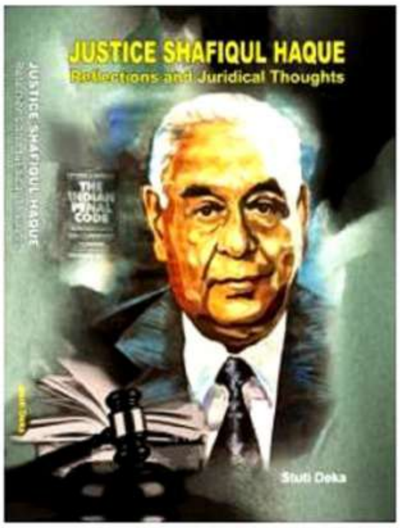

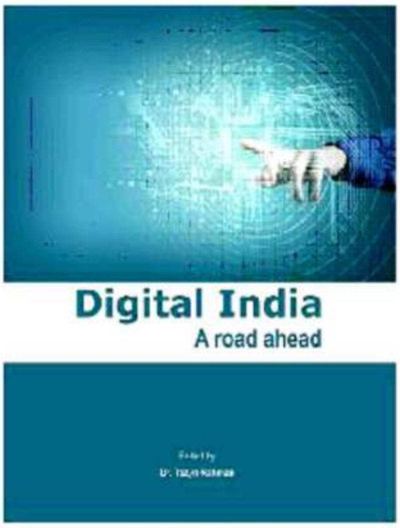

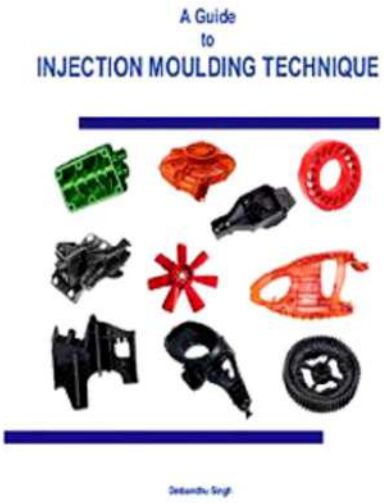

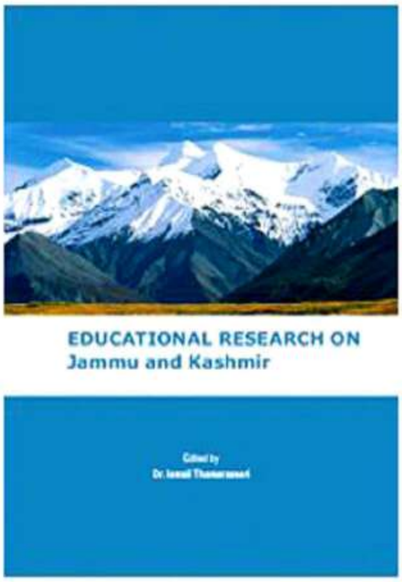

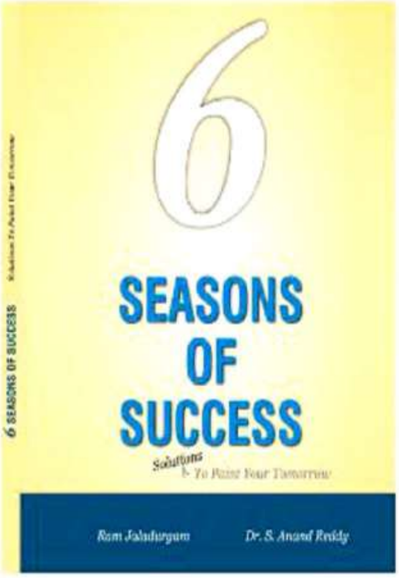

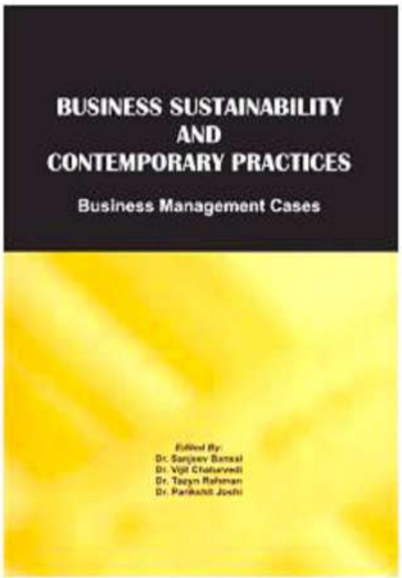

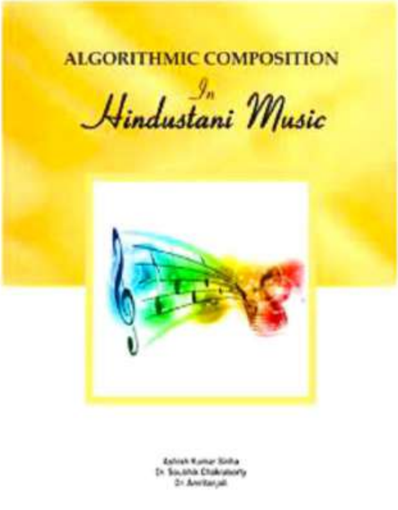

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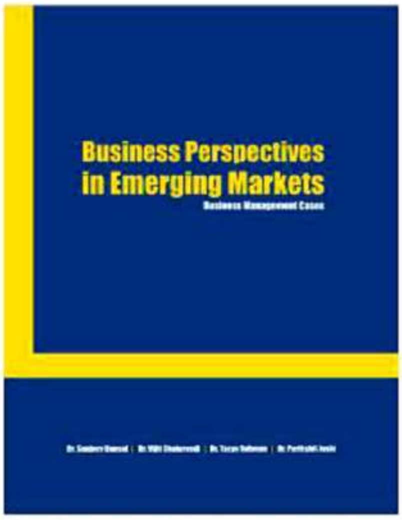

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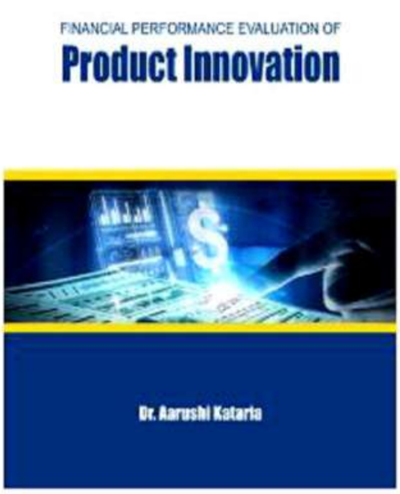

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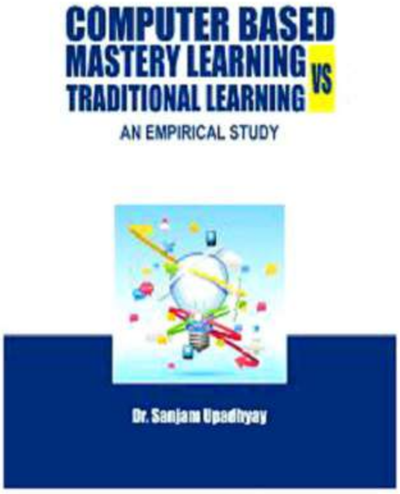

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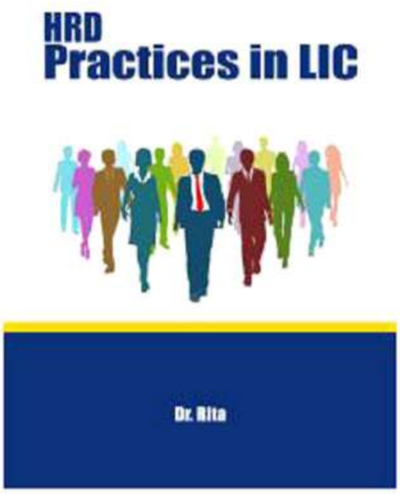

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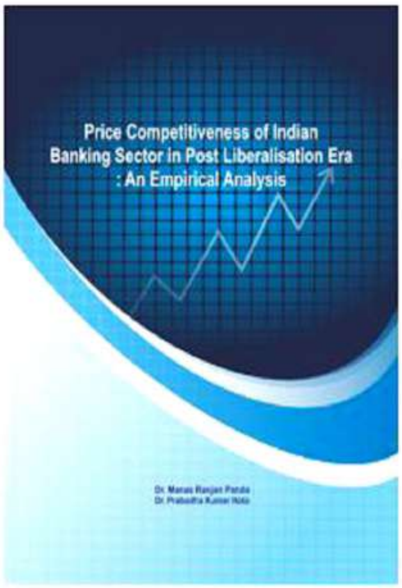

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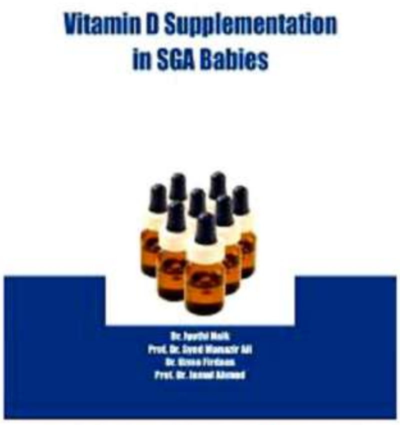

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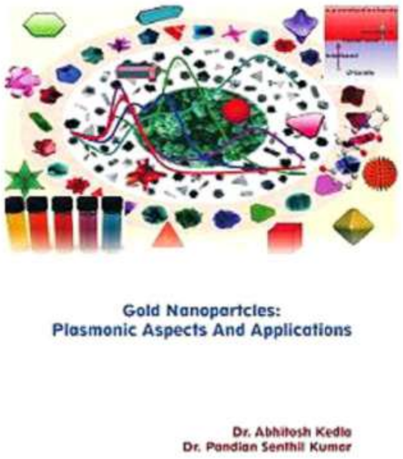

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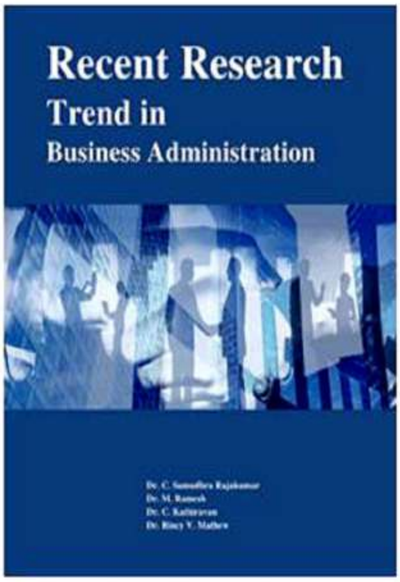

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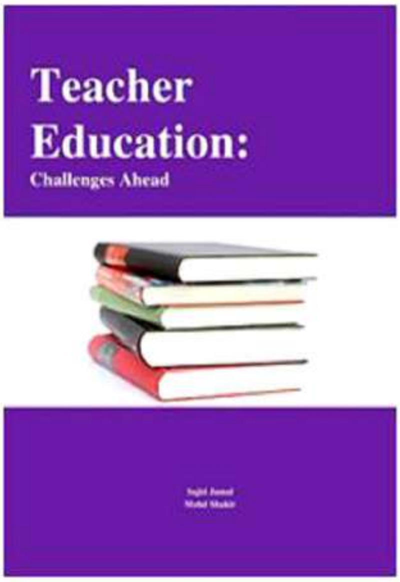

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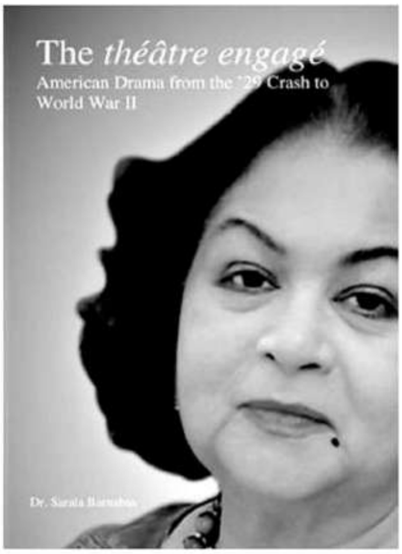

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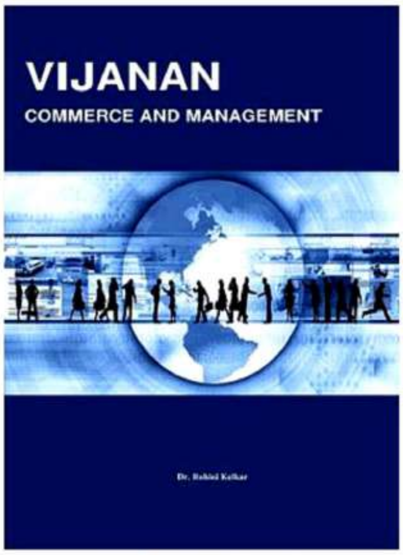

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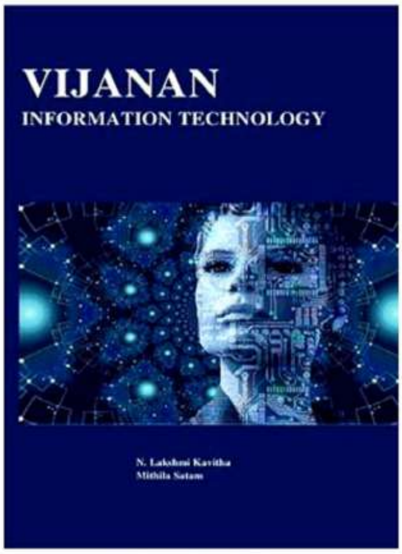

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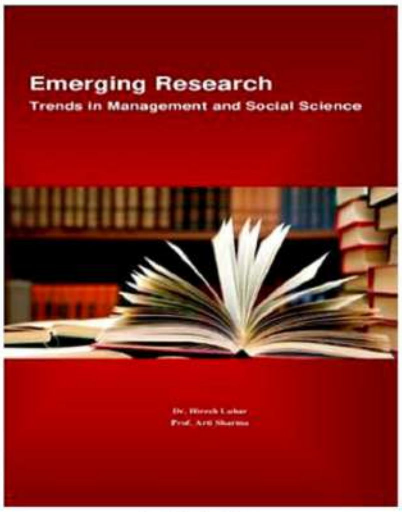

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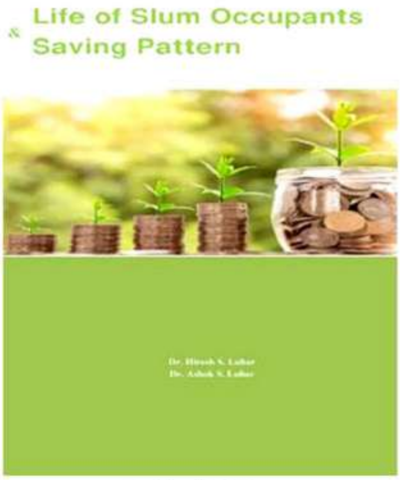

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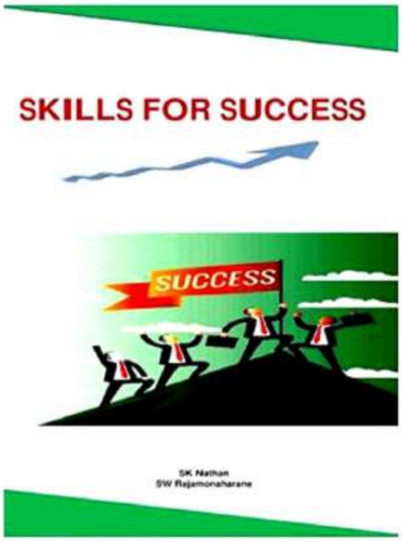

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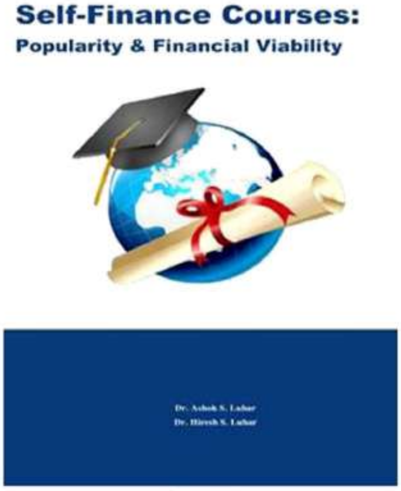

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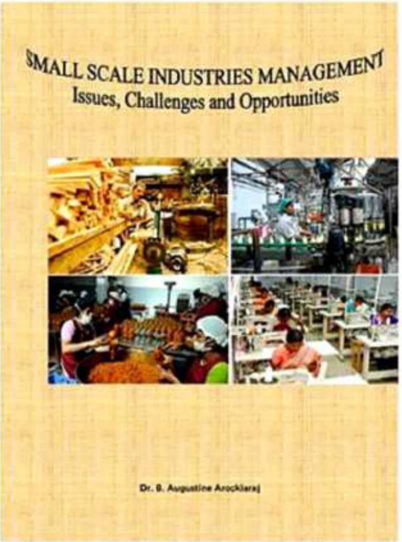

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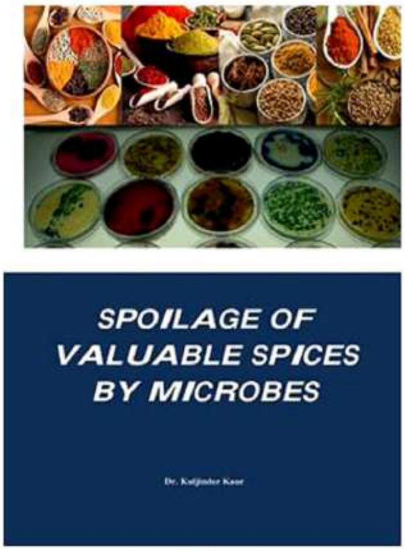

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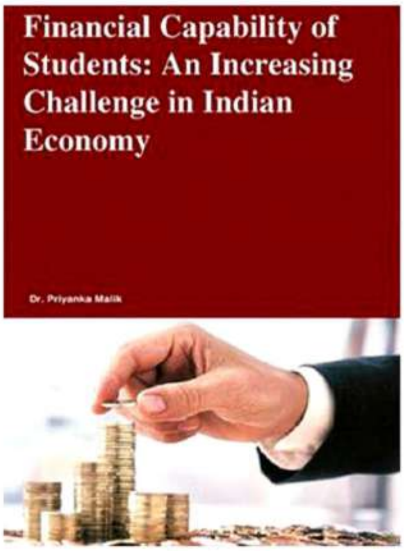

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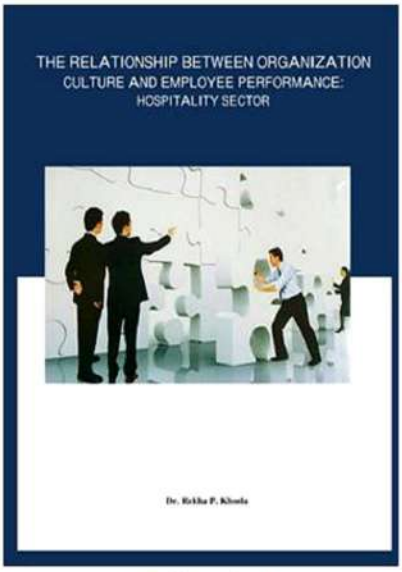

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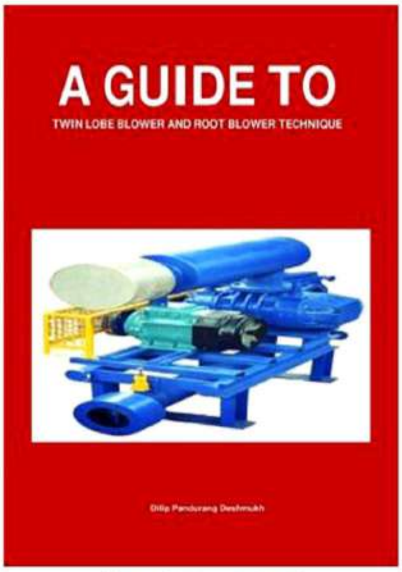

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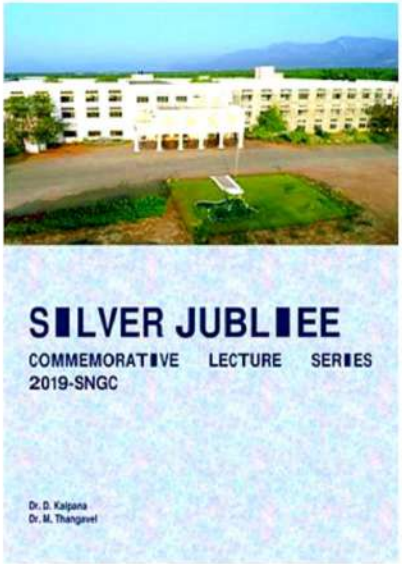

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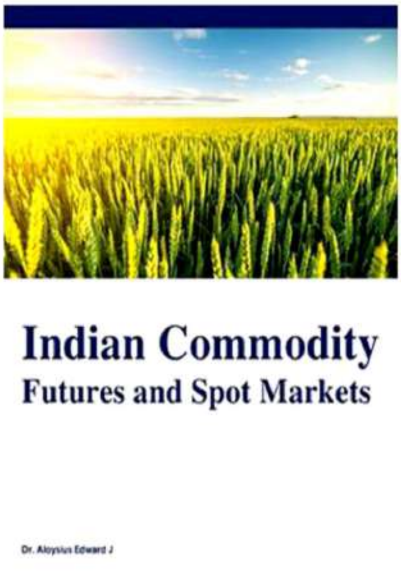
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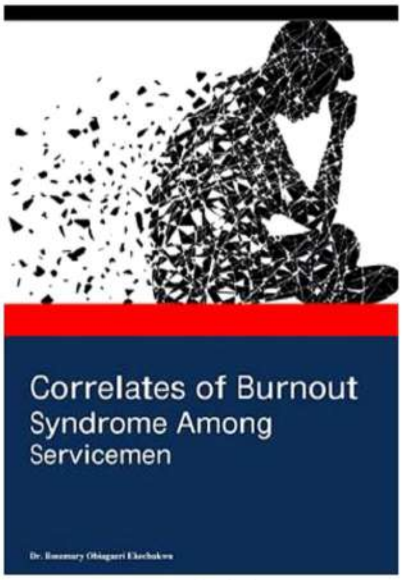
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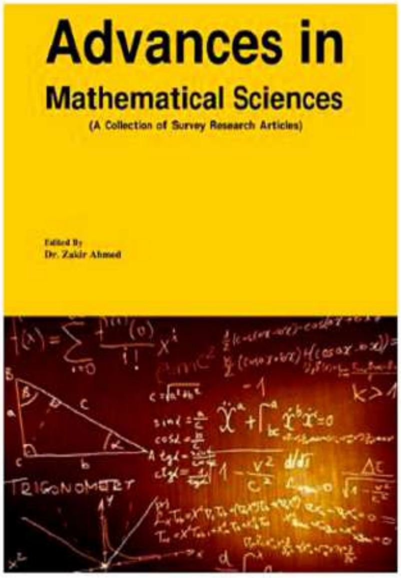
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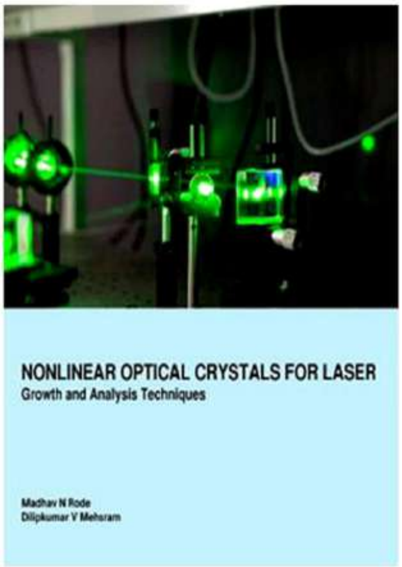
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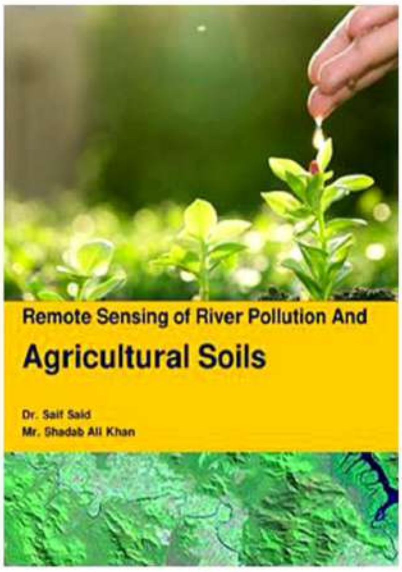
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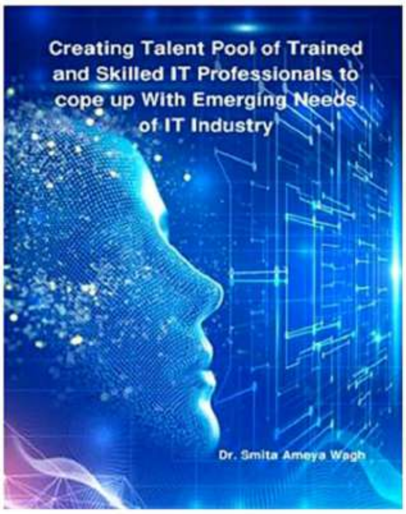
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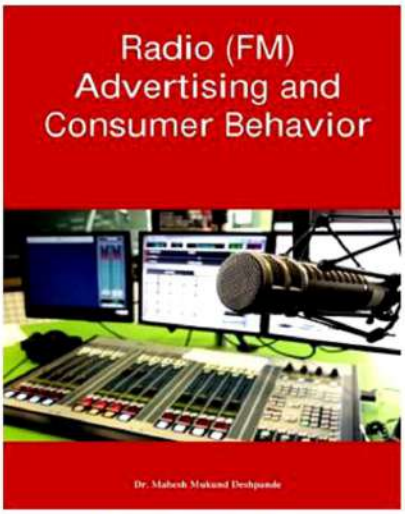
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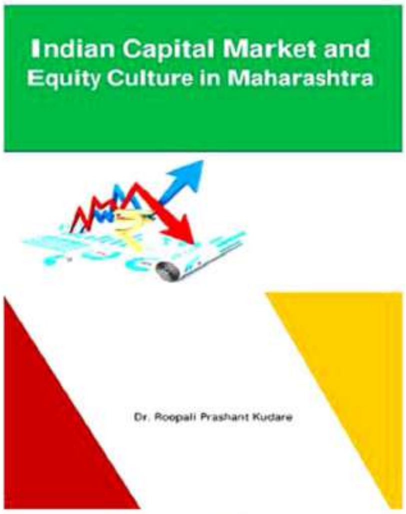
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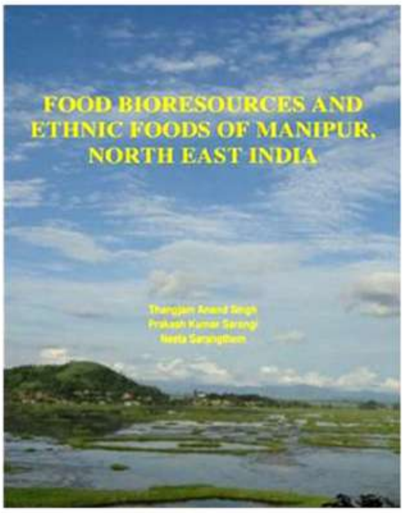

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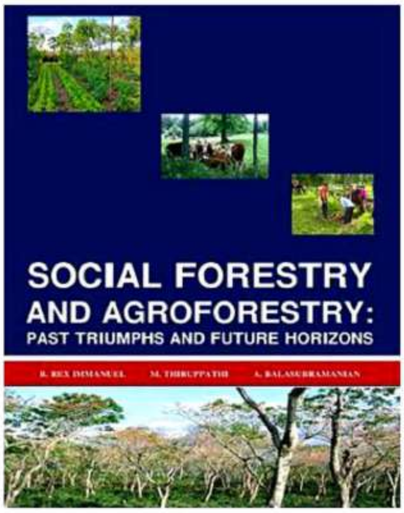

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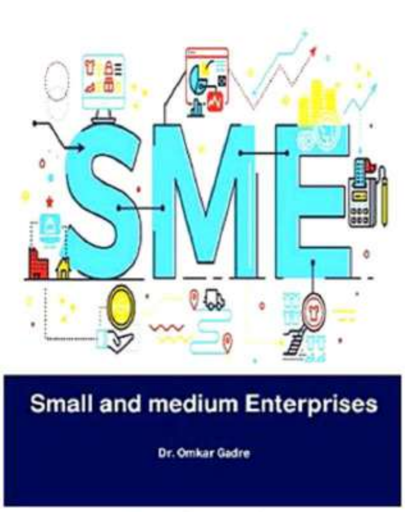

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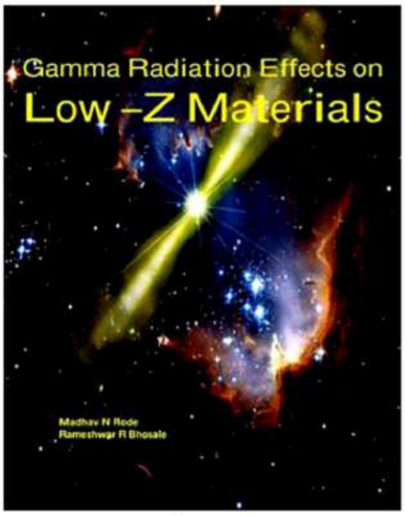

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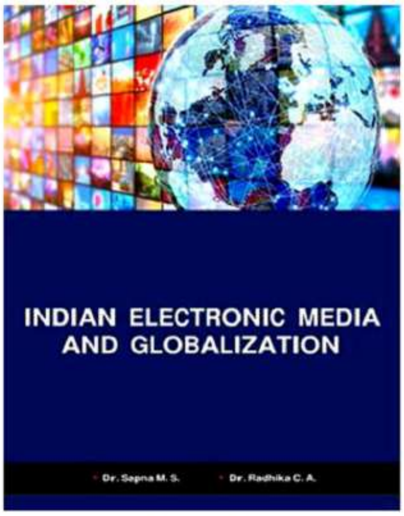

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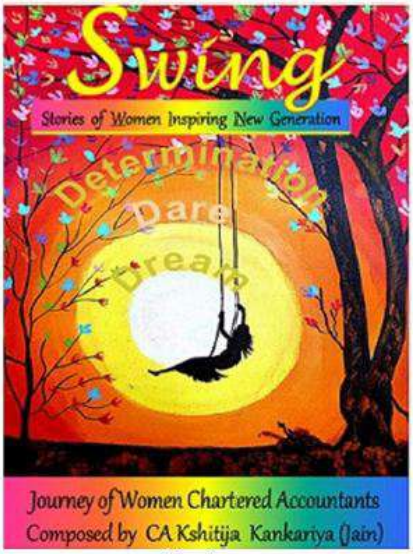

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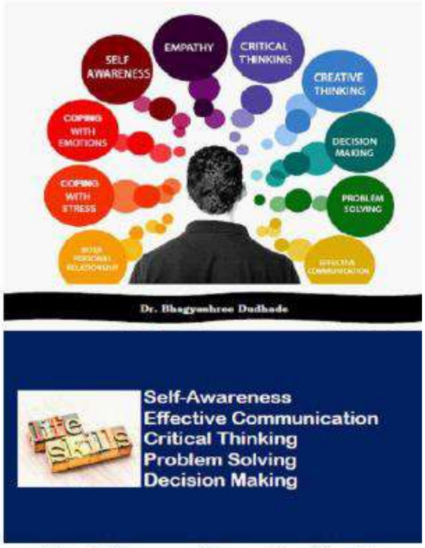

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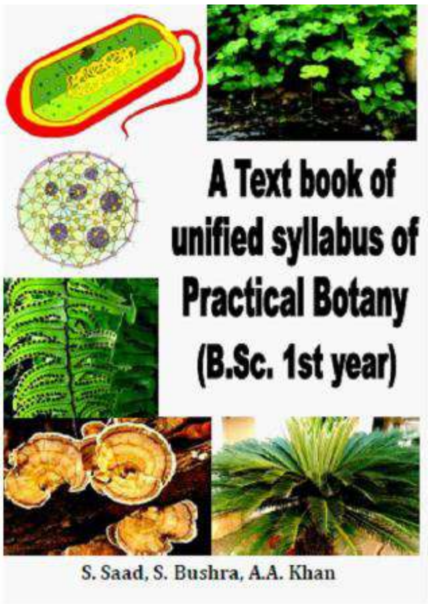

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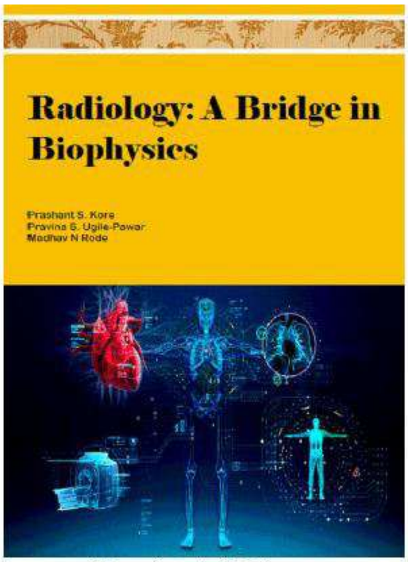

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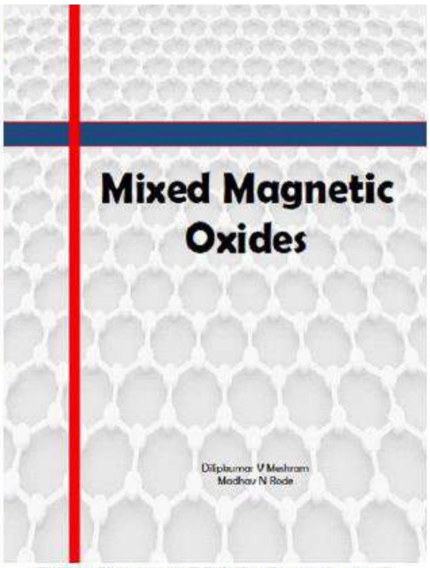

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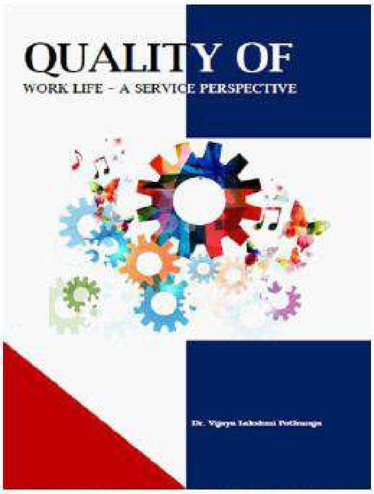

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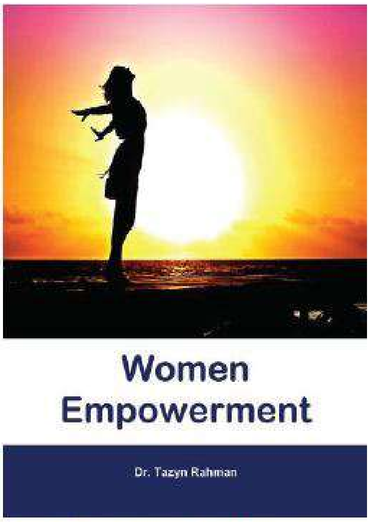

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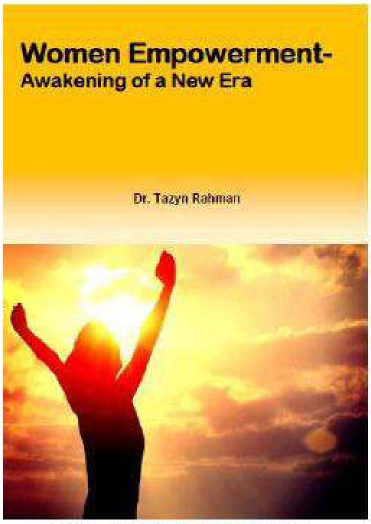

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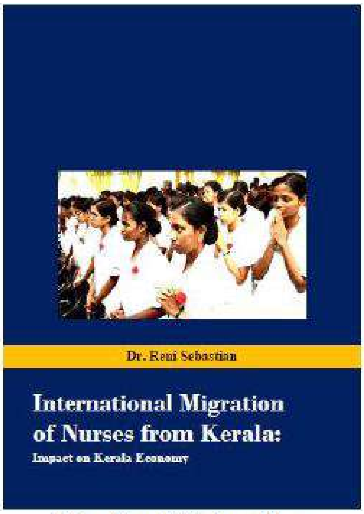

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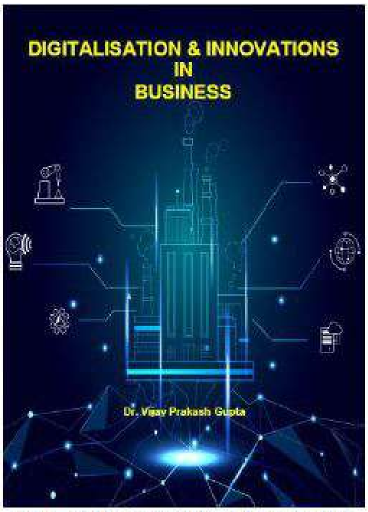

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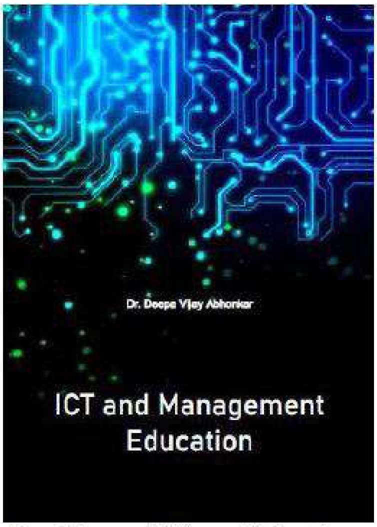

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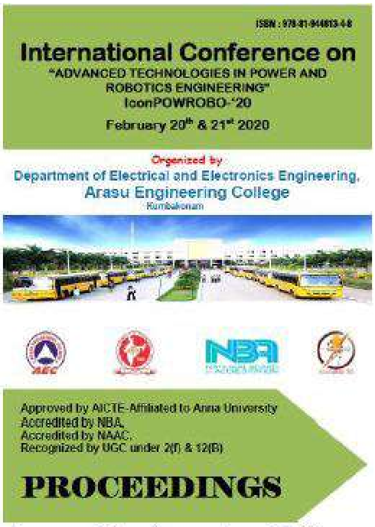

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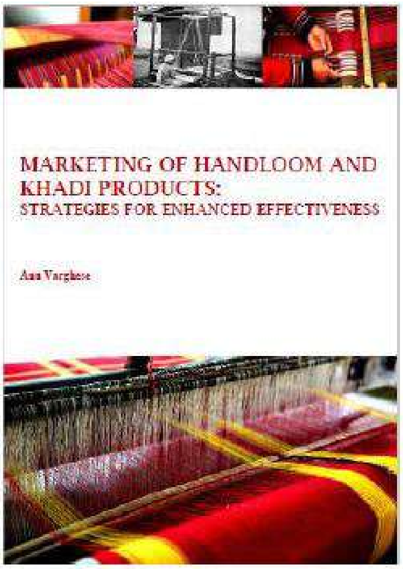

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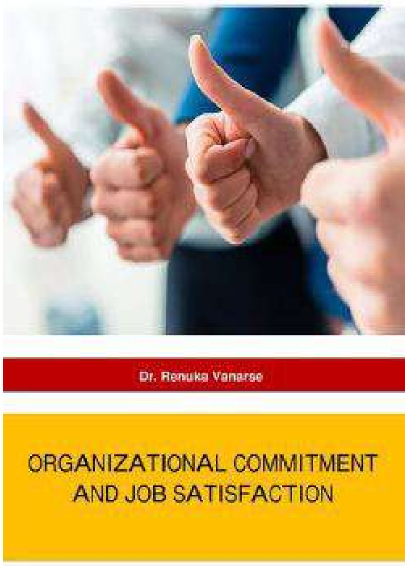

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